ERRATA CORRECTION TO DENTABILITY, TREES, AND DUNFORD-PETTIS OPERATORS ON L_1

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Abstract. A Banach space has the complete continuity property if all its bounded subsets are midpoint Bocce dentable. We show that a lemma used in the original proposed proof of this result if false; however, we give a proof to show that the result is indeed true.

1. INTRODUCTION

Throughout this paper, $\mathfrak{X}$ denotes an arbitrary Banach space, $\mathfrak{X}^*$ the dual space of $\mathfrak{X}$, $B(\mathfrak{X})$ the closed unit ball of $\mathfrak{X}$, and $S(\mathfrak{X})$ the unit sphere of $\mathfrak{X}$. The triple (Ω, Σ, μ) refers to the Lebesgue measure space on $[0,1]$, Σ^+ to the sets in Σ with positive measure, and L_1 to $L_1(\Omega, \Sigma, \mu)$. The σ-field generated by a partition π of $[0,1]$ is $\sigma(\pi)$. The conditional expectation of $f \in L_1$ given a σ-field $\mathcal{B}$ is $E(f|\mathcal{B})$.

A Banach space $\mathfrak{X}$ has the *complete continuity property* (CCP) if each bounded linear operator from L_1 into $\mathfrak{X}$ is *Dunford-Pettis* (i.e. carries weakly convergent sequences onto norm convergent sequences). Since a representable operator is Dunford-Pettis, the CCP is a weakening of the Radon-Nikodým property (RNP). Recall that a Banach space has the RNP if and only if all its bounded subsets are dentable. A subset D of $\mathfrak{X}$ is *dentable* if for each $\epsilon > 0$ there is x in D such that $x \notin \overline{\text{co}}\, (\{y \in D : \| x - y \| \geq \epsilon\})$. Midpoint Bocce dentability is a weakening of dentability. The subset D is *midpoint Bocce dentable* if for each $\epsilon > 0$ there is a finite subset F of D such that for each x^* in $B(\mathfrak{X}^*)$ there is x in F satisfying:

$$\text{if} \quad x = \tfrac{1}{2}z_1 + \tfrac{1}{2}z_2 \quad \text{with } z_i \in D \quad \text{then} \quad |\, x^*(x - z_1)\, | \equiv |\, x^*(x - z_2)\, | < \epsilon.$$

The following theorem is presented in [G1].

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THEOREM 1. *$\mathfrak{X}$ has the CCP if each bounded subset of $\mathfrak{X}$ is midpoint Bocce dentable.*

Our purpose in writing this note is to show that Lemma 2.9 in [G1] (which was used in [G1] to prove Theorem 1) is false and to provide a proof of the theorem. Lemma 2.9 asserts that if A is in Σ^+ and f in $L_\infty(\mu)$ is not constant a.e. on A, then there is an increasing sequence $\{\pi_n\}$ of positive finite measurable partitions of A such that $\sigma(\cup\pi_n) = \Sigma \cap A$ and for each n

$$\mu\left(\bigcup\left\{E \quad : \quad E \in \pi_n \quad \text{and} \quad \frac{\int_E f\,d\mu}{\mu(E)} \geq \frac{\int_A f\,d\mu}{\mu(A)}\right\}\right) = \frac{\mu(A)}{2}\ .$$

Example 2 shows that Lemma 2.9 is false.

EXAMPLE 2. Let $f = 3\chi_{[0,\frac{1}{4})} - \chi_{[\frac{1}{4},1]}$. Then $\int_\Omega f\,d\mu = 0$. Suppose that $\{\pi_n\}$ is an increasing sequence of positive finite measurable partitions of $[0,1]$ such that for each n

$$\mu\left(\bigcup\left\{E \quad : \quad E \in \pi_n \quad \text{and} \quad \frac{\int_E f\,d\mu}{\mu(E)} \geq 0\right\}\right) = \frac{1}{2}\ .$$

Then $\sigma(\cup\pi_n) \neq \Sigma$.

PROOF. Consider the martingale $\{f_n\}$ given by

$$f_n(\cdot) \;=\; E(f|\sigma(\pi_n)) \;=\; \sum_{E\in\pi_n} \frac{\int_E f\,d\mu}{\mu(E)}\,\chi_E\,(\cdot)\ .$$

For each $n \in \mathbb{N}$ put

$$P_n \;=\; \bigcup\left\{E \;:\; E \in \pi_n \text{ and } \int_E f\,d\mu \geq 0\right\} \quad \text{and} \quad Q_n \;=\; P_n \cap (\tfrac{1}{4}, 1]\ .$$

Since $\mu(P_n) = \frac{1}{2}$, we have that $\mu(Q_n) \geq \frac{1}{4}$. Thus

$$\int_\Omega |\, f_n - f\,|\; d\mu \;\geq\; \int_{Q_n} |\, f_n - f\,|\; d\mu \;\geq\; \int_{Q_n} (f_n - -1)\,d\mu$$
$$\geq\; \int_{Q_n} 1\,d\mu \;=\; \mu(Q_n) \;\geq\; \frac{1}{4}\ .$$

We know that such a martingale $E(f|\sigma(\pi_n))$ converges in L_1 norm to $E(f|\sigma(\cup\pi_n))$. But $E(f|\Sigma) = f$. Thus $\sigma(\cup\pi_n) \neq \Sigma$. □

The error in the proof of Lemma 2.9 occurred in assuming that if A is in Σ^+ and $\{\pi_n\}$ is an increasing sequence of positive measurable partitions of A such that for each n and each E in π_n the $\mu(E) \le \epsilon_n$ with $\lim_n \epsilon_n = 0$, then $\sigma(\cup\pi_n) = \Sigma \cap A$. This seemingly sound assertion is not true as shown by the following counterexample.

EXAMPLE 3. For $n \in \mathbb{N}$ and $1 \le i \le 2^n$, define

$$E_i^n = \left[\frac{i-1}{2^{n+1}}, \frac{i}{2^{n+1}}\right) \cup \left[\frac{1}{2} + \frac{i-1}{2^{n+1}}, \frac{1}{2} + \frac{i}{2^{n+1}}\right)$$

and

$$\pi_n = \{E_i^n : 1 \le i \le 2^n\} .$$

Clearly $\{\pi_n\}$ is an increasing sequence of positive measurable partitions of $[0,1]$ such that for each n and each E in π_n the $\mu(E) = 2^{-n}$. Let $f = \chi_{[0,\frac{1}{2})}$. An easy computation shows that $E(f|\sigma(\pi_n)) = \frac{1}{2}\chi_{[0,1]}$. We know that such a martingale $E(f|\sigma(\pi_n))$ converges in L_1 norm to $E(f|\sigma(\cup\pi_n))$. But $E(f|\Sigma) = f$. Thus $\sigma(\cup\pi_n) \ne \Sigma$. $\square$

2. PROOF OF THEOREM

Our proof of Theorem 1 uses the following observations. For f in L_1 and A in Σ, the average value and the Bocce oscillation of f on A respectively are

$$m_A(f) \equiv \frac{\int_A f\,d\mu}{\mu(A)}$$

and

$$\text{Bocce-osc } f\big|_A \equiv \frac{\int_A |f - m_A(f)|\,d\mu}{\mu(A)}$$

observing the convention that $0/0$ is 0.

LEMMA 4. Fix A in Σ and f in L_1. There is a subset E of A with $2\,\mu(E) = \mu(A)$ and

$$\frac{1}{2} \text{ Bocce-osc } f\big|_A \le |\, m_E(f) - m_A(f)\,| \ .$$

Furthermore, for each subset E of A with $2\,\mu(E) = \mu(A)$,

$$|\, m_E(f) - m_A(f)\,| \le \text{ Bocce-osc } f\big|_A \ .$$

PROOF. Without loss of generality, $A = \Omega$ and $\int_\Omega f\,d\mu = 0$ and $\int_\Omega |\, f\,|\,d\mu = 1$. With this normalization, Bocce-osc $f\big|_A = 1$ and $|\, m_E(f) - m_A(f)\,| = |\, m_E(f)\,|$. Let $P = [f \ge 0]$ and $N = [f < 0]$.

The first claim now reads that $\frac{1}{2} \leq 2 \mid \int_E f\,d\mu \mid$ for some subset E of measure one half. Wlog $\mu(P) \geq \frac{1}{2}$. Partition P into 2 sets, P_1 and P_2, of equal measure such that $\int_{P_2} f\,d\mu \leq \int_{P_1} f\,d\mu$. Note that

$$\begin{aligned} 1 &= \int_\Omega \mid f \mid \, d\mu = \int_P f\,d\mu + \int_N -f\,d\mu \\ &= 2 \int_P f\,d\mu = 2 \left[\int_{P_1} f\,d\mu + \int_{P_2} f\,d\mu \right] \leq 4 \int_{P_1} f\,d\mu \,. \end{aligned}$$

Since $\mu(P_1) \leq \frac{1}{2} \leq \mu(P)$, we can find a set E such that $P_1 \subset E \subset P$ and $\mu(E) = \frac{1}{2}$. For such a set E

$$\frac{1}{4} \leq \int_{P_1} f\,d\mu \leq \int_E f\,d\mu \,,$$

as needed.

Normalized, the second claim reads that for each subset E of measure $\frac{1}{2}$

$$2 \left| \int_E f\,d\mu \right| \leq 1 \,.$$

Fix a subset E of measure $\frac{1}{2}$. Wlog $\int_{E\cap N} -f\,d\mu \leq \int_{E \cap P} f\,d\mu$. So

$$\left| \int_E f\,d\mu \right| = \left| \int_{E\cap P} f\,d\mu + \int_{E\cap N} f\,d\mu \right| \leq \left| \int_{E \cap P} f\,d\mu \right| \leq \int_P \mid f \mid \, d\mu = \tfrac{1}{2} \,,$$

as needed. □

A subset K of L_1 satisfies the *Bocce criterion* if for each $\epsilon > 0$ and B in Σ^+ there is a finite collection $\mathcal{F}$ of subsets of B each with positive measure such that for each f in K there is an A in $\mathcal{F}$ satisfying

$$(*) \qquad \text{Bocce-osc}\, f\big|_A < \epsilon \,.$$

Lemma 4 provides an equivalent formulation of the Bocce criterion; namely we can replace condition $(*)$ by the condition

$(**)$

if the subset E of A has half the measure of A, then $\mid m_E(f) - m_A(f) \mid < \epsilon$.

We now attack the proof of Theorem 1. Our proof follows mainly the proof in [G1].

PROOF OF THEOREM 1. Let all bounded subsets of $\mathfrak{X}$ be midpoint Bocce dentable. Fix a bounded linear operator T from L_1 into $\mathfrak{X}$. It suffices to show that the subset $T^*(B(\mathfrak{X}^*))$ of L_1 satisfies the Bocce criterion (this is a necessary and sufficient condition for T to be Dunford-Pettis [G2]). To this end, fix $\epsilon > 0$ and B in Σ^+.

Consider the vector measure F from Σ into $\mathfrak{X}$ given by $F(E) = T(\chi_E)$. For $x^* \in \mathfrak{X}^*$

$$m_E(T^*x^*) \;=\; \frac{x^*F(E)}{\mu(E)}$$

since $\int_E (T^*x^*)\,d\mu \;=\; x^*T(\chi_E) \;=\; x^*F(E)$.

Since the subset $\{\frac{F(E)}{\mu(E)} : E \subset B \text{ and } E \in \Sigma^+\}$ of $\mathfrak{X}$ is bounded, it is midpoint Bocce dentable. Accordingly, there is a finite collection $\mathcal{F}$ of subsets of B each in Σ^+ such that for each $x^* \in B(\mathfrak{X}^*)$ there is a set A in $\mathcal{F}$ such that if

$$\frac{F(A)}{\mu(A)} \;=\; \frac{1}{2}\,\frac{F(E_1)}{\mu(E_1)} \;+\; \frac{1}{2}\,\frac{F(E_2)}{\mu(E_2)}$$

for some subsets E_i of B with $E_i \in \Sigma^+$, then

$$\left|\frac{x^*F(E_1)}{\mu(E_1)} - \frac{x^*F(A)}{\mu(A)}\right| \;=\; \left|\frac{x^*F(E_2)}{\mu(E_2)} - \frac{x^*F(A)}{\mu(A)}\right| \quad < \quad \epsilon\ .$$

Fix $x^* \in B(\mathfrak{X}^*)$ and find the associated A in $\mathcal{F}$.

At this point we turn to our new formulation of the Bocce criterion (whereas [G1] used the old formulation and Lemma 2.9).

This $A \in \mathcal{F}$ satisfies the condition $(**)$. For consider a subset E of A with $\mu(E) = \frac{1}{2}\mu(A)$. Since

$$\frac{F(A)}{\mu(A)} \;=\; \frac{1}{2}\,\frac{F(E)}{\mu(E)} \;+\; \frac{1}{2}\,\frac{F(A\setminus E)}{\mu(A\setminus E)}$$

we have that

$$|\; m_E(T^*x^*) - m_A(T^*x^*) \;| \equiv \left|\frac{x^*F(E)}{\mu(E)} - \frac{x^*F(A)}{\mu(A)}\right| \quad < \quad \epsilon\ .$$

Thus $T^*(B(\mathfrak{X}^*))$ satisfies the Bocce criterion, as needed. $\square$

3. CLOSING COMMENTS

A relatively weakly compact subset of L_1 is relatively norm compact if and only if it satisfies the Bocce criterion [G2]. Thus our new formulation of the Bocce criterion provides another (perhaps at times more useful) method for testing for norm compactness in L_1 .

Fix A in Σ^+ and f in L_1. Put

$$M_A(f) \;=\; \sup\,\{|\, m_E(f) - m_A(f)\,|\,:\; E \subset A \quad \text{and} \quad 2\,\mu(E) = \mu(A)\}\ .$$

This supremum is obtained. For just normalize so that $A = \Omega$ and $\int_\Omega f\,d\mu = 0$ and $\int_\Omega \mid f \mid\, d\mu = 1$. As Ralph Howard pointed out, next find disjoint subsets E_1 and E_2 of measure $\frac{1}{2}$ and $a \in \mathbb{R}$ such that

$$E_1 \subset [f \le a] \qquad \text{and} \qquad E_2 \subset [f \ge a]\ .$$

Then $M_A(f)$ will be the larger of $|\, m_{E_1}(f)\,|$ and $|\, m_{E_2}(f)\,|$.

Basically, our lemma 4 says that

$$\frac{1}{2}\ \text{Bocce-osc}\ f\big|_A \;\le\; M_A(f) \;\le\; \text{Bocce-osc}\ f\big|_A\ .$$

These bounds are the best possible.

For the second inequality, consider the function defined on $A \;\equiv\; [0,1]$ by

$$f \;=\; \chi_{[0,\frac{1}{2})} \;-\; \chi_{[\frac{1}{2},1]}\ .$$

Straightforward calculations show that $m_{[0,\frac{1}{2}]}(f) = 1$ and that Bocce-osc $f\big|_A = 1$. Thus

$$M_A(f) \;=\; \text{Bocce-osc}\ f\big|_A\ .$$

As for the first inequality, consider the family of functions defined on $A \;\equiv\; [0,1]$ by

$$f_\delta \;=\; \frac{\delta - 1}{\delta}\ \chi_{[0,\delta)} \;+\; \chi_{[\delta,1]}$$

for $0 < \delta < \frac{1}{2}$. Straightforward calculations show that

$$M_A(f_\delta) \;=\; \frac{1}{2\ (1-\delta)}\ \text{Bocce-osc}\ f_\delta\big|_A\ .$$

Actually $M_A(f) \;=\; \frac{1}{2}$ Bocce-osc $f\big|_A$ if and only if f is the zero function on A.

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