AN APPLICATION OF A PISIER FACTORIZATION THEOREM TO THE PETTIS INTEGRAL

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There are several generalizations of the space $L_1(\mathbf{R})$ of Lebesgue integrable functions taking values in the real numbers $\mathbf{R}$ (and defined on the usual Lebesgue measure space (Ω, Σ, μ) on $[0,1]$) to a space of strongly-measurable "integrable" (suitably formulated) functions taking values in a Banach space $\mathfrak{X}$.

The most common generalization is the space $L_1(\mathfrak{X})$ of Bochner-Lebesgue integrable functions. Using the fact [P1, Theorem 1.1] that a strongly-measurable function is essentially separably-valued, one can easily extend Lebesgue's Differentiation Theorem from $L_1(\mathbf{R})$ to $L_1(\mathfrak{X})$. Specifically [B; cf. DU, Theorem II.2.9], if $f \in L_1(\mathfrak{X})$, then

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} \|f(\omega) - f(t)\| \; d\mu(\omega) \;=\; 0$$

and so

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} f(\omega)\, d\mu(\omega) \;=\; f(t)$$

for almost all t in Ω.

Another generalization of $L_1(\mathbf{R})$ is the space $\mathcal{P}_1(\mathfrak{X})$ of strongly-measurable Pettis integrable functions. A function $f : \Omega \to \mathfrak{X}$ is Pettis integrable if for each $E \in \Sigma$ there is an element $x_E \in \mathfrak{X}$ satisfying

$$x^*(x_E) \;=\; \int_E x^* f \, d\mu$$

for each x^* in the dual space $\mathfrak{X}^*$ of $\mathfrak{X}$. The element x_E is called the Pettis integral of f over E and we write

$$\mathcal{P} - \int_E f \, d\mu \;=\; x_E \;.$$

It is clear that $L_1(\mathfrak{X}) \subset \mathcal{P}_1(\mathfrak{X})$, while the reverse inclusion holds if and only if $\mathfrak{X}$ is finite dimensional (see e.g. [DG]).

If $f \in \mathcal{P}_1(\mathfrak{X})$, then for each $x^* \in \mathfrak{X}^*$ the function $x^* f \in L_1(\mathbf{R})$ and so there exists a set A (which depends on x^*) of full measure such that

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} x^* f(\omega)\, d\mu(\omega) \;=\; x^* f(t)$$

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for each $t \in A$. In his paper [P1] introducing the Pettis integral, Pettis phrased this by saying that the Pettis integral of a function in $\mathcal{P}_1(\mathfrak{X})$ is *pseudo-differentiable*. He closed his paper by asking whether the Pettis integral of a function f in $\mathcal{P}_1(\mathfrak{X})$ enjoys the stronger property of being *a.e. weakly differentiable*; that is, does there necessarily exist a set A (*independent* of x^*) of full measure such that

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} x^* f(\omega)\, d\mu(\omega) \;=\; x^* f(t)$$

for each $t \in A$ and $x^* \in \mathfrak{X}^*$, or such that (which is the same thing of course)

$$\text{weak} - \lim_{h\to 0} \frac{1}{h}\, \mathcal{P} - \int_t^{t+h} f(\omega)\, d\mu(\omega) \;=\; f(t)$$

for each $t \in A$.

If $\mathfrak{X}$ is finite dimensional, then the Pettis integral of a function in $\mathcal{P}_1(\mathfrak{X})$ is a.e. weakly differentiable. R.S. Phillips [Ph] (for $\mathfrak{X} = \ell_2$) and M.E. Munroe [M] (for $\mathfrak{X} = C[0,1]$) each constructed an example of a function in $\mathcal{P}_1(\mathfrak{X})$ whose Pettis integral is not a.e. weakly differentiable. G.E.F. Thomas [T, p. 131] conjectured that such a function in $\mathcal{P}_1(\mathfrak{X})$ exists for every infinite-dimensional Banach space $\mathfrak{X}$.

At the recent May 1993 Kent State University Functional Analysis Conference, Joe Diestel requested a further investigation into Pettis's question. Independently, V. Kadets [K] recently constructed, for each infinite-dimensional Banach space $\mathfrak{X}$, a function in $\mathcal{P}_1(\mathfrak{X})$ whose Pettis integral fails to be a.e. weakly differentiable; specifically, it fails to be weakly differentiable on a set of positive, but not full, measure.

The first theorem of this paper constructs, for each infinite-dimensional Banach space $\mathfrak{X}$, a function in $\mathcal{P}_1(\mathfrak{X})$ whose Pettis integral is *nowhere* weakly differentiable. This theorem also addresses the degree of nondifferentiability of the Pettis integral. Our second theorem shows, for *arbitrary* Banach spaces, that the functions which we construct are close to being optimal with respect to their degree of nondifferentiability. The proof of this theorem uses a factorization theorem of Pisier [P]. From these two theorems it follows (Corollaries 3 and 4) that the cotype of a space is closely tied to the degree of nondifferentiability of the Pettis integral.

Theorem 2 was shown to us by Nigel Kalton in answer to a question posed in a preliminary version of this paper. We are grateful to him for permission to include this result here.

To state our first result we introduce the collection Ψ of all increasing functions $\psi\colon\ [0,\infty) \to [0,\infty)$ satisfying the growth condition

$$\sum_{n=1}^{\infty} \psi(2^{-p_{n-1}})\, \sqrt{2^{p_n}} \;<\; \infty\,, \tag{$\dagger$}$$

for some increasing sequence $\{p_n\}_{n=0}^{\infty}$ of integers. Examples of functions in Ψ are

$$\psi(s) = s^{\frac{1}{2}+\epsilon},$$

$$\psi(s) = s^{\frac{1}{2}} \left[\frac{1}{\log(1/s)}\right]^{1+\epsilon} \qquad \text{and} \qquad \psi(s) = s^{\frac{1}{2}} \left[\frac{1}{\log(1/s)}\right] \left[\frac{1}{\log\log(1/s)}\right]^{1+\epsilon}$$

for $p_n = n$ and any $\epsilon > 0$.

Theorem 1. *Let $\mathfrak{X}$ be an infinite-dimensional Banach space. For each $\psi \in \Psi$, there exists $f \in \mathcal{P}_1(\mathfrak{X})$ such that*

$$\left\| \mathcal{P} - \int_I f \, d\mu \right\|_{\mathfrak{X}} \geqslant \psi(\mu(I)) \tag{‡}$$

for each interval I contained in $[0,1]$.

Remark. Taking $\psi(t) = t^{\frac{3}{4}}$ gives a Pettis integrable function f such that for each $t \in \Omega$,

$$\lim_{h \to 0} \left\| \frac{1}{h}\, \mathcal{P} - \int_t^{t+h} f(\omega)\, d\mu(\omega) \right\|_{\mathfrak{X}} = \infty .$$

If the Pettis integral of this f were weakly differentiable at t, then the above limit would be finite.

Sketch of Proof. Let $\{I_k^n : n = 0, 1, \ldots ; k = 1, \ldots, 2^n\}$ be the dyadic intervals on $[0,1]$, i.e.

$$I_k^n = \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right) .$$

Define inductively a collection $\{A_k^n : n = 0, 1, \ldots ; k = 1, \ldots, 2^n\}$ of disjoint sets of strictly positive measure such that $A_k^n \subset I_k^n$ (e.g. appropriately chosen "fat Cantor" sets).

Fix $K > 1$. By a theorem of Mazur there is a basic sequence $\{x_n\}$ in $\mathfrak{X}$ with basis constant at most K. Take a blocking $\{F_n\}$ of the basis with each subspace F_n of large enough dimension to find (using the finite-dimensional version of Dvoretzky's Theorem [D]) a 2^n-dimensional subspace E_n of F_n such that the Banach-Mazur distance between E_n and $\ell_2^{2^n}$ is less than 2. Note that $\{E_n\}$ forms a finite-dimensional decomposition. Next find operators $T_n \colon \ell_2^{2^n} \to E_n$ such that $\|T_n\| \leqslant 2$ and $\|T_n^{-1}\| = 1$. Let $\{u_k^n \colon k = 1, \ldots 2^n\}$ be the standard unit vectors of $\ell_2^{2^n}$ and let $e_k^n \equiv T_n u_k^n$.

By the growth condition (†) on ψ, there is an increasing sequence $\{p_n\}_{n=0}^{\infty}$ of integers, with $p_0 = 0$, satisfying

$$\sum_{n=1}^{\infty} \psi(4 \cdot 2^{-p_{n-1}}) \sqrt{2^{p_n}} < \infty .$$

Define $f\colon\ [0,1] \to \mathfrak{X}$ by

$$f(\omega) \;=\; \sum_{n=1}^{\infty}\sum_{k=1}^{2^n} \; c_n \; \frac{1_{A_k^n}(\omega)}{\mu(A_k^n)} \; e_k^n \;,$$

where

$$c_m \;=\; 2K \; \left[\psi\left(4 \;\cdot\; 2^{-p_{n-1}}\right)\right] \cdot \; \delta_{m,p_n} \;,$$

(here $\delta_{j,k}$ is the usual Kronecker delta symbol). Clearly, f is strongly measurable.

It is straightforward to verify that the Pettis integral of f is

$$\mathcal{P} - \int_E f\,d\mu \;=\; \sum_{n=1}^{\infty}\sum_{k=1}^{2^n} \; c_n \; \left(\int_E \frac{1_{A_k^n}}{\mu(A_k^n)}\,d\mu\right) \; e_k^n \;. \tag{$*$}$$

Next fix an interval $I \in \Sigma$. Find a dyadic interval $I_j^m \subset I$ such that $4\;\mu(I_j^m) \geqslant \mu(I)$ and then find n such that $p_{n-1} \leqslant m < p_n$. Let P be the natural projection from $\sum \oplus E_j$ onto E_{p_n}. It follows that

$$2K \; \left\|\mathcal{P} - \int_I f\,d\mu\right\|_{\mathfrak{X}} \;\geqslant\; c_{p_n} \left[\sum_{k=1}^{2^{p_n}} \left|\int_I \frac{1_{A_k^{p_n}}}{\mu(A_k^{p_n})}\,d\mu\right|^2\right]^{\frac{1}{2}} \;,$$

and so since $A_k^{p_n} \subset I_k^{p_n} \subset I_j^m \subset I$ for some k,

$$2K \; \left\|\mathcal{P} - \int_I f\,d\mu\right\|_{\mathfrak{X}} \;\geqslant\; c_{p_n} \;=\; 2K \; \psi\left(4 \;\cdot\; 2^{-p_{n-1}}\right) \;.$$

But ψ is increasing and $4 \;\cdot\; 2^{-p_{n-1}} \;\geqslant\; 4 \;\cdot\; 2^{-m} \;\geqslant\; \mu(I)$ and so (‡) holds. Thus f satisfies the statement of the theorem. ■

The functions in Ψ can be viewed as indicators of the degree of nondifferentiability (i.e. the poor "averaging behavior") of the indefinite Pettis integral. For instance, taking

$$\psi(s) = s^{\frac{1}{2}} \left[\frac{1}{\log\left(1/s\right)}\right]^{1+\epsilon} ,$$

we deduce from Theorem 1 that there exists $f \in \mathcal{P}_1(\mathfrak{X})$ such that, not only do we have

$$\lim_{h\to 0} \; \left\|\frac{1}{h}\;\mathcal{P} - \int_t^{t+h} f(\omega)\,d\mu(\omega)\right\|_{\mathfrak{X}} \;=\; \infty \;,$$

but even worse,

$$\lim_{h\to 0} \; h^{\frac{1}{2}} \;\cdot\; \left[\log\left(\frac{1}{h}\right)\right]^{1+\epsilon} \; \left\|\frac{1}{h}\;\mathcal{P} - \int_t^{t+h} f(\omega)\,d\mu(\omega)\right\|_{\mathfrak{X}} \;=\; \infty$$

for all $t \in \Omega$.

The next theorem shows that Theorem 1 comes close to describing the *worst* type of averaging behavior of the Pettis integral that can occur in an *arbitrary* infinite-dimensional Banach space. In particular, it shows that, for spaces on which the identity operator is $(2,1)$-summing (i.e., spaces with the Orlicz property), Theorem 1 fails to hold for the function $\psi(s) = s^{\frac{1}{2}}$. Thus, the growth condition (†) on $\psi \in \Psi$ can *not* be replaced by $\psi(s) = O(s^{\frac{1}{2}})$ as $s \to 0$. We do not know, however, whether it can be replaced by $\psi(s) = o(s^{\frac{1}{2}})$ as $s \to 0$.

Theorem 2. *If the identity operator on an infinite-dimensional Banach space $\mathfrak{X}$ is $(q,1)$-summing for some $2 \leqslant q < \infty$, then, for every $f \in \mathcal{P}_1(\mathfrak{X})$,*

$$\left\| \mathcal{P} - \int_t^{t+h} f \, d\mu \right\|_{\mathfrak{X}} = o\left(h^{\frac{1}{q}}\right)$$

as $h \to 0^+$ for μ-a.e. t.

The proof below, which uses a factorization theorem of Pisier [P], was pointed out to us by Nigel Kalton.

Sketch of Proof. Fix $f \in \mathcal{P}_1(\mathfrak{X})$ for an infinite-dimensional Banach space $\mathfrak{X}$. Consider the operator $K \colon L_\infty \to \mathfrak{X}$ given by

$$K(g) = \mathcal{P} - \int_\Omega g(\omega) f(\omega) \, d\mu(\omega) \; .$$

We need to show that

$$\left\| K\left(1_{[0,t+h]}\right) - K\left(1_{[0,t]}\right) \right\|_{\mathfrak{X}} = o\left(h^{\frac{1}{q}}\right)$$

as $h \to 0^+$ for μ-a.e. t. Fix $\epsilon > 0$.

Since K is compact and since the dual of L_∞ has the approximation property, there is [e.g. DU, Thm. VIII.3.6] a decomposition $K = K_1 + K_2$, with $K_i \in \mathcal{L}\left(L_\infty, \mathfrak{X}\right)$, such that K_1 has finite rank and K_2 has norm at most ϵ^2. It is enough to show that there is some constant A, which depends only on $\mathfrak{X}$ and q, such that for each i,

$$\limsup_{h \to 0^+} \; h^{-\frac{1}{q}} \; \left\| K_i\left(1_{[0,t+h]}\right) - K_i\left(1_{[0,t]}\right) \right\|_{\mathfrak{X}} \leqslant A \, \epsilon \; , \qquad (\lozenge)$$

on a set of μ-measure at least $1 - \epsilon^q$.

Towards this, consider [see e.g. R] the natural surjective isometry $\tau \colon L_\infty \to C(\Delta)$ for the appropriate extremally disconnected compact Hausdorff space Δ. Recall that τ takes an indicator function of a Borel set in $[0,1]$ to an indicator function of a clopen set in Δ, say $\tau(1_A) = 1_{\widehat{A}}$ in such a way that if $A \subset B \subset \Omega$, then $\widehat{A} \subset \widehat{B} \subset \Delta$ and $\widehat{B \setminus A} = \widehat{B} \setminus \widehat{A}$. Let $\widehat{K}_i$ be the composite map:

$$\widehat{K}_i \quad : \quad C(\Delta) \quad \xrightarrow{\tau^{-1}} \quad L_\infty \quad \xrightarrow{K_i} \quad \mathfrak{X} \; .$$

It can be shown that K_1 satisfies

$$\left\| K_1\left(1_{[0,t+h]}\right) - K_1\left(1_{[0,t]}\right) \right\|_{\mathfrak{X}} \; = \; O(h) \qquad \mu\text{–a.e.}$$

and so $(\lozenge)$ holds for any $q > 1$.

Now we deal with K_2. Fix $2 \leqslant q < \infty$. If the identity operator on $\mathfrak{X}$ is $(q,1)$-summing, then [P, Cor. 2.7] there is a probability measure ν on the Borel sets of Δ such that the operator $\widehat{K}_2$ admits a factorization of the form

$$\begin{tikzcd} C(\Delta) \arrow[rr, "\widehat{K}_2"] \arrow[dr, "J"'] & & \mathfrak{X} \\ & L_{q,1}(\nu) \arrow[ur, "T"'] & \end{tikzcd}$$

where J is the natural inclusion map and T is a bounded linear operator with operator norm at most $C\|\widehat{K}_2\| \leq C\epsilon^2$, where C depends only on $\mathfrak{X}$ and q. Here, $L_{q,1}(\nu)$ is the usual Lorentz space of all real-valued ν-measurable functions f on Δ for which the norm $\|f\|_{q,1}$ is finite, where

$$\|f\|_{q,1} \; = \; \int_0^\infty t^{\frac{1}{q}-1} f^*(t)\, dt$$

and f^* is the non-increasing rearrangement of $|f|$. As above

$$\begin{aligned} \left\| K_2\left(1_{[0,t+h]}\right) - K_2\left(1_{[0,t]}\right) \right\|_{\mathfrak{X}} \; &= \; \left\| K_2\left(1_{(t,t+h]}\right) \right\|_{\mathfrak{X}} \\ &= \; \left\| \widehat{K}_2\left(1_{\widehat{(t,t+h]}}\right) \right\|_{\mathfrak{X}} \\ &\leqslant \; C\epsilon^2 \left\| J\left(1_{\widehat{(t,t+h]}}\right) \right\|_{L_{q,1}(\nu)} \; . \end{aligned}$$

Since the non-increasing rearrangement of $J\left(1_{\widehat{(t,t+h]}}\right)$ is just the indicator function of the set $\left[0, \nu\left(\widehat{(t,t+h]}\right)\right)$, we have

$$\left\| J\left(1_{\widehat{(t,t+h]}}\right) \right\|_{L_{q,1}(\nu)} \; = \; q \; \left[\nu\left(\widehat{(t,t+h]}\right)\right]^{\frac{1}{q}} \; ,$$

and so

$$h^{-\frac{1}{q}} \left\| K_2\left(1_{[0,t+h]}\right) - K_2\left(1_{[0,t]}\right) \right\|_{\mathfrak{X}} \; \leqslant \; Cq\epsilon^2 \left[\frac{|\beta(t+h) - \beta(t)|}{h}\right]^{\frac{1}{q}} \; ,$$

where $\beta\colon [0,1] \to \mathbb{R}$ is given by $\beta(t) = \nu\left(\widehat{[0,t]}\right)$. The function β is increasing and hence differentiable μ-almost everywhere. Thus

$$\limsup_{h \to 0^+} \; h^{-\frac{1}{q}} \; \left\| K_2\left(1_{[0,t+h]}\right) - K_2\left(1_{[0,t]}\right) \right\|_{\mathfrak{X}} \;\leqslant\; C \; q \; \epsilon^2 \;\; [\beta'(t)]^{\frac{1}{q}}$$

for μ-a.e. t. From $\int_0^1 \beta'(t)\, dt \leqslant \beta(1) - \beta(0) \leqslant 1$, it follows that $\mu\left[\beta'(t) \geqslant \epsilon^{-q}\right] \leqslant \epsilon^q$. Thus, on a set of measure at least $1 - \epsilon^q$,

$$\limsup_{h \to 0^+} \; h^{-\frac{1}{q}} \; \left\| K_2\left(1_{[0,t+h]}\right) - K_2\left(1_{[0,t]}\right) \right\|_{\mathfrak{X}} \;\leqslant\; C \; q \; \epsilon \; ,$$

which implies ($\Diamond$) for K_2. ■

Recall that the identity operator on a space with finite cotype q is $(q,1)$-summing. Indeed, cotype plays a major rôle in the unfolding drama. To see this, consider a space $\mathfrak{X}$ which contains a finite-dimensional decomposition $\sum \oplus E_n$ where the Banach-Mazur distance between E_n and $\ell_p^{2^n}$ is less than M for each n for some fixed $1 \leqslant p \leqslant \infty$ and $M > 1$. By modifying Mazur's construction [see e. g. LT] of a basic sequence and using the fact (a simple compactness argument suffices) that finite representability of ℓ_p is inherited by subspaces of finite codimension, it is possible to construct such a finite-dimensional decomposition in $\mathfrak{X}$ whenever ℓ_p is finitely representable in $\mathfrak{X}$. By the Maurey-Pisier Theorem [MP], ℓ_{q_0} is finitely representable in $\mathfrak{X}$ where $2 \leqslant q_0 \leqslant \infty$ and

$$q_0 \;=\; \inf\,\{q\colon \mathfrak{X} \text{ has cotype } q\} \; .$$

In the same spirit as in the proof of Theorem 1 (and with similar notation), for $1 \leqslant p \leqslant \infty$ let Ψ_p be the collection of all increasing functions $\psi\colon\; [0,\infty) \to [0,\infty)$ satisfying the growth condition

$$\sum_{n=1}^{\infty} \psi\left(2^{-p_{n-1}}\right) \;\; [2^{p_n}]^{\frac{1}{p}} \;<\; \infty \qquad (\dagger_p)$$

for some increasing sequence $\{p_n\}_{n=0}^{\infty}$ of integers (following the convention that $1/\infty$ is 0). For $1 \leqslant p < \infty$, a typical function in Ψ_p is

$$\psi(s) = s^{\frac{1}{p}+\epsilon}$$

with $p_n = n$ and for any $\epsilon > 0$. For $p = \infty$, ($\dagger_p$) reduces to the condition

$$\lim_{s \to 0^+} \psi(s) = 0 \; .$$

Fix $\psi \in \Psi_p$ and find an increasing sequence $\{p_n\}_{n=0}^{\infty}$ of integers, with $p_0 = 0$, satisfying

$$\sum_{n=1}^{\infty} \psi(4 \;\cdot\; 2^{-p_{n-1}}) \;\; [2^{p_n}]^{\frac{1}{p}} \;<\; \infty$$

(again, $1/\infty$ is 0). Define $f\colon\ [0,1] \to \mathfrak{X}$ by

$$f(\omega) \ = \ \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} \ c_n \ \frac{1_{A_k^n}(\omega)}{\mu(A_k^n)} \ e_k^n \ ,$$

where

$$c_m \ = \ 2K \ \left[\psi\left(4 \ \cdot \ 2^{-p_{n-1}}\right)\right] \cdot \ \delta_{m,p_n} \ ,$$

where K is the finite-dimensional decomposition constant. Minor variations of the proof of Theorem 1 show that this function f satisfies

$$\left\|\mathcal{P} - \int_I f \, d\mu\right\|_{\mathfrak{X}} \ \geqslant \ \psi\left(\mu\left(I\right)\right)$$

for each interval I contained in $[0,1]$.

Theorems 1 and 2, along with the above observations, give the following corollaries.

Corollary 3. *Let $\mathfrak{X}$ be an infinite-dimensional Banach space with finite cotype and let $q_0 \ = \ \inf\{q\colon \mathfrak{X}_0$ has cotype $q\}$. Then the following hold.*

(1) *If $p > q_0$, then for each $f \in \mathcal{P}_1(\mathfrak{X})$, we have*

$$\left\|\mathcal{P} - \int_t^{t+h} f \, d\mu\right\|_{\mathfrak{X}} \ = \ o\left(h^{\frac{1}{p}}\right)$$

as $h \to 0^+$ for μ-a.e. t.

(2) *If $p < q_0$, then there is an $f \in \mathcal{P}_1(\mathfrak{X})$ such that*

$$\left\|\mathcal{P} - \int_t^{t+h} f \, d\mu\right\|_{\mathfrak{X}} \ \geqslant \ h^{\frac{1}{p}}$$

for all $t \in [0,1]$.

Corollary 4. *For an infinite-dimensional Banach space $\mathfrak{X}$, the following are equivalent.*

(1) *$\mathfrak{X}$ fails cotype.*

(2) *For each $\psi \in \Psi_\infty$, there exists $f \in \mathcal{P}_1(\mathfrak{X})$ such that*

$$\left\|\mathcal{P} - \int_I f \, d\mu\right\|_{\mathfrak{X}} \ \geqslant \ \psi\left(\mu\left(I\right)\right)$$

for each interval I contained in $[0,1]$.

Remark. Note that Corollary 4 proves the existence of a *reflexive* Banach space for which the Pettis integral has essentially no kind of differentiability property whatsoever.

For further details and results along these lines, we refer the interested reader to [DG2].

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