WEAK VS. NORM COMPACTNESS IN L_1: THE BOCCE CRITERION

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ABSTRACT. We present a new simple proof that if a relatively weakly compact subset of L_1 satisfies the Bocce criterion (an oscillation condition), then it is relatively norm compact. The converse of this fact is easy to verify. A direct consequence is that, for a bounded linear operator T from L_1 into a Banach space $\mathfrak{X}$, T is Dunford-Pettis if and only if the subset $T^*(B(\mathfrak{X}^*))$ of L_1 satisfies the Bocce criterion.

A relatively weakly compact subset of L_1 is relatively norm compact if and only if it satisfies the Bocce criterion (an oscillation condition) [G1]. We shall present a new simple proof that if a relatively weakly compact subset of L_1 satisfies the Bocce criterion, then it is relatively norm compact. The converse is easy to verify.

Recall that a Banach space $\mathfrak{X}$ has the *complete continuity property* (CCP) if each bounded linear operators from L_1 into $\mathfrak{X}$ is *Dunford-Pettis* (i.e. maps weakly convergent sequences onto norm convergent sequences). The CCP is a weakening of the Radon-Nikodým property and of strong regularity. Since a bounded linear operator T from L_1 into $\mathfrak{X}$ is Dunford-Pettis if and only if the subset $T^*(B(\mathfrak{X}^*))$ of L_1 is relatively norm compact, the above fact gives that T is Dunford-Pettis if and only if $T^*(B(\mathfrak{X}^*))$ satisfies the Bocce criterion. This oscillation characterization of Dunford-Pettis operators leads to dentability and tree characterizations of the CCP [G2]. Namely, $\mathfrak{X}$ has the CCP if and only if all bounded subsets of $\mathfrak{X}$ are weak-norm-one dentable. Also, $\mathfrak{X}$ has the CCP if and only if no bounded separated δ-trees grow in $\mathfrak{X}$, or equivalently, no bounded δ-Rademacher trees grow in $\mathfrak{X}$.

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Throughout this note, $\mathfrak{X}$ denotes an arbitrary Banach space. The triple (Ω, Σ, μ) refers to the Lebesgue measure space on $[0,1]$, Σ^+ to the sets in Σ with positive measure, and L_1 to $L_1(\Omega, \Sigma, \mu)$. All notation and terminology, not otherwise explained, are as in [DU].

[G1] introduces the following definitions.

Definitions. For f in L_1 and A in Σ^+, the *Bocce oscillation of f on A* is given by

$$\text{Bocce-osc}\ f\big|_A \ \equiv\ \frac{\int_A |f - \frac{\int_A f\,d\mu}{\mu(A)}|\,d\mu}{\mu(A)} \quad .$$

A subset K of L_1 satisfies the *Bocce criterion* if for each $\epsilon > 0$ and B in Σ^+ there is a finite collection $\mathcal{F}$ of subset of B each with positive measure such that for each f in K there is an A in $\mathcal{F}$ satisfying $\text{Bocce-osc}\ f\big|_A < \epsilon$.

This note's main purpose is to present a new proof to the theorem below. The author is grateful to Michel Talagrand for his helpful discussions concerning this theorem and proof.

Theorem. *If a relatively weakly compact subset of L_1 satisfies the Bocce criterion, then it is relatively L_1-norm compact.*

We need the following lemma which we shall verify after the proof of the Theorem.

Lemma. *If a subset of L_1 satisfies the Bocce criterion, then the translate of that set by a L_1-function also satisfies the Bocce criterion.*

Proof of the Theorem. Assume that the relatively weakly compact subset K of L_1 is not relatively norm compact. We shall show that K does not satisfy the Bocce criterion.

Since K is not relatively norm compact but is relatively weakly compact, there is a sequence $\{f_n\}$ in a translate $\tilde{K}$ of K satisfying

(1) $\{f_n\}$ has no L_1-convergent subsequence

(2) $\{f_n\}$ converges weakly in L_1 to 0

(3) $\{\,|\,f_n\,|\,\}$ converges weakly in L_1, say to f

(4) $\int f\,d\mu\ \geq 4\epsilon$ for some $\epsilon > 0$.

Set $B\ =\ [\ f \geq 3\epsilon\ ]$. Condition (4) guarantees that $B \in \Sigma^+$.

Let $\mathcal{F}$ be a finite collection of subsets of B, each with positive measure. Choose N such that for each $A \in \mathcal{F}$

(5) $|\int_A f_N\,d\mu\,| < \ \epsilon\ \mu(A)$ (possible by (2))

(6) $|\int_A f\,d\mu\ -\ \int_A |\,f_N\,|\ d\mu\,| < \ \epsilon\ \mu(A)$ (possible by (3)).

Then for each $A \in \mathcal{F}$ we have that

$$\begin{aligned}\text{Bocce-osc } f_N\big|_A \ &\equiv\ \frac{\int_A |f_N - \frac{\int_A f_N\,d\mu}{\mu(A)}|\,d\mu}{\mu(A)} \qquad \geq\ \frac{\int_A |f_N|\,d\mu}{\mu(A)} - \frac{|\int_A f_N\,d\mu|}{\mu(A)} \\ &\geq\ \frac{\int_A f\,d\mu\ -\ \epsilon\mu(A)}{\mu(A)} - \frac{\epsilon\mu(A)}{\mu(A)} \geq \frac{3\epsilon\mu(A)}{\mu(A)} - \epsilon - \epsilon = \epsilon\ .\end{aligned}$$

Thus, $\tilde{K}$ does not satisfy the Bocce criterion and so K also does not satisfy the Bocce criterion. ■

Proof of the Lemma. Let the subset K of L_1 satisfies the Bocce criterion and $f \in L_1$. We need to show that the set $K+f \equiv \{\,g+f\ :\ g \in K\,\}$ satisfies the Bocce criterion. Towards this end, fix $\epsilon > 0$ and $B \in \Sigma^+$. Find $B_0 \subset B$ with $B_0 \in \Sigma^+$ such that f is bounded on B_0.

Approximate $f\chi_{B_0}$ in L_∞-norm within $\frac{\epsilon}{4}$ by a simple function $\tilde{f}$. Find $C \subset B_0$ with $C \in \Sigma^+$ such that $\tilde{f}$ is constant on C. Since K satisfies the Bocce criterion, we can find a finite collection $\mathcal{F}$ of subsets corresponding to $\frac{\epsilon}{2}$ and C.

Fix $g + f \in K + f$. Find $A \in \mathcal{F}$ such that Bocce-osc $g\big|_A < \frac{\epsilon}{2}$. Note that since $\tilde{f}$ is constant on A, Bocce-osc $g\big|_A$ = Bocce-osc $(g + \tilde{f})\big|_A$. Now,

$$\begin{aligned} \text{Bocce-osc } (g+f)\big|_A &\leq \text{Bocce-osc } (g+\tilde{f})\big|_A + \text{Bocce-osc } (\tilde{f}-f)\big|_A \\ &\leq \text{Bocce-osc } g\big|_A + 2 \, \| (\tilde{f}-f)\, \chi_A \|_{L_\infty} < \epsilon \ . \end{aligned}$$

Thus $K + f$ satisfies the Bocce criterion. ■

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