FROM WEAK TO STRONG TYPES OF $\mathcal{L}^1_E$-CONVERGENCE BY THE BOCCE-CRITERION

Erik J. Balder   Maria Girardi*   Vincent Jalby**

14 February 1994

Studia Math. **111** *(1994) 241–262*

Abstract. Necessary and sufficient oscillation conditions are given for a weakly convergent sequence (resp. relatively weakly compact set) in the Bochner-Lebesgue space $\mathcal{L}^1_E$ to be norm convergent (resp. relatively norm compact), thus extending the known results for $\mathcal{L}^1_{\mathbb{R}}$. Similarly, necessary and sufficient oscillation conditions are given to pass from weak to limited (and also to Pettis-norm) convergence in $\mathcal{L}^1_E$. It is shown that tightness is a necessary and sufficient condition to pass from limited to strong convergence. Other implications between several modes of convergence in $\mathcal{L}^1_E$ are also studied.

1. Introduction

Vaguely speaking, a relatively weakly compact set in $\mathcal{L}^1_{\mathbb{R}}$ is relatively norm compact if the functions in the set do not oscillate too much. Specifically, a relatively weakly compact subset of $\mathcal{L}^1_{\mathbb{R}}$ is relatively norm compact if and only if it satisfies the Bocce criterion (an oscillation condition) [G1, G2]. However, the set of constant functions of norm at most one in $\mathcal{L}^1_E$ already shows that (for a reflexive infinite-dimensional Banach space E), in the Bochner-Lebesgue space $\mathcal{L}^1_E$, more care is needed in order to pass from weak to strong compactness. In Section 2, we extend from $\mathcal{L}^1_{\mathbb{R}}$ to $\mathcal{L}^1_E$ the above weak-to-norm result, along with the sequential analogue. In Section 3, limited convergence (a weakening of strong convergence [B1,B2]) is examined. Limited convergence provides an extension of the Lebesgue Dominated Convergence Theorem to $\mathcal{L}^1_E$. Necessary and sufficient conditions to pass from weak to limited convergence are given. In Section 4, the concept of tightness helps to extend the results from the previous two sections. In Section 5, convergence in the Pettis norm, a weakening of strong convergence along lines distinct from limited convergence, is examined. Similarly, necessary and sufficient conditions to

1991 *Mathematics Subject Classification.* 28A20, 28A99, 46E30.

* Supported in part by NSF grant DMS-9204301 and DMS-9306460.

** Part of this paper is contained in the doctoral dissertation of this author, who is grateful for the guidance of his advisor Professor C. Castaing.

pass from weak to Pettis-norm convergence are given. In the study, implications between several modes of convergence on $\mathcal{L}^1_E$ are examined.

Throughout this paper $(E, \|\cdot\|)$ is a Banach space with dual E^* and B_E is the closed unit ball of E. The triple $(\Omega, \mathcal{F}, \mu)$ is a finite measure space. Without loss, we take μ to be a probability measure. For $B \in \mathcal{F}$, we often examine the collection $\mathcal{F}^+(B)$ of all measurable subsets of B with (strictly) positive measure and denote $\mathcal{F}^+(\Omega)$ by just $\mathcal{F}^+$. By $\mathcal{L}^1_E$ we denote the (prequotient) space of all Bochner μ-integrable functions from Ω into E. On this space the classical $\mathcal{L}^1_E$-seminorm is given by $\|f\|_1 := \int_\Omega \|f\| d\mu$ and convergence in this seminorm is called *strong* convergence.

Recall [IT] that the dual of $(\mathcal{L}^1_E, \|\cdot\|_1)$ is the (prequotient) space $\mathcal{L}^\infty_{E^*}[E]$ of *scalarly* measurable bounded functions from Ω into E^*. The subspace $\mathcal{L}^\infty_{E^*}$ of $\mathcal{L}^\infty_{E^*}[E]$ consisting of the *strongly* measurable functions actually coincides with $\mathcal{L}^\infty_{E^*}[E]$ if and only if E^* has the *Radon Nikodym property* (RNP); cf. [DU, IT]. Convergence in the corresponding weak topology $\sigma(\mathcal{L}^1_E, \mathcal{L}^\infty_{E^*}[E])$ is called *weak* convergence. We will also consider the $\sigma(\mathcal{L}^1_E, \mathcal{L}^\infty_{E^*})$-topology on $\mathcal{L}^1_E$.

Also recall that a subset K of $\mathcal{L}^1_E$ functions is *uniformly integrable* if

$$\lim_{c\to\infty} \sup_{f\in K} \int_{[||f||\geq c]} ||f||\, d\mu = 0 \ .$$

It is well known [N] that K is uniformly integrable if and only if it is bounded (i.e. $\sup_{f\in K} ||f||_{\mathcal{L}^1_E}$ is finite) and equi-integrable, i.e.

$$\lim_{\mu(A)\to 0^+} \sup_{f\in K} \int_A ||f||\, d\mu = 0 \ .$$

All notations and terminology, not otherwise explained, are as in [DU, IT, or N].

2. Weak vs. Strong Convergence in $\mathcal{L}^1_E$

Our goal is to determine precisely when (via an oscillation condition) a weakly convergent sequence is also strongly convergent, along with the nonsequential analogue.

For $f \in \mathcal{L}^1_E$ and $A \in \mathcal{F}$, the *average value* and the *Bocce oscillation* of f over A are (respectively)

$$m_A(f) \ := \ \frac{\int_A f\, d\mu}{\mu(A)}$$

$$\text{Bocce-osc } f\big|_A \ := \ \frac{\int_A ||f - m_A(f)||\, d\mu}{\mu(A)}$$

observing the convention that 0/0 is 0. The following elementary inequalities are useful

$$\left| \text{Bocce-osc } f\big|_A \Leftrightarrow \text{Bocce-osc } g\big|_A \right| \leq \text{Bocce-osc } (f \Leftrightarrow g)\big|_A$$

$$\text{Bocce-osc } (f+g)\big|_A \leq \text{Bocce-osc } f\big|_A + \text{Bocce-osc } g\big|_A$$

$$\mu(A) \text{ Bocce-osc } f\big|_A \leq 2 \int_A \|f\| \, d\mu \ . \tag{2.1}$$

In the spirit of [G1], we consider the following oscillation conditions.

Definition 2.1. [B3] A sequence $(f_k)_{k=1}^{\infty}$ of functions in $\mathcal{L}^1_E$ satisfies the *sequential Bocce criterion* if for each subsequence (f_{k_j}) of (f_k), each $\epsilon > 0$, and each B in $\mathcal{F}^+$ there is a set A in $\mathcal{F}^+(B)$ such that $\liminf_j \text{Bocce-osc } f_{k_j}\big|_A < \epsilon$.

Definition 2.2. [G1] A subset K of $\mathcal{L}^1_E$ satisfies the *Bocce criterion* if for each $\epsilon > 0$ and each B in $\mathcal{F}^+$ there is a finite collection $\mathcal{A}$ of sets in $\mathcal{F}^+(B)$ such that for each f in K there is a set A in $\mathcal{A}$ satisfying Bocce-osc $f\big|_A < \epsilon$.

It is known [G1, G2] that a relatively weakly compact subset of $L^1_{\mathbb{R}}$ is relatively norm compact if and only if it satisfies the Bocce criterion. We now extend to $\mathcal{L}^1_E$.

Theorem 2.3. *A sequence (f_k) in $\mathcal{L}^1_E$ converges* strongly *to f_0 in $\mathcal{L}^1_E$ if and only if*

(1) *(f_k) converges* weakly *to f_0 in $\mathcal{L}^1_E$*
(2) *(f_k) satisfies the sequential Bocce criterion*
(3) *$\Delta_B := \{m_B(f_k) : k \in N\}$ is relatively norm compact in E for each $B \in \mathcal{F}^+$.*

Condition (1) may be replaced with

(1') *(f_k) converges to f_0 in $\mathcal{L}^1_E$ in the $\sigma(\mathcal{L}^1_E, \mathcal{L}^{\infty}_{E^*})$-topology.*

Also, condition (3) may be replaced with

(3') $\lim_k \|m_B(f_k) \Leftrightarrow m_B(f_0)\| = 0$ *for each $B \in \mathcal{F}^+$.*

Note that Theorem 2.3 need not hold if one replaces condition (1) (resp. (1')) with (f_k) is Cauchy in the weak (resp. $\sigma(\mathcal{L}^1_E, \mathcal{L}^{\infty}_{E^*})$-) topology since $\mathcal{L}^1_E$ need not be sequentially complete in this topology. Recall that $\mathcal{L}^1_E$ is weakly sequentially complete if and only if E is [T]; on the other hand, $\mathcal{L}^1_E$ is $\sigma(\mathcal{L}^1_E, \mathcal{L}^{\infty}_{E^*})$-sequentially complete if and only if E is weakly sequentially complete and has the RNP (cf. [BH1], [SW]).

There is a set-analogue of Theorem 2.3:

Theorem 2.4. *A subset K of $\mathcal{L}^1_E$ is relatively* norm *compact if and only if*

(1) *K is relatively weakly compact*
(2) *K satisfies the Bocce criterion*
(3) *$\Delta_B := \{m_B(f) : f \in K\}$ is relatively norm compact in E for each $B \in \mathcal{F}^+$.*

Condition (1) *may be replaced with*

(1') K *is relatively compact in the* $\sigma(\mathcal{L}^1_E, \mathcal{L}^\infty_{E^*})$*-topology.*

Note that the above condition (3) is indispensable, as shown by Example 3.2 to come. In general, if (f_k) is weakly convergent (resp. K is relatively weakly compact), then the corresponding sets Δ_B are relatively weakly compact in E. Thus if E is finite-dimensional, then condition (3) in the above theorems is not necessary.

It is possible to prove Theorems 2.3 and 2.4 by using methods similar to those in [G2]. Here ideas from both [B3] and [G2] are combined. The following elementary lemmas are useful.

Lemma 2.5. *If* f *is in* $\mathcal{L}^1_E$, *then for each* $\epsilon > 0$ *and* $B \in \mathcal{F}^+$ *there is a set* A *in* $\mathcal{F}^+(B)$ *such that* Bocce-osc $f\big|_{A_0} < \epsilon$ *for each subset* A_0 *of* A.

Proof. By strong measurability of f in $\mathcal{L}^1_E$ and Egorov's Theorem, there exists a sequence of simple functions converging almost uniformly to f. In combination with (2.1), the remainder of the proof is clear. □

Lemma 2.6. *Let* $\phi\colon \Omega \to [0, +\infty]$ *be measurable. If for each* $\epsilon > 0$ *and each* B *in* $\mathcal{F}^+$ *there exists a set* A *in* $\mathcal{F}^+(B)$ *such that* $m_A(\phi) < \epsilon$, *then* $\phi(\omega) = 0$ *for a.e.* ω.

Proof. Fix $\epsilon > 0$. Let B be the set of all $\omega \in \Omega$ with $\phi(\omega) \geq 2\epsilon$. If $B \in \mathcal{F}^+$, then for the corresponding set A in $\mathcal{F}^+(B)$ we would have $2\epsilon\mu(A) < \epsilon\mu(A)$, which cannot be. So B must be a null set. □

Proof of Theorem 2.3. Consider a sequence (f_k) in $\mathcal{L}^1_E$ which converges strongly to f_0. Conditions (1) and (3) follow immediately. Also, by (2.1) one has that

$$\mu(A)\left|\text{Bocce-osc } f_k\big|_A \Leftrightarrow \text{Bocce-osc } f_0\big|_A\right| \leq 2\int_\Omega \|f_k \Leftrightarrow f_0\|\, d\mu \to 0$$

for each A in $\mathcal{F}^+$. By Lemma 2.5 the singleton $\{f_0\}$ satisfies the Bocce criterion. Thus condition (2) also holds.

As for sufficiency of (1), (2), and (3), note that to prove strong convergence it is enough to show that any subsequence (f_n) of (f_k) contains a further subsequence which converges strongly to f_0. By condition (1) the set (f_k) is uniformly integrable; hence $(\|f_k \Leftrightarrow f_0\|)$ must also be uniformly integrable. So the subsequence (f_n) contains a further subsequence (f_{n_j}) such that $(\|f_{n_j} \Leftrightarrow f_0\|)$ converges weakly to some (nonnegative) function ϕ in $\mathcal{L}^1_{\mathbb{R}}$. We shall show that Lemma 2.6 applies to ϕ; this then gives $\phi = 0$ a.e., which finishes the proof. To show that the lemma applies, first note that by Lemma 2.5 (applied to f_0) and the given Bocce property (2), the sequence $(f_k \Leftrightarrow f_0)$ also satisfies the sequential Bocce criterion. Now fix $\epsilon > 0$ and B

in $\mathcal{F}^+$. Let A in $\mathcal{F}^+(B)$ be as in Definition 2.1 applied to the subsequence $(f_{n_j} \Leftrightarrow f_0)$ of $(f_k \Leftrightarrow f_0)$, thus

$$\liminf_j \text{Bocce-osc } (f_{n_j} \Leftrightarrow f_0)\big|_A < \epsilon.$$

But by the triangle inequality

$$\text{Bocce-osc } (f_{n_j} \Leftrightarrow f_0)\big|_A \geq \frac{1}{\mu(A)} \int_A \left[\|f_{n_j} \Leftrightarrow f_0\| \Leftrightarrow \|m_A(f_{n_j} \Leftrightarrow f_0)\|\right] \, d\mu \ ,$$

so by weak convergence of $(\|f_{n_j} \Leftrightarrow f_0\|)$ to ϕ and by the given property (3), this leads us to $m_A(\phi) < \epsilon$, which is precisely what is needed to apply Lemma 2.6. □

A close look at the proof reveals that the conditions may be slightly weakened. Using terminology and results to come in Section 3, note that condition (1) may be replaced with the two conditions that (f_k) is uniformly integrable and that (f_k) converges scalarly weakly (see Definition 3.3) to f_0 in $\mathcal{L}^1_E$. These two conditions are equivalent to (1'), as noted in Remark 3.7. Also, condition (3') is equivalent to the two conditions that (f_k) converges scalarly weakly to f_0 and condition (3). Thus, under condition (1) or (1'), condition (3) is equivalent to (3').

Proof of Theorem 2.4. It is well-known and easy to check that a subset K of $\mathcal{L}^1_E$ is relatively strongly compact if and only if it satisfies condition (3) and for each $\eta > 0$ there is a finite measurable partition π of Ω such that $\int_\Omega \|f \Leftrightarrow E_\pi(f)\| d\mu < \eta$ for each f in K. Here $E_\pi(f)$ denotes the conditional expectation of f relative to the finite algebra generated by π.

Consider a relatively strongly compact subset K of $\mathcal{L}^1_E$. Clearly conditions (1) and (3) are satisfied. To see that condition (2) holds, fix $\epsilon > 0$ and $B \in \mathcal{F}^+$. Next, from the above observation, find the partition $\pi := \{A_1, \cdots, A_N\}$ corresponding to $\eta := \epsilon\mu(B)$. Put $\mathcal{A} = \{A_i \cap B \in \mathcal{F}^+ \colon A_i \in \pi\}$. Fix f in K. Since

$$\begin{aligned}\sum_i \mu(A_i \cap B)\text{Bocce-osc } f\big|_{A_i \cap B} &\leq \sum_i \mu(A_i)\text{Bocce-osc } f\big|_{A_i} \\ &= \int_\Omega \|f \Leftrightarrow \sum_i m_{A_i}(f) 1_{A_i}\| d\mu < \epsilon\mu(B) \ ,\end{aligned}$$

for at least one $A_i \cap B \in \mathcal{A}$ we have Bocce-osc $f\big|_{A_i \cap B} < \epsilon$.

As for the sufficiency of (1), (2), and (3), note that it is enough to show relative strong *sequential* compactness of K. So consider a sequence (f_k) in K. By condition (1), there is a subsequence (f_{k_j}) of (f_k) that converges weakly to some function f_0 in $\mathcal{L}^1_E$ while condition (2) implies that (f_{k_j}) satisfies the sequential Bocce criterion. Now an appeal to Theorem 2.3 shows that (f_{k_j}) converges strongly, as needed.

As for replacing (1) with (1'), recall [BH2] that for the $\sigma(\mathcal{L}^1_E, \mathcal{L}^\infty_{E^*})$-topology, relatively compact sets and relatively sequentially compact sets coincide. □

Section 5 gives several variations of the Bocce criterion which also provide necessary and sufficient conditions to pass from weak to strong convergence (resp. compactness).

3. Limited Convergence

This section examines limited convergence, a weakening of strong convergence [B1]. Limited convergence provides an extension to $\mathcal{L}^1_E$ of the Vitali Convergence Theorem (VCT), thus also of the Lebesgue Dominated Convergence Theorem (LDCT). Furthermore, it extends the previous section's results. In the next section, a tightness condition ties together limited and strong convergence and thus extends the results of this section.

Let $\mathcal{G}$ be the collection of all functions $g\colon \Omega \times E \to \mathbb{R}$ satisfying

(i) $g(\omega, 0) = 0$ for each ω in Ω
(ii) $g(\omega, \cdot)$ is weakly $\sigma(E, E^*)$-continuous for each ω in Ω
(iii) $|g(\omega, \cdot)| \leq C\|\cdot\| + \phi(\omega)$ for each ω in Ω, for some $C > 0$ and ϕ in $\mathcal{L}^1_{\mathbb{R}}$
(iv) $g(\cdot, f(\cdot))$ is $\mathcal{F}$-measurable for each f in $\mathcal{L}^1_E$.

An example of such a function g in $\mathcal{G}$ is given by $g(\omega, x) = \sum_{i=1}^n |x_i^*(x)|\, 1_{A_i}(\omega)$ where $x_i^* \in E^*$ and $A_i \in \mathcal{F}$. The function g given by $g(\omega, x) = ||x||$ is in $\mathcal{G}$ if E is finite-dimensional (for only then does g satisfies (ii)). The class $\mathcal{G}$ serves as a "test class" for limited convergence (see Remark 3.9).

Definition 3.1. A sequence (f_k) of functions in $\mathcal{L}^1_E$ converges *limitedly* to f_0 in $\mathcal{L}^1_E$ if $\lim_k \int_\Omega g(\omega, f_k(\omega) \Leftrightarrow f_0(\omega))\, d\mu(\omega) = 0$ for each $g \in \mathcal{G}$.

Strong convergence implies limited convergence. For first note that a sequence converges limitedly to f if each subsequence has a further subsequence which converges limitedly to f. Next note that a strongly convergent sequence has the property that each subsequence has a further subsequence which is pointwise a.e. strongly convergent. Lastly note that any uniformly integrable sequence (f_k) which is a.e. weakly null (i.e. there is a set A of full measure such that if $x^* \in E^*$ and $\omega \in A$ then $x^* f_k(\omega)$ converges to zero) converges limitedly. To see this, fix $g \in \mathcal{G}$ and put $h_k(\omega) = g(\omega, f_k(\omega))$. Condition (iii) gives that the set (h_k) is uniformly integrable. Conditions (i) and (ii) give that (h_k) is a.e.-convergent to 0. So (h_k) converges strongly to zero and so (f_k) converges limitedly.

If E is finite-dimensional then strong and limited convergence coincide (consider $g \in \mathcal{G}$ given by $g(\omega, x) = ||x||$). However, as seen by modifying the next example, for any infinite-dimensional reflexive space E there is a sequence of $\mathcal{L}^1_E$ functions which converges limitedly but not strongly.

Example 3.2 (limited $\not\Rightarrow$ strong). Take $(\Omega, \mathcal{F}, \mu)$ to be the interval $[0, 1]$, equipped with the Lebesgue σ-algebra and measure and $E := \ell^2$. Setting f_k identically equal to the k-th unit vector e_k in ℓ^2 gives a sequence (f_k) which converges limitedly but not strongly to the null function.

Limited convergence implies weak convergence since for each $b \in \mathcal{L}_{E^*}^{\infty}[E]$ the function g defined by $g(\omega, x) = \langle x, b(\omega)\rangle$ is in $\mathcal{G}$. As for the converse implication, even for finite-dimensional E weak convergence does not imply limited convergence.

Towards a variant of the VCT–LDCT for a sequence (f_k) in $\mathcal{L}_E^1$, we examine the corresponding sequences $(x^*(f_k))$ in $\mathcal{L}_{\mathbb{R}}^1$ for x^* in E^*.

Definition 3.3. A sequence (f_k) of functions in $\mathcal{L}_E^1$ converges *scalarly strongly* (resp. *scalarly in measure, scalarly weakly*) to f_0 in $\mathcal{L}_E^1$ if the corresponding sequence $(x^*(f_k))$ in $\mathcal{L}_{\mathbb{R}}^1$ converges *strongly* (resp. *in measure, weakly*) to $x^*(f_0)$ for each x^* in E^*.

Note the following chain of *strict* implications:

$$\text{(3.1)} \qquad \textit{strong} \Rightarrow \textit{limited} \Rightarrow \textit{scalarly strong} \Rightarrow \textit{scalarly in measure}\,.$$

Since for $x^* \in E^*$ functions of the form $g(\omega, \cdot) = |x^*(\cdot)|\, 1_\Omega(\omega)$ are in $\mathcal{G}$, limited convergence implies scalarly strong convergence. The other implications in (3.1) are clear.

Furthermore, the implications are strict. Example 3.2 showed the first one is not reversible. The last implication is not reversible even for $E = \mathbb{R}$. The next example shows that the second implication is also strict.

Example 3.4 (scalarly strong $\not\Rightarrow$ limited). Take $(\Omega, \mathcal{F}, \mu)$, E, and (e_k) as in Example 3.2. Let I_i^j be the dyadic interval $[(i \Leftrightarrow 1)2^{-j}, i2^{-j})$ for $j \in \mathbb{N}$ and $i = 1, \cdots, 2^j$. Consider the sequence (f_k) of the functions $f_k\colon [0,1] \to \ell^2$ given by $f_k(\omega) = 1_{I_1^k}(\omega) 2^k e_k$. Since for every $y^* := (y^j)_j$ in $E^* \approx \ell^2$

$$\int_\Omega |y^*(f_k(\omega))|\; d\mu(\omega) \;=\; |y^k|\;,$$

(f_k) converges scalarly strong to the null function. But for the test function $g(\omega, (x_j)) = \sum_{j=1}^{\infty} x_j\; 1_{I_2^{j+1}}(\omega)$ in $\mathcal{G}$

$$\int_\Omega g(\omega, f_k(\omega))\; d\mu(\omega) = \int_{I_1^k} 2^k\; 1_{I_2^{k+1}}\, d\mu = \frac{1}{2}\;.$$

So (f_k) does not converge limitedly to the null function.

Note that a scalarly strongly convergent sequence need not be uniformly integrable (as Example 3.4 shows). However, a limitedly convergent sequence, being also weakly convergent, is necessarily uniformly integrable.

Limited convergence provides the following extension of the VCT–LDCT to $\mathcal{L}_E^1$.

Theorem 3.5. *Let E^* have the RNP. If a uniformly integrable sequence (f_k) converges scalarly in measure to f_0 in $\mathcal{L}^1_E$, then it also converges limitedly to f_0.*

The necessity of the uniform integrability condition has already been noted while the necessity of E^* having the RNP follows from Remark 5.4. The proof of Theorem 3.5 uses the following lemma.

Lemma 3.6. *A uniformly integrable sequence (f_k) of $\mathcal{L}^1_E$ functions converges limitedly to the null function provided that, for each $N \in \mathbb{N}$, the sequence $(f_k^N)_k$ of truncated functions converges limitedly to the null function, where $f_k^N := f_k \; 1_{[||f_k|| \leq N]}$.*

Proof. Fix $g \in \mathcal{G}$ with $|g(\omega, \cdot)| \leq C\| \cdot \| + \phi(\omega)$. Now

$$\left| \int_\Omega \left(g\left(\omega, f_k\left(\omega\right)\right) \Leftrightarrow g\left(\omega, f_k^N\left(\omega\right)\right)\right) \; d\mu \right| \; = \; \left| \int_{[||f_k||>N]} g(\omega, f_k(\omega)) \; d\mu \right|$$

$$\leq C \int_{[||f_k||>N]} ||f_k|| \; d\mu \; + \; \int_{[||f_k||>N]} \phi \; d\mu \; ,$$

so by uniform integrability of (f_k) it follows that

$$\lim_{N\to\infty} \sup_k \left| \int_\Omega \left(g\left(\omega, f_k\left(\omega\right)\right) \Leftrightarrow g\left(\omega, f_k^N\left(\omega\right)\right)\right) \; d\mu \right| \; = \; 0 \; .$$

The lemma now follows with ease. □

Proof of Theorem 3.5. Without loss of generality, we assume that $f_0 = 0$ and (using the previous lemma) that the f_k's are uniformly bounded. Note that we may also assume that E^* is separable. Indeed, by the Pettis measurability theorem [DU, Theorem II.1.2], there is a separable subspace E_0 of E such that the f_k's are essentially valued in E_0. Because E^* has the RNP, E_0^* must be separable [DU, Corollary VII.2.8]. Moreover, if (f_k) converges limitedly to 0 in $\mathcal{L}^1_{E_0}$, then it also does so in $\mathcal{L}^1_E$.

As noted earlier, it is enough to show that every subsequence of (f_k) has a further subsequence that is a.e. weakly null. We assume (w.l.o.g.) that this former subsequence is actually the entire sequence (f_k).

Now let (x_i^*) be a countable dense subset of E^*. For each i the sequence $(x_i^*(f_k))$ converges in measure to zero. So there exists a subsequence (f_{k_j}) such that for a.e. ω

$$\lim_j \; x_i^*(f_{k_j}(\omega)) \; = \; 0 \; .$$

By a Cantor diagonalization argument there is a set A of full measure and a subsequence (f_{k_p}) such that $\lim_p x_i^*(f_{k_p}(\omega)) = 0$ for each fixed i and each ω in A. Since the f_k's are uniformly bounded and (x_i^*) are dense in E^*, this pointwise limit property extends so that $\lim_p x^*(f_{k_p}(\omega)) = 0$ for each fixed x^* in E^* and each ω in A. Thus, (f_{k_p}) is a.e. weakly null, as needed. □

Limited convergence also provides an extension of the results from the previous section; namely, it is possible to pass from weak to limited convergence via an oscillation condition. The following string of *strict* implications summarizes the ideas thus far.

$$\textit{scalarly strong} \Rightarrow \textit{scalarly weak} \Leftarrow \sigma(\mathcal{L}^1_E, \mathcal{L}^\infty_{E^*})\textit{-topology} \Leftarrow \textit{weak} .$$

Remark 3.7. A scalarly weakly convergent sequence converges in the $\sigma(\mathcal{L}^1_E, \mathcal{L}^\infty_{E^*})$-topology if and only if it is uniformly integrable. (Recall that the simple functions are not dense in $\mathcal{L}^\infty_{E^*}$ for infinite-dimensional E.) Convergence in the $\sigma(\mathcal{L}^1_E, \mathcal{L}^\infty_{E^*})$-topology implies weak convergence if and only if E^* has the RNP [cf. DG] .

In the light of these observations and Theorem 3.5, we have the following variant of Theorem 2.3 for limited convergence.

Theorem 3.8. *Let E^* have the RNP. A sequence (f_k) of $\mathcal{L}^1_E$ functions converges* limitedly *to f_0 in $\mathcal{L}^1_E$ if and only if*

(1) *(f_k) converges* weakly *to f_0 in $\mathcal{L}^1_E$*
(2) *$(x^*(f_k))$ satisfies the sequential Bocce criterion for each x^* in E^*.*

Condition (1) *may be replaced with*

(1') *(f_k) converges to f_0 in $\mathcal{L}^1_E$ in the $\sigma(\mathcal{L}^1_E, \mathcal{L}^\infty_{E^*})$-topology.*

Remark 3.9. Limited convergence for separable reflexive E was introduced in [B1, B2]. There, the condition (iv) is replaced with

(iv') g is $\mathcal{F} \otimes \mathcal{B}(E)$-measurable,

Of course (iv') always implies (iv). To see that (iv) implies (iv') if E is separable, consider a function g which satisfies (iv). For each $k \in \mathbb{N}$, write $1_E = \sum_n 1_{E^k_n}$ where $E^k_n \in \mathcal{B}(E)$ and the diameter of E^k_n is less than $\frac{1}{k}$. Choose $x^k_n \in E^k_n$. Define $g_k \colon \Omega \times E \to \mathbb{R}$ by

$$g_k(\omega, x) = \sum_n g(\omega, x^k_n)\ 1_{E^k_n}(x) .$$

Since each g_k is $\mathcal{F} \times \mathcal{B}(E)$-measurable and g_k converges to g almost everywhere, g is also $\mathcal{F} \otimes \mathcal{B}(E)$-measurable.

4. The Tightness Connection

The concept of tightness links strong and limited convergence. In this section, we assume that E is a separable Banach space. Tightness is considered here with respect to the norm topology on E and only for functions. The following formulation of tightness is given in [B4].

Definition 4.1. A subset L of $\mathcal{L}^1_E$ is *tight* if there exists an $\mathcal{F} \otimes \mathcal{B}(E)$-measurable function $h\colon \Omega \times E \to [0, +\infty]$ such that

$$\sup_{f \in L} \int_\Omega h(\omega, f(\omega))\, d\mu(\omega) \ < \ +\infty$$

and such that $\{x \in E : h(\omega, x) \leq \beta\}$ is compact for each $\omega \in \Omega$ and each $\beta \in \mathbb{R}$.

In [Jaw], the following equivalent formulation of tightness is observed.

Definition 4.1′. A subset L of $\mathcal{L}^1_E$ is *tight* if for each $\epsilon > 0$ there exists a measurable multifunction F_ϵ from Ω to the compact subsets of E such that

$$\mu\left(\{\omega \in \Omega\colon f(\omega) \notin F_\epsilon(\omega)\}\right) \leq \epsilon$$

for each $f \in L$. We say that such a multifunction F_ϵ is measurable (i.e. graph-measurable) if its graph $\{(\omega, x) \in \Omega \times E : x \in F_\epsilon(\omega)\}$ is an $\mathcal{F} \otimes \mathcal{B}(E)$-measurable subset of $\Omega \times E$.

To see the equivalence in one direction, denote the supremum in Definition 4.1 by σ and define $F_\epsilon(\omega)$ as the set of all $x \in E$ for which $h(\omega, x) \leq \sigma/\epsilon$. In the other direction, one obtains a sequence (F_n) of compact-valued multifunctions by letting F_n correspond to $\epsilon = 3^{-n}$ in Definition 4.1′. Without loss of generality we may suppose that $(F_n(\omega))$ is nondecreasing (rather than taking finite unions $\cup_{m \leq n} F_m$). Now a function h satisfying the requirements of Definition 4.1 is obtained by setting $h(\omega, x) := 2^n$ for $x \in F_n(\omega) \backslash F_{n-1}(\omega)$ with $F_0(\omega) := \emptyset$ and $h(\omega, x) := +\infty$ for $x \in E \setminus \cup_n F_n(\omega)$.

In Definition 4.1′ we may assume without loss of generality that $F_\epsilon(\omega)$ is convex and contains 0 for each ω in Ω by consider the corresponding multifunction $\omega \mapsto \overline{\text{co}}(F_\epsilon(\omega) \cup \{0\})$. The measurability of this new map follows from [CV, Theorem III.40] and [HU, Remark (1), p. 163]. Therefore, if L is tight and $(B_f)_{f \in L}$ is a family of sets from $\mathcal{F}$, then the set $\{f\ 1_{B_f}\colon f \in L\}$ is also tight. Note that a bounded sequence in $\mathcal{L}^1_E$ is tight if E is finite dimensional (simply take $h(\omega, x) := ||x||$ in Definition 4.1). For further details on tightness see [B4, B5].

Recall the following fact [ACV, Théorème 6].

Fact 4.2. *A uniformly integrable tight subset of $\mathcal{L}^1_E$ is relatively weakly compact.*

Although weak compactness is not sufficient to guarantee that the corresponding subset Δ_B are relatively norm compact (consider Example 3.2), the following generalization of a result of Castaing [C1] shows that uniform integrability plus tightness is sufficient.

Lemma 4.3. *Let L be a tight uniformly integrable subset of $\mathcal{L}^1_E$. Then $\Delta_B := \{m_B(f) : f \in L\}$ is relatively norm compact in E for each B in $\mathcal{F}^+$.*

Proof. Let the subset L of $\mathcal{L}^1_E$ be uniformly integrable and tight. Since for each $B \in \mathcal{F}^+$ the set $\{f\ 1_B\colon f \in L\}$ is also uniformly integrable and tight, it is enough to show that Δ_Ω is relatively norm compact. Arguing as in Remark (1) on p. 163 of [HU], we may suppose without loss of generality that $\mathcal{F}$ is complete.

Fix $\delta > 0$. By the uniform integrability of L, there exist $\alpha > 0$ and $\epsilon > 0$ such that for each set A of measure at most ϵ we have that

$$\sup_{f\in L}\int_A ||f||\,d\mu \;\le\; \delta/2 \quad\text{and}\quad \sup_{f\in L}\int_{[||f||>\alpha]} ||f||\,d\mu \;\le\; \delta/2\;.$$

Let F_ϵ be a multifunction given by Definition 4.1$'$ and $G_\epsilon^\alpha = F_\epsilon \cap \alpha B_E$ (i.e. $G_\epsilon^\alpha(\omega) = F_\epsilon(\omega)\cap\alpha B_E$, $\forall\omega\in\Omega$). Since G_ϵ^α is convex compact valued and integrably bounded (that means $||G_\epsilon^\alpha|| = \sup\{||x||\colon x\in G_\epsilon^\alpha(\omega)\}\in L^1_{\mathbb{R}_+}$), the subset $K_\epsilon^\alpha = \{\int_\Omega G_\epsilon^\alpha\,d\mu\}$ is convex and compact in E [CV, Theorem V.15]. Let now A_f^ϵ be the set of all $\omega\in\Omega$ with $f(\omega)\in F_\epsilon(\omega)$. Note that $\mu(\Omega\setminus A_f^\epsilon)\le\epsilon$. Since for each $f\in L$

$$\int_{[||f||\le\alpha]} f\,1_{A_f^\epsilon}\,d\mu \in K_\epsilon^\alpha\;,$$

the set $\Delta_\Omega^{\epsilon,\alpha} := \{\int_{[||f||\le\alpha]} f\,1_{A_f^\epsilon}\,d\mu\colon f\in L\}$ is relatively compact in E. Moreover, the distance between $\Delta_\Omega^{\epsilon,\alpha}$ and Δ_Ω is at most δ since

$$||\;\int_\Omega f\,d\mu \Leftrightarrow \int_{[||f||\le\alpha]} f\,1_{A_f^\epsilon}\,d\mu\;||\;\le\;\int_{[||f||>\alpha]}||f||\,d\mu + \int_\Omega ||f\,1_{\Omega\setminus A_f^\epsilon}||\,d\mu\;\le\;\delta$$

for each $f\in L$. Thus Δ_Ω is relatively compact. □

Measure convergent sequences enjoy tightness.

Lemma 4.4. *A sequence in $\mathcal{L}^1_E$ which converges in measure is tight.*

Proof. Consider a sequence (f_k) in $\mathcal{L}^1_E$ which converges in measure to f_0. For each natural number k, let λ_k be the bounded non-negative image measure on E induced by μ and the measurable function $f_k\colon\Omega\to E$. Since E is a Radon space (thanks to the separability assumption), λ_k is a Radon (or tight) measure. For each bounded continuous function $\phi\in\mathcal{C}^b(E)$, we have

$$\lambda_k(\phi) = \int_\Omega \phi(f_k(\omega))\,d\mu(\omega)\;. \tag{4.1}$$

It is easy to see that the measure convergence of (f_k) to f_0 in $\mathcal{L}^1_E$ implies the narrow convergence (or weak convergence in the $\sigma(\mathcal{M}^b(E),\mathcal{C}^b(E))$-topology) of (λ_k) to λ_0. For otherwise there would exist $\phi\in\mathcal{C}^b(E)$ and a subsequence (f_{k_j}) converging almost everywhere to f_0 and such that $(\lambda_{k_j}(\phi))$ does not converge to $\lambda_0(\phi)$. But by (4.1), this contradicts the Lebesgue Dominated Convergence Theorem. Therefore, the sequence (λ_k) is tight in $\mathcal{M}^b(E)$ in the classical sense [S, Appendix Theorem 4], which implies that (f_k) is tight in the sense of Definitions 4.1 and 4.1$'$. □

From Lemmas 4.3 and 4.4, the following reformulation of Theorem 2.3 follows with ease.

Theorem 4.5. *A sequence (f_k) in $\mathcal{L}^1_E$ converges* strongly *to f_0 in $\mathcal{L}^1_E$ if and only if*

(1) *(f_k) converges* weakly *to f_0 in $\mathcal{L}^1_E$*
(2) *(f_k) satisfies the sequential Bocce criterion*
(3) *(f_k) is tight.*

Tightness connects strong and limited convergence.

Theorem 4.6. *A sequence (f_k) of $\mathcal{L}^1_E$ converges strongly to f_0 if and only if (f_k) is tight and converges limitedly to f_0.*

Before proceeding with the proof of Theorem 4.6, we note some immediate corollaries.

Theorem 3.5 - revisited. *Let E^* have the RNP and E be separable. If a uniformly integrable tight sequence (f_k) converges scalarly in measure to f_0 in $\mathcal{L}^1_E$, then it also converges strongly to f_0.*

Theorem 3.8 - revisited. *Let E^* have the RNP and E be separable. A sequence (f_k) of $\mathcal{L}^1_E$ functions converges* strongly *to f_0 in $\mathcal{L}^1_E$ if and only if*

(1) *(f_k) converges* weakly *to f_0 in $\mathcal{L}^1_E$*
(2) *$(x^*(f_k))$ satisfies the sequential Bocce criterion for each x^* in E^**
(3) *(f_k) is tight.*

Condition (1) may be replaced with

(1') *(f_k) converges to f_0 in $\mathcal{L}^1_E$ in the $\sigma(\mathcal{L}^1_E, \mathcal{L}^\infty_{E^*})$-topology.*

The proof of Theorem 4.6 uses the following standard fact (compare with Lemma 3.6).

Fact 4.7. *A uniformly integrable sequence (f_k) of $\mathcal{L}^1_E$ functions converges strongly to the null function provided that, for each $N \in \mathbb{N}$, the sequence $(f_k^N)_k$ of truncated functions converges strongly to the null function, where $f_k^N := f_k \, 1_{[||f_k|| \le N]}$.*

Proof of Theorem 4.6. The implication in one direction follows from our previous work. As for the other direction, let (f_k) be a tight sequence in $\mathcal{L}^1_E$ which converges limitedly to f_0. Because the image measure of μ under f_0 is a Radon measure on E, the singleton $\{f_0\}$ must be tight. Since the union of two tight sets is again tight, we have that the set $\{f_k : k \in \mathbb{N} \cup \{0\}\}$ is also tight; let h correspond to this set as in Definition 4.1. Without loss of generality, we assume that f_0 is the null function and that the f_k's are uniformly bounded (in $\mathcal{L}^\infty_E$) by some $M > 0$. To avoid the non-metrizability of the $\sigma(E, E^*)$-topology, we use ideas from [B5]. By well-known facts about Suslin spaces [S, Corollary 2 of Theorem II.10], there exists a metric d on E defining a topology τ_d weaker than the weak topology $\sigma(E, E^*)$ and such that

(E, τ_d) is a Suslin space. Define $\phi\colon \Omega \times E \to \mathbb{R}$ by $\phi(\omega, x) := \max(\Leftrightarrow\|x\|, \Leftrightarrow M)$. For each $\epsilon > 0$, consider the function $\phi_\epsilon : \Omega \times E \to \mathbb{R}$ given by

$$\phi_\epsilon(\omega, x) := \phi(\omega, x) + \epsilon h(\omega, x).$$

From the inf-compactness property of h (see Definition 4.1) it follows that $\phi_\epsilon(\omega, \cdot)$ is also inf-compact on E for each $\omega \in \Omega$ and $\epsilon > 0$; in turn, this implies inf-compactness of the same functions for the weak topology $\sigma(E, E^*)$ and hence for τ_d. Moreover, the $\mathcal{F} \otimes \mathcal{B}(E)$-measurability[1] of ϕ_ϵ is evident.

For each $\epsilon > 0$ and $p \in \mathbb{N}$ we define the approximate function $\phi^p_\epsilon\colon \Omega \times E \to \mathbb{R}$ by

$$\phi^p_\epsilon(\omega, x) = \inf_{y \in E} \{\phi_\epsilon(\omega, y) + p\, d(x, y)\}.$$

Evidently, for each $\epsilon > 0$ the sequence (ϕ^p_ϵ) is (pointwise) nondecreasing. It is well-known [B4,V1] that ϕ^p_ϵ has an $\mathcal{F} \otimes \mathcal{B}(E)$-measurable modification ψ^p_ϵ (i.e., $\psi^p_\epsilon(\omega, \cdot) = \phi^p_\epsilon(\omega, \cdot)$ a.e.) such that for each $\omega \in \Omega$ the function $\psi^p_\epsilon(\omega, \cdot)$ is d-Lipschitz continuous on E and therefore is $\sigma(E, E^*)$-continuous. Furthermore, as a well-known property of this approximation, by τ_d-lower semicontinuity and boundedness below of $\phi_\epsilon(\omega, \cdot)$, we have

$$\phi_\epsilon(\omega, x) = \lim_p \uparrow \psi^p_\epsilon(\omega, x)$$

for a.e. ω and each $x \in E$. We now set $\widehat{\psi}^p_\epsilon(\omega, x) = \min(\psi^p_\epsilon(\omega, x) \Leftrightarrow \psi^p_\epsilon(\omega, 0), p)$. Note that $\Leftrightarrow M \Leftrightarrow \epsilon h(\omega, 0) \le \widehat{\psi}^p_\epsilon(\omega, \cdot) \le p$ for a.e. ω, where $\omega \mapsto h(\omega, 0)$ is integrable in view of Definition 4.1 and $f_0 \equiv 0$. For each $\epsilon > 0$ and $p \in \mathbb{N}$, the function $\widehat{\psi}^p_\epsilon$ satisfies the conditions (i) to (iv) for the test functions of $\mathcal{G}$. Therefore, by the limited convergence of the f_k's, we have

$$\lim_{k \to \infty} \int_\Omega \widehat{\psi}^p_\epsilon(\omega, f_k(\omega))\, d\mu(\omega) = 0.$$

It follows that

$$\liminf_{k \to \infty} \int_\Omega \psi^p_\epsilon(\omega, f_k(\omega))\, d\mu(\omega) \ge \int_\Omega \psi^p_\epsilon(\omega, 0)\, d\mu(\omega)\ .$$

Thus, for each $\epsilon > 0$ and $p \in \mathbb{N}$

$$\liminf_{k \to \infty} \int_\Omega \phi_\epsilon(\omega, f_k(\omega))\, d\mu(\omega) \ge \liminf_{k \to \infty} \int_\Omega \psi^p_\epsilon(\omega, f_k(\omega))\, d\mu(\omega) \ge \int_\Omega \psi^p_\epsilon(\omega, 0)\, d\mu(\omega)\ .$$

[1] For any of the three topologies E is a Suslin space; hence, it has the same Borel σ-algebra $\mathcal{B}(E)$.

The monotone convergence theorem gives, for each $\epsilon > 0$

$$\begin{aligned}\alpha_\epsilon := \liminf_{k\to\infty} \int_\Omega \phi_\epsilon(\omega, f_k(\omega))\, d\mu(\omega) \;&\geq\; \lim_p \uparrow \int_\Omega \psi_\epsilon^p(\omega, 0)\, d\mu(\omega) \\ &= \int_\Omega \phi_\epsilon(\omega, 0)\, d\mu(\omega) \;=\; \epsilon \int_\Omega h(\omega, 0) d\mu(\omega)\ ,\end{aligned}$$

thus

$$0 \leq \alpha_\epsilon \leq \liminf_{k\to\infty} \int_\Omega \phi(\omega, f_k(\omega))\, d\mu(\omega) + \epsilon\, \sup_k \int_\Omega h(\omega, f_k(\omega))\, d\mu(\omega)\ .$$

Since $\phi(\omega, f_k(\omega)) = \Leftrightarrow ||f_k(\omega)||$, by our initial assumption, the proof is finished by letting ϵ go to zero. □

Fact 4.2 and Theorem 4.5 gives that a uniformly integrable tight sequence in $\mathcal{L}^1_E$ which satisfies the sequential Bocce criterion has a strongly convergent subsequence. Recall that a sequence (f_k) is said to be bounded if $\sup_k ||f_k||_{\mathcal{L}^1_E}$ is finite. In the above, if we relax uniform integrability to boundedness, we need not have strong subsequential convergence (just consider the sequence $(n\, 1_{[0,1/n]})_n$ in $\mathcal{L}^1_{\mathbb{R}}$) but we do have measure subsequential convergence. We can state this result as a strong Biting lemma.

Theorem 4.8. *Let (f_k) be a bounded tight sequence in $\mathcal{L}^1_E$ satisfying the sequential Bocce criterion. Then there exist a subsequence, say (f_n), of (f_k) and an increasing sequence (A_n) in $\mathcal{F}$ such that*

(1) $\lim_{n\to\infty} \mu(A_n) = \mu(\Omega)$
(2) *the sequence $(f_n\, 1_{A_n})$ converges strongly in $\mathcal{L}^1_E$*
(3) *the sequence $(f_n\, 1_{\Omega\setminus A_n})$ converges to 0 in measure.*

Therefore, the subsequence (f_n) converges in measure.

The proof uses Gaposhkin's Biting lemma [Ga, Lemma C], which is also referred to as Slaby's Biting lemma [cf. C2].

Biting lemma. *Let (f_k) be a bounded sequence in $\mathcal{L}^1_E$. Then there exist a subsequence, say (f_n), of (f_k) and an increasing sequence (A_n) in $\mathcal{F}$ such that*

(1) $\lim_{n\to\infty} \mu(A_n) = \mu(\Omega)$
(2) *the sequence $(f_n\, 1_{A_n})$ is uniformly integrable in $\mathcal{L}^1_E$.*

Note that (1) implies that the sequence $(f_n\, 1_{\Omega\setminus A_n})$ converges to 0 in measure.

Proof of Theorem 4.8. Consider a bounded tight sequence (f_k) in $\mathcal{L}^1_E$ which satisfies the sequential Bocce criterion. Apply the Biting lemma to find the corresponding subsequence, say (f_n), of (f_k) and sequence (A_n) in $\mathcal{F}$. Since $(f_n\, 1_{A_n})$ is uniformly integrable and tight, it is relatively weakly sequentially compact. By passing to a further subsequence we can assume that $(f_n\, 1_{A_n})$ converges weakly in $\mathcal{L}^1_E$. Since

(f_k) satisfies the sequential Bocce criterion, using condition (1) it is easy to check that $(f_n 1_{A_n})$ also satisfies the sequential Bocce criterion (in the definition, for a fixed $B \in \mathcal{F}^+(\Omega)$, apply the criterion to $B_0 := B \cap A_N$ for a sufficiently large N). Theorem 4.5 gives that $(f_n 1_{A_n})$ converges strongly. □

5. Pettis Norm

This section examines Pettis norm convergence in light of the previous sections.

Definition 5.1. A strongly measurable function $f\colon \Omega \to E$ is *Pettis integrable* if $x^*(f)$ belongs to $\mathcal{L}_{\mathbb{R}}^1$ for every x^* in E^* and if for every B in $\mathcal{F}$ there exists x_B in E such that

$$\int_B x^*(f)d\mu = x^*(x_B) \quad \text{for all } x^* \in E^*.$$

The space $\mathcal{P}_E^1$ of (equivalence classes of) all strongly measurable Pettis integrable functions forms a normed linear space under the *Pettis (semi)norm*

$$||f||_{\text{Pettis}} = \sup_{x^* \in B_{E^*}} \int_\Omega |x^*(f)|\, d\mu \; .$$

Clearly $\mathcal{P}_E^1$ contains $\mathcal{L}_E^1$, to which we restrict considerations.

In general, Pettis norm convergence on $\mathcal{L}_E^1$ is incomparable with limited convergence but is comparable with the other modes of convergence in chain (3.1). A parallel chain of *strict* implications is

$$\textit{strong} \Rightarrow \textit{Pettis} \Rightarrow \textit{scalarly strong} \; . \tag{5.1}$$

Note that when E is finite-dimensional, the two chains (3.1) and (5.1) merge into

$$\textit{strong} \Leftrightarrow \textit{Pettis} \Leftrightarrow \textit{limited} \Leftrightarrow \textit{scalarly strong} \; .$$

The implications in chain (5.1) are clear; the following two examples show that they are strict.

Example 5.2 (scalarly strong $\not\Rightarrow$ Pettis). Example 3.2 suffices here but, for later use, we consider the following variation. Take $(\Omega, \mathcal{F}, \mu)$, $E := \ell^2$, and (e_k) as in Example 3.2. Consider the Rademacher-type functions $f_k\colon [0,1] \to \ell^2$ defined by $f_k(\omega) := e_k r_k(\omega)$ where r_k is the k-th Rademacher function. Clearly, (f_k) converges scalarly strong to the null function yet the Pettis norm of each f_k is one.

Example 5.3 (Pettis $\not\Rightarrow$ strong). [P] Take $(\Omega, \mathcal{F}, \mu)$, E, (e_k), and (I_i^j) as in Example 3.4. Consider the sequence (f_k) of the integrable functions $f_k\colon \Omega \to \ell^2$

given by $f_k(\omega) := \sum_{i=1}^{2^k} 1_{I_i^k}(\omega) e_{2^k+i}$. To see that (f_k) converges in the Pettis norm to the null function, fix $y^* := (y^i)_i \in B_{\ell^2}$. Put $\overline{y}^* := (|y^i|)_i$ and note that

$$\int_\Omega |y^*(f_k)|\, d\mu \;=\; \sum_{i=1}^{2^k} |y^{2^k+i}|\; \mu(I_i^k) \;=\; 2^{-k}\; \overline{y}^* \left(\sum_{i=1}^{2^k} e_{2^k+i} \right)$$
$$\le\; 2^{-k}\; \Big\| \sum_{i=1}^{2^k} e_{2^k+i} \Big\|_{\ell^2} \;=\; 2^{-\frac{k}{2}} \;.$$

Thus $||f_k||_{\text{Pettis}} \to 0$. But (f_k) does not converge strongly since $\int_\Omega ||f_k||_{\ell^2}\, d\mu \;=\; 1$.

Example 5.3 illustrates (consider $g_k := 2^{\frac{k}{4}} f_k$) that a Pettis-norm convergent sequence need not be uniformly integrable. Example 3.2 shows that a limitedly convergent sequence need not converge in the Pettis norm. Theorem 3.5 gives that if E^* has the RNP, then a uniformly integrable Pettis-norm convergent sequence in $\mathcal{L}_E^1$ also converges limitedly. The following remark shows the necessity of E^* having the RNP.

Remark 5.4 [DG]. A uniformly integrable Pettis-norm convergent sequence also converges in the $\sigma(\mathcal{L}_E^1, \mathcal{L}_{E^*}^\infty)$-topology and, if furthermore E^* has the RNP, then also weakly. But if E^* fails the RNP, then there is an essentially bounded sequence which converges in the Pettis norm but not weakly (thus not limitedly).

In the case that $E = \ell^1$, this sequence is easy to construct.

Example 5.5 (Pettis $\not\Rightarrow$ limited). Let $(\Omega, \mathcal{F}, \mu)$ be as in Example 3.2 and let $E = \ell^1$. Consider the sequence (f_k) in $\mathcal{L}_E^1$ given by $f_k(\omega) := \frac{1}{k} \sum_{i=1}^k r_i(\omega) e_i$, where e_i is the i-th unit vector in ℓ^1 and r_i is the i-th Rademacher function. Note that (f_k) is essentially bounded. As for the Pettis norm of f_k, fix $y^* = (y^i)_i \in E^* = \ell^\infty$. Since

$$\int_\Omega |y^*(f_k)|\, d\mu \;=\; \frac{1}{k} \int_\Omega \left| \sum_{i=1}^k y^i\, r_i(\omega) \right| \mu(d\omega) \;,$$

Khintchine's inequality [cf. D1] shows that $||f_k||_{\text{Pettis}}$ behaves like $\frac{1}{\sqrt{k}}$ and so $||f_k||_{\text{Pettis}} \to 0$. Thus (f_k) converges scalarly weakly to the null function and so if it also converges limitedly or weakly, it does so to the null function. But consider $b \in \mathcal{L}_{E^*}^\infty[E] \approx {\mathcal{L}_E^1}^*$ given by $b(\omega) := (1_{[r_i=1]}(\omega))_i$, along with the corresponding test function $g(\omega, x) := \langle x, b(\omega) \rangle$. Since $\int_\Omega \langle f_k(\omega), b(\omega) \rangle \; d\mu(\omega) \;=\; \frac{1}{2}$ we see that (f_k) does not converge limitedly nor weakly.

At this time there is no analogue to Theorem 3.5 which would allow one to pass from scalarly in measure convergence to Pettis-norm convergence when E^* has the RNP. Note that if the sequence (f_k) is Cauchy in the Pettis norm, then the corresponding subsets Δ_B of E are relatively norm compact for each $B \in \mathcal{F}^+$.

But even for an essentially bounded (thus uniformly integrable) sequence (f_k) for which the Δ_B are all relatively norm compact, the implication *scalarly in measure* $\Rightarrow$ *Pettis* does not hold in general, as shown by Example 5.2.

It is possible in certain situations to pass from weak to Pettis-norm convergence. For this, a measurement of the oscillation relative to the Pettis norm is needed.

Definition 5.6. For $f \in \mathcal{L}_E^1$ and $A \in \mathcal{F}$ the *Pettis Bocce oscillation* of f over A is

$$\text{Pettis-Bocce-osc } f\big|_A := \sup_{x^* \in B_{E^*}} \text{Bocce-osc } x^*(f)\big|_A .$$

Since Bocce-osc $x^* f\big|_A$ is at most $\|x^*\|$ Bocce-osc $f\big|_A$, the Pettis-Bocce-osc $f\big|_A$ is at most Bocce-osc $f\big|_A$.

Definition 5.7. A sequence (f_k) of functions in $\mathcal{L}_E^1$ satisfies the *sequential Pettis Bocce criterion* if for each subsequence (f_{k_j}) of (f_k), each $\epsilon > 0$, and each B in $\mathcal{F}^+$, there is a set A in $\mathcal{F}^+(B)$ such that $\liminf_j$ Pettis-Bocce-osc $f_{k_j}\big|_A < \epsilon$.

Definition 5.8. A subset K of $\mathcal{L}_E^1$ is *Pettis uniformly integrable* if the corresponding subset $\widetilde{K} := \{x^* f \colon x^* \in B_{E^*}, f \in K\}$ of $\mathcal{L}_{\mathbb{R}}^1$ is uniformly integrable.

Clearly, K is Pettis uniformly integrable if and only if it is Pettis-norm bounded and the corresponding set $\widetilde{K}$ is equi-integrable.

The following variants of Lemma 2.5 and Lemma 2.6, respectfully, are useful.

Lemma 5.9. *The sequential Pettis Bocce criterion is translation invariant.*

Proof. Let the sequence (f_k) satisfy the sequential Pettis Bocce criterion and fix $f \in \mathcal{L}_E^1$. The fact that $(f_k + f)$ also satisfies the Pettis Bocce criterion follows directly from the definition, Lemma 2.5, and the observation that (cf. inequalities 2.1)

$$\text{Pettis-Bocce-osc } (f_k + f)\big|_A \leq \text{Pettis-Bocce-osc } f_k\big|_A + \text{Pettis-Bocce-osc } f\big|_A . \qquad \square$$

Lemma 5.10. *Let (f_k) be a Pettis uniformly integrable sequence in $\mathcal{L}_E^1$. If for each subsequence (f_{k_j}) of (f_k), each $\epsilon > 0$, and each B in $\mathcal{F}^+$, there exists a subset A in $\mathcal{F}^+(B)$ such that*

$$\liminf_j \sup_{x^* \in B_{E^*}} \frac{\int_A |x^*(f_{k_j})| d\mu}{\mu(A)} < \epsilon$$

then (f_k) converges to 0 in the Pettis norm.

Proof. Assume (f_k) is Pettis uniformly integrable but does not converge to 0 in the Pettis norm. Since (f_k) is Pettis uniformly integrable, the subset $\{|x^*(f_k)| : x^* \in B_{E^*}, k \in \mathbb{N}\}$ of $\mathcal{L}_{\mathbb{R}}^1$ is relatively weakly compact. So there exists $\epsilon > 0$, a subsequence (f_{k_j}) of (f_k), a sequence $(x_{k_j}^*)$ in B_{E^*}, and g in $\mathcal{L}_{\mathbb{R}}^1$ such that $2\epsilon < \int_\Omega |x_{k_j}^*(f_{k_j})| d\mu$

and $|x^*_{k_j}(f_{k_j})| \to g$ weakly in $\mathcal{L}^1_{\mathbb{R}}$. Since $2\epsilon \leq \int_\Omega g\, d\mu$, the set $B := [g > \epsilon]$ is in $\mathcal{F}^+$. For any subset A of B with positive measure

$$\liminf_j \sup_{x^* \in B_{E^*}} \frac{\int_A |x^*(f_{k_j})| d\mu}{\mu(A)} \geq \liminf_j \frac{\int_A |x^*_{k_j}(f_{k_j})| d\mu}{\mu(A)} = \frac{\int_A g\, d\mu}{\mu(A)} > \epsilon.$$

Thus the lemma holds. □

The Pettis-norm analogue to Theorems 2.3 and 3.8 now follows with ease.

Theorem 5.11. *A sequence (f_k) in $\mathcal{L}^1_E$ converges in the Pettis norm to f_0 in $\mathcal{L}^1_E$ if and only if*

(1) *(f_k) is Pettis uniformly integrable*
(2) *(f_k) satisfies the sequential Pettis Bocce criterion*
(3) $\lim_k \|m_B(f_k) \Leftrightarrow m_B(f_0)\| = 0$ *for each* $B \in \mathcal{F}^+$.

Proof. Consider a sequence (f_k) that converges in the Pettis norm to f_0 in $\mathcal{L}^1_E$. It is easy to check that conditions (1) and (3) hold. Since for x^* in B_{E^*} and A in $\mathcal{F}^+$

$$\left| \text{Bocce-osc } x^*(f_k)\big|_A \Leftrightarrow \text{Bocce-osc } x^*(f_0)\big|_A \right| \leq \frac{2}{\mu(A)} \|f_k \Leftrightarrow f_0\|_{\text{Pettis}}$$

and Bocce-osc $x^*(f_0)\big|_A \leq$ Bocce-osc $f_0\big|_A$, from Lemma 2.5 we see that (f_k) satisfies the sequential Pettis Bocce criterion.

As for the other implication, consider a sequence (f_k) which satisfies conditions (1), (2) and (3). To show that $f_k \to f_0$ in the Pettis norm, we will show that $(f_k \Leftrightarrow f_0)$ satisfies the conditions of Lemma 5.10. First note that condition (1) gives that $(f_k \Leftrightarrow f_0)$ is Pettis uniformly integrable. Fix $\epsilon > 0$ and B in $\mathcal{F}^+$. Consider a subsequence (f_{k_j}) of (f_k). Since $(f_k \Leftrightarrow f_0)$ satisfies the sequential Pettis Bocce criterion, there is a set A in $\mathcal{F}^+(B)$ such that $\liminf_j$ Pettis-Bocce-osc $(f_{k_j} \Leftrightarrow f_0)\big|_A < \epsilon$. Since

$$\sup_{x^* \in B_{E^*}} \frac{\int_A |x^*(f_{k_j} \Leftrightarrow f_0)| d\mu}{\mu(A)} \Leftrightarrow \|m_A(f_{k_j} \Leftrightarrow f_0)\| \leq \text{Pettis-Bocce-osc } (f_{k_j} \Leftrightarrow f_0)\big|_A,$$

using (3) we see that

$$\liminf_j \sup_{x^* \in B_{E^*}} \frac{\int_A |x^*(f_{k_j} \Leftrightarrow f_0)| d\mu}{\mu(A)} < \epsilon$$

as needed. Thus $f_k \to f_0$ in the Pettis norm. □

Remark 5.4 ties weak convergence into Theorem 5.11.

Corollary 5.12. *A sequence* (f_k) *in* $\mathcal{L}_E^1$ *converges in the Pettis norm to* f_0 *in* $\mathcal{L}_E^1$ *and is uniformly integrable if and only if*

(1) (f_k) *converges to* f_0 *in the* $\sigma(\mathcal{L}_E^1, \mathcal{L}_{E^*}^\infty)$*-topology*
(2) (f_k) *satisfies the sequential Pettis Bocce criterion*
(3) $\Delta_B := \{m_B(f_k) : k \in N\}$ *is relatively norm compact in* E *for each* B *in* $\mathcal{F}^+$.

Condition (3) *may be replaced by*

(3') $\lim_k ||m_B(f_k) \Leftrightarrow m_B(f_0)|| = 0$ *for each* B *in* $\mathcal{F}^+$.

Furthermore, if E^* *has the RNP, then* (1) *is equivalent to*

(1') (f_k) *converges to* f_0 *weakly in* $\mathcal{L}_E^1$.

Note that under (1), conditions (3) and (3') are equivalent.

Since a Pettis convergent sequence need not be tight (consider Example 5.5 along with Fact 4.2), there is no Pettis-analogue to Theorem 4.5.

6. Variation of the Bocce Criterion

As noted in this section, several variations of the sequential Bocce criterion also provided necessary and sufficient conditions to pass from weak to strong convergence. For a sequence (f_k) of functions in $\mathcal{L}_E^1$, consider the following Bocce-like oscillation conditions.

The sequence (f_k) satisfies oscillation condition (B0) if for each $\epsilon > 0$ and each B in $\mathcal{F}^+$ there is a set C in $\mathcal{F}^+(B)$ and $N \in \mathbb{N}$ such that

$$\text{Bocce-osc } f_k\big|_C < \epsilon$$

for each $k \geq N$.

The sequence (f_k) satisfies oscillation condition (B1) if for each $\epsilon > 0$ there is a finite measurable partition $\pi = (A_i)_{i=0}^p$ of Ω with $\mu(A_0) < \epsilon$ and $N \in \mathbb{N}$ such that

$$\text{Bocce-osc } f_k\big|_{A_i} < \epsilon$$

for each $k \geq N$ and $1 \leq i \leq p$.

The sequence (f_k) satisfies oscillation condition (B2) if for each $\epsilon > 0$ there is a finite measurable partition $\pi = (A_i)_{i=0}^p$ of Ω with $\mu(A_0) < \epsilon$ such that for each collection $(B_i)_{i=1}^p$ of sets with B_i in $\mathcal{F}^+(A_i)$ there is $N \in \mathbb{N}$ such that

$$\text{Bocce-osc } f_k\big|_{B_i} < \epsilon$$

for each $k \geq N$ and $1 \leq i \leq p$.

The above 3 oscillation conditions have appeared in the literature [V2, B3, J] under various names. In [J], it is shown that

$$(B2) \Rightarrow (B1) \Leftrightarrow (B0) \ .$$

The proof that (B2) implies (B0) and the proof that (B1) implies (B0) are both straightforward while the proof that (B0) implies (B1) involves an exhaustion argument. It is straightforward [cf. G1] to show that (B1) implies the sequential Bocce criterion.

If the sequence (f_k) in $\mathcal{L}^1_E$ converges strongly then it satisfies (B2). This follows from minor variations of earlier arguments and noting that Lemma 2.5 may be strengthened.

Lemma 2.5 – revisited. *Let f be in $\mathcal{L}^1_E$. For every $\epsilon > 0$ there is a finite measurable partition $\pi = (A_i)_{i=0}^p$ of Ω with $\mu(A_0) < \epsilon$ such that for each collection $(B_i)_{i=1}^p$ of sets with B_i in $\mathcal{F}^+(A_i)$*

$$\text{Bocce-osc } f\big|_{B_i} < \epsilon$$

for each $1 \le i \le p$.

Thus in Theorem 2.3 (and thus also in the related theorems) oscillation condition (2) may be replaced with the condition that (f_k) satisfies either oscillation condition (B2), (B1), or (B0).

As for the subset analogue, recall [G1] that a subset K of $\mathcal{L}^1_E$ is a set of small Bocce oscillation if for each $\epsilon > 0$ there is a finite measurable partition $\pi = (A_i)_{i=1}^p$ of Ω such that for each f in K

$$\sum_{i=1}^{p} \mu(A_i) \text{ Bocce-osc } f\big|_{A_i} \ < \ \epsilon \ .$$

As in the $\mathcal{L}^1_{\mathbb{R}}$ case [G1], a relatively strongly compact set is a set of small Bocce oscillation and a set of small Bocce oscillation satisfies the Bocce criterion. Thus in Theorem 2.4 the oscillation condition (2) may be replaced by the condition that K be a set of small Bocce oscillation.

Acknowledgment

The authors express our appreciation to the Department of Mathematics at the University of South Carolina for providing the financial support which enabled us to start this work. We are grateful to M. Valadier and M. Moussaoui for their suggestions and discussions concerning the Section 4 and for the anonymous referee's careful suggestions that helped clarify many fine details. B. Dawson [Da] has independently obtained results in the spirit of this paper's Theorem 2.4.

References

[ACV]. A. Amrani, C. Castaing, and M. Valadier, *Méthodes de troncature appliquées à des problèmes de convergence faible ou forte dans L^1*, Arch. Rational Mech. Anal. **117** (1992), 167–191.

[B1]. E.J. Balder, *On weak convergence implying strong convergence in L_1-spaces*, Bull. Austral. Math. Soc. **33** (1986), 363–368.

[B2]. E.J. Balder, *On equivalence of strong and weak convergence in L_1-spaces under extreme point conditions*, Isr. J. Math. **75** (1991), 1–23.

[B3]. E.J. Balder, *From weak to strong L_1-convergence by an oscillation restriction criterion of BMO type*, Preprint No. 666, Dept. of Math., University of Utrecht, 1991.

[B4]. E.J. Balder, *A general approach to lower semicontinuity and lower closure in optimal control theory*, SIAM J. Control Optim. **22** (1984), 570–598.

[B5]. E.J. Balder, *On Prohorov's theorem for transition probabilities*, Sém. Anal. Convexe **19** (1989), 9.1–9.11.

[BH1]. J. Batt and W. Hiermeyer, *Weak compactness in the space of Bochner integrable functions*, Unpublished manuscript (1980).

[BH2]. J. Batt and W. Hiermeyer, *On compactness in $L_p(\mu, X)$ in the weak topology and in the topology $\sigma(L_p(\mu, X), L_q(\mu, X'))$*, Math. Z. **182** (1983), 409–423.

[BS]. J. Batt and G. Schlüchtermann, *Eberlein Compact in $L_1(X)$*, Studia Math. **83** (1986), 239–250.

[BD]. J.K. Brooks and N. Dinculeanu, *Weak compactness in spaces of Bochner integrable functions*, Adv. Math. **24** (1977), 172–188.

[C1]. C. Castaing, *Un résultat de compacité lié à la propriété des ensembles Dunford-Pettis dans $L^1_F(\Omega, \mathcal{A}, \mu)$*, Sém. Anal. Convexe **9** (1979), 17.1–17.7.

[C2]. C. Castaing, *Sur la décomposition de Slaby. Applications aux problèmes de convergences en probabilités. Economie mathématique. Théorie du contrôle. Minimisation*, Sém. Anal. Convexe **19** (1989), 3.1–3.35.

[CV]. C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Math., vol. 580, Springer-Verlag, Berlin, 1977.

[Da]. B. Dawson, *Convergence of Conditional Expectation Operators and the Compact Range Property*, Ph. D. Dissertation, University of North Texas, 1992.

[D1]. J. Diestel, *Sequences and Series in Banach Spaces*, Graduate Texts in Math., vol. 92, Springer-Verlag, New York-Berlin, 1984.

[D2]. J. Diestel, *Uniform integrability: an introduction*, School on Measure Theory and Real Analysis. Grado (Italy), October 14-25, 1991, Rend. Istit. Mat. Univ. Trieste **XXIII** (1991), 41–80.

[DU]. J. Diestel and J.J. Uhl, *Vector Measures*, Amer. Math. Soc., Providence, 1977.

[DG]. S.J. Dilworth and M. Girardi, *Bochner vs. Pettis norms: examples and results*, Banach Spaces (Bor-Luh Lin and W. B. Johnson, eds.), Contemp. Math., vol. 144, American Mathematical Society, Providence, Rhode Island, 1993, pp. 69–80.

[Ga]. V.F. Gaposhkin, *Convergence and limit theorems for sequences of random variables*, Theory Prob. and Appl. **17** (1972), 379–400.

[G1]. M. Girardi, *Compactness in L_1, Dunford-Pettis operators, geometry of Banach spaces*, Proc. Amer. Math. Soc. **111** (1991), 767–777.

[G2]. M. Girardi, *Weak vs. norm compactness in L_1; the Bocce criterion*, Studia Math. **98** (1991), 95–97.

[HU]. F. Hiai and H. Umegaki, *Integrals, Conditional Expectations, and Martingales of Multivalued Functions*, J. Multivariate Anal. **7** (1977), 149–182.

[IT]. A. and C. Ionescu-Tulcea, *Topics in the Theory of Lifting*, Springer-Verlag, Berlin, 1969.

[J]. V. Jalby, *Contribution aux problèmes de convergence des fonctions vectorielles et des intégrales fonctionnelles*, Thèse de Doctorat, Université Montpellier II, 1993.

[Jaw]. A. Jawhar, *Mesures de transition et applications*, Sém. Anal. Convexe **14** (1984), 13.1–13.62.

[N]. J. Neveu, *Mathematical Foundations of the Calculus of Probability*, Holden-Day, San Fransisco, 1965.

[P]. B.J. Pettis, *On integration in vector spaces*, Trans. Amer. Math. Soc. **44** (1938), 277–304.

[SW]. G. Schlüchtermann and R.F. Wheeler, *On strongly WCG Banach spaces*, Math. Z. **199** (1988), 387–398.

[S]. L. Schwartz, *Radon Measures*, Oxford University Press, London, 1973.

[T]. M. Talagrand, *Weak Cauchy sequences in $L_1(E)$*, Amer. J. Math. **106** (1984), 703–724.

[V1]. M. Valadier, *Young Measures*, Methods of Nonconvex Analysis (A. Cellina, ed.), Lecture Notes in Mathematics, vol. 1446, Springer-Verlag, Berlin and New York, 1990, pp. 152–188.

[V2]. M. Valadier, *Oscillations et compacité forte dans L_1*, Sém. Anal. Convexe **21** (1991), 7.1–7.10.

Mathematical Institute, University of Utrecht, P.O. Box 80.010, 3508 TA Utrecht, Netherlands

E-mail address: balder@math.ruu.nl

Department of Mathematics, University of South Carolina, Columbia, South Carolina 29208, U.S.A.

E-mail address: girardi@math.scarolina.edu

Département de Mathématiques, Université Montpellier II, 34095 Montpellier Cedex 05, France

Current address: Département de Mathématiques, Université de Limoges, 87060 Limoges Cedex, France

E-mail address: jalby@math.univ-montp2.fr