Contents

Chapter 1
Normed Spaces

1.1 Examples of Normed Spaces

In the following, vector spaces over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ are considered. For sanity reasons, unless otherwise stated, the trival vector space $\{0\}$ is excluded. So to consider both cases simultaneously, $\mathbb{K}$-vector space denotes both real and complex vector spaces.

Definition 1.1.1. Let X be a $\mathbb{K}$-vector space. A mapping $p\colon X \to [0,\infty)$ is a *semi-norm* provided

(a) $\quad p(\lambda x) = |\lambda|\, p(x) \quad \forall \lambda \in \mathbb{K}, x \in X$

(b) $\quad p(x+y) \leq p(x) + p(y) \quad \forall x, y \in X$.

Condition (b) is the *triangle inequality*. If, in addition,

(c) $\quad p(x) = 0 \;\Rightarrow\; x = 0$

then p is a *norm*. Norms are usually denoted by $\|\cdot\|$ instead of p. The pair (X, p) is a *(semi)normed space*. For convenience, (X, p) is denoted by just X if the (semi)norm p, with which X is endowed, is understood by context.

Note that (a) implies $p(0) = 0$; indeed, apply (a) with $\lambda = 0$ and $x = 0$ to see that

$$p(0) = p(0 \cdot 0) = 0 \cdot p(0) = 0 \,.$$

A normed space (X, p) induces a natural metric, namely

$$d(x, y) = \|x - y\| \quad \forall x, y \in X \,,$$

since, as it is easy to verify, the following axioms hold for each $x, y, z \in X$.

(a) $\quad d(x, y) \geq 0$

(b) $\quad d(x, y) = d(y, x)$

(c) $\quad d(x, z) \leq d(x, y) + d(y, z)$

(d) $d(x,y) = 0 \iff x = y$.

For example, (c) follows from

$$\begin{aligned} d(x,z) &= \|x-z\| = \|(x-y)+(y-z)\| \\ &= \|x-y\| + \|y-z\| = d(x,y) + d(y,z) \ . \end{aligned}$$

Similarly, a seminorm induces a semimetric, whereas in (d) only the $\Leftarrow$ implication is valid.

Thus, in a (semi)normed space, one naturally has topological concepts (such as: convergent sequences, Cauchy sequences, convergence, compactness, etc.), which we are soon to explore. (Fundamental facts on metric spaces are gathered in Appendix B). For the time being, recall that a sequence (x_n) of elements from a (semi)normed space X is a Cauchy sequence provided

$$\forall \varepsilon > 0 \ \exists N(\varepsilon) \in \mathbb{N} \ \forall n,m \geq N(\varepsilon) \quad \|x_n - x_m\| < \varepsilon \ .$$

Also (x_n) converges to $x \in X$ (denoted by $\lim_{n\to\infty} x_n = x$) provided

$$\forall \varepsilon > 0 \ \exists N(\varepsilon) \in \mathbb{N} \ \forall n \geq N(\varepsilon) \quad \|x_n - x\| < \varepsilon$$

(where "$\leq \varepsilon$" is also allowed).

Clearly, a convergent sequence is a Cauchy sequence. For sequences from $\mathbb{K}^n$, equipped with its natural metric, the equivalence of Cauchy and convergent sequences is well-known; however, this equivalence need to hold in a general normed space (see p. 5).

Definition 1.1.2. A metric space in which each Cauchy sequence converges is called *complete*. A complete normed space is called a *Banach space*.

A incomplete normed space X can always be "embedded" into a Banach space. To see this, consider the set $CF(X)$ of all Cauchy sequences in X, form an equivalence relation $\sim$ on $CF(X)$ by

$$(x_n) \sim (y_n) \iff \|x_n - y_n\| \to 0 \ ,$$

and let $\widehat{X}$ be the set of all equivalence classes. There is a natural induced vector space structure on $\widehat{X}$ as well as a norm

$$\|[(x_n)]\| = \lim_{n\to\infty} \|x_n\| \ ,$$

which is complete. Identify X with the constant sequences in $\widehat{X}$ so that X densely embeds into $\widehat{X}$ in a natural way. $\widehat{X}$ is called the completion of X; furthermore, one can show that such a completion is essentially (i.e. up to an isometry) uniquely determined. This procedure reminds one of the construction of $\mathbb{R}$ as the completion of $\mathbb{Q}$. A more elegant method of forming the completion of a normed space is presented in Chapter 3.

Along our path, we will see that completeness is essential for the proof of many nontrivial results (cf. Chapter 4).

The remainder of this section discusses a few examples of normed spaces.

Example 1. $(\mathbb{K}^n, \|\cdot\|)$
In the analysis of $\mathbb{K}^n$, several norms are considered, e.g.:

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}$$

$$\|x\|_\infty = \max_{1,\dots,n} |x_i|$$

where $x = (x_1, \dots, x_n) \in \mathbb{K}^n$. These norms are the same in the sense that if a sequence converges with respect to one of the norms, then it converges with respect to each of these norms.

Example 2. $(\ell^\infty(T), \|\cdot\|_\infty)$ for a set T.
$\ell^\infty(T)$ is the vector space (!) of all bounded functions from T into $\mathbb{K}$. For $x \in \ell_\infty(T)$, set

$$\|x\|_\infty = \sup_{t \in T} |x(t)| \quad (< \infty) \; .$$

This norm $\|\cdot\|_\infty$ is called the *supremum norm*. Furthermore

- $(\ell^\infty(T), \|\cdot\|_\infty)$ is a Banach space.

To see this, first note that $\|\cdot\|_\infty$ is indeed a norm. Only the triangle inequality is not obvious. So let $x, y \in \ell_\infty(T)$ and fix $t_0 \in T$. Then

$$\begin{aligned} |x(t_0) + y(t_0)| \; &\le \; |x(t_0)| + |y(t_0)| \; \le \; \sup_{t \in T} |x(t)| \; + \; \sup_{t \in T} |y(t)| \\ &= \; \|x\|_\infty + \|y\|_\infty \; . \end{aligned}$$

Now taking the supremum over all $t_0 \in T$ gives that

$$\|x + y\|_\infty \; \le \; \|x\|_\infty + \|y\|_\infty \; .$$

Thus $\|\cdot\|_\infty$ is indeed a norm.

Towards showing the asserted completeness, let (x_n) be a Cauchy sequence in $\ell_\infty(T)$. We need to find an element $x \in \ell_\infty(T)$ such that $\|x_n - x\|_\infty \to \infty$. The inequality $|y(t)| \le \|y\|_\infty$, for each $t \in T$ and $y \in \ell_\infty(T)$ is useful. It implies, for each $t \in T$, that $(x_n(t))$ is a $\mathbb{K}$-valued Cauchy (!) sequence, so, by the completeness of the scalar field, it has a limit, which we denote by $x(t)$. In this manner we define a function $x \colon T \to \mathbb{K}$. We now show that x is bounded and that (x_n) converges in the supremum norm to x. First, for an arbitrary $\varepsilon > 0$, choose a natural number N with

$$\|x_n - x_m\|_\infty \leq \varepsilon \qquad \forall\, n,m \geq N \; .$$

Let $t \in T$. Because $x_n(t) \to x(t)$, there exists $m_0 = m_0(\varepsilon,t)$ so that

$$\left|x_{m_0}(t) - x(t)\right| \leq \varepsilon \; .$$

Without loss of generality, take $m_0 \geq N$ and so, for each $n \geq N$,

$$\begin{aligned} |x_n(t) - x(t)| &\leq \left|x_n(t) - x_{m_0}(t)\right| + \left|x_{m_0}(t) - x(t)\right| \\ &\leq \left\|x_n - x_{m_0}\right\|_\infty + \varepsilon \\ &\leq 2\varepsilon \; . \end{aligned}$$

Thus, for any $t \in T$,

$$|x(t)| \leq |x_N(t)| + |x(t) - x_N(t)| \leq \|x_N\|_\infty + 2\varepsilon$$

and so x is bounded. Furthermore,

$$\|x_n - x\|_\infty \leq 2\varepsilon \qquad \forall\, n \geq N$$

and so $\lim_{n\to\infty} x_n = x$.

Convergence in the supremum norm is equivalent to uniform convergence. Later, even more convergence notions, in connection with other norms, will be discussed.

$\ell_\infty(T)$ is the largest space of functions on T for which the supremum norm is defined. It is, thus, not suprising that $\ell_\infty(T)$ is complete (for otherwise what would be lacking?). Later we will examine vector *sub*spaces. The following lemma will be useful in this examination.

Lemma 1.1.3. *Let U be a vector subspace of a normed space X.*

(a) If X is a Banach space and U is (topologically) closed, then U is complete.

(b) If U is complete, then U is (topologically) closed.

Proof. (a) Let (x_n) be a Cauchy sequence in U. Since X is complete, its limit exists in X. Thus $\lim_{n\to\infty} x_n := x \in X$. Since U is closed, x must also be in U.

(b) Consider a sequence (x_n) from U that converges to some $x \in X$. Since it converges, (x_n) is a Cauchy sequence in U. Since U is complete, (x_n) converges to some point in U. By uniqueness of a limit, $x \in U$, as needed. □

This lemma reduces the problem of showing the completeness of a normed space U to identifying U as a closed subspace of a known Banach space. Instead of having to prove the *existence* of a limit of a Cauchy sequence from U, one just needs to verify that U is a closed subspace of a Banach space X, i.e. *when* the limit $\lim_{n\to\infty} x_n := x$ exists for some sequence (x_n) from U, then $x \in U$. We illustrate this principle in the following Examples 3–6.

Example 3. The space of continuous functions
Let T be a metric (or merely a topological) space. (For an easy example, think of T as a subset of $\mathbb{R}$.) Denote by $C^b(T)$ the set of all bounded continuous functions from T into $\mathbb{K}$. Since the sums and products of continuous functions are continuous, $C^b(T)$ is a vector subspace of $\ell_\infty(T)$. We now show that:

- $C^b(T)$, endowed with the supremum norm, is a Banach space.

By Lemma 1.1.3(a), it suffices to show that $C^b(T)$ is a closed subspace of $\ell_\infty(T)$. In other words, it suffices to show that, if (x_n) is a sequence of bounded continuous functions that converges uniformly to a bounded function x, then x is likewise continuous. This is already a well-know theorem from analysis! For the sake of completeness, following is the proof, a standard $\frac{\varepsilon}{3}$-argument.

For an $\varepsilon > 0$ choose $N \in \mathbb{N}$ so that $\|x_N - x\|_\infty \le \frac{\varepsilon}{3}$. Fix $t_0 \in T$. By the continuity of x_N there exists $\delta > 0$ so that (d denotes the metric on T)

$$d(t, t_o) < \delta \quad \Rightarrow \quad |x_N(t) - x_N(t_0)| \le \frac{\varepsilon}{3}\,.$$

Thus, for each t with $d(t, t_o) < \delta$

$$\begin{aligned} |x(t) - x(t_0)| &\le |x(t) - x_N(t)| + |x_N(t) - x_N(t_0)| + |x_N(t_0) - x(t_0)| \\ &\le 2\,\|x - x_N\|_\infty + |x_N(t) - x_N(t_0)| \\ &\le 2\frac{\varepsilon}{3} + \frac{\varepsilon}{3} \;=\; \varepsilon, \end{aligned}$$

and so x is continuous at t_0.

In the case that T is a *compact* metric (or *compact* topological) space, then each continuous function from T into $\mathbb{K}$ is bounded; and so in this case, one writes just $C(T)$ instead of $C^b(T)$ By the way, the symbol C comes from the French word for continuous, *continu*.

Also of interest is the space $C_0(T)$ (particually $C_0(\mathbb{R}^n)$) of continuous functions f on a locally compact space T that "vanish at infinity" (i.e. for each $\varepsilon > 0$ the set $\{t \in T : \; |f(t)| \ge \varepsilon\}$ is compact). It is easy to see that $C_0(T)$, endowed with the supremum norm, is a closed subset of $C^b(T)$ and so is also a Banach space.

Example 4. The space of differentiable functions
$C^1[a,b]$ is the vector space of continuously differentiable [1] functions defined on the compact interval $[a,b]$. Clearly $C^1[a,b]$ is also a vector subspace of $C[a,b]$. However, $C^1[a,b]$ is not closed in the supremum norm. (Indeed, for instance if a=-1 and b=1, a sequence (x_n) from $C^1[a,b]$ that converges uniformly to a nondifferentiable function is given by $x_n(t) = \left(t^2 + \frac{1}{n}\right)^{1/2}$.) So by Lemma 1.1.3(b), $\left(C^1[a,b], \|\cdot\|_\infty\right)$ is no Banach space. So now consider, for each $x \in C^1[a,b]$, the norms

[1] A function $f\colon [a,b] \to \mathbb{K}$ is called *continuously differentiable* provided it has a continuous derivative on $[a,b]$.

$$\|x\| \;=\; \sup_{t\in[a,b]} \max\left\{|x(t)|,\left|x'(t)\right|\right\} \;=\; \max\left\{\|x\|_\infty,\left\|x'\right\|_\infty\right\}$$
$$|||x||| \;=\; \|x\|_\infty + \left\|x'\right\|_\infty \;.$$

It is easy to see that $\|\cdot\|$ and $|||\cdot|||$ are norms; i.e., the triangle inequality for $|||\cdot|||$ follows from

$$\begin{aligned} |||x+y||| \;&=\; \|x+y\|_\infty + \left\|(x+y)'\right\|_\infty \\ &=\; \|x+y\|_\infty + \left\|x'+y'\right\|_\infty \\ &\le\; \|x\|_\infty + \|y\|_\infty + \left\|x'\right\|_\infty + \left\|y'\right\|_\infty \\ &=\; |||x||| + |||y||| \;. \end{aligned}$$

Furthermore, clearly one has the inequality

$$\|x\| \le |||x||| \le 2\,\|x\| \quad \forall x\in C^1[a,b]\;; \tag{1.1}$$

thus, $\|\cdot\|$ and $|||\cdot|||$ are *equivalent* norms (more on this topic in Section 1.2). The inequality (1.1) shows that a sequence is a Cauchy (resp. convergent) sequence with respect to $\|\cdot\|$ if it is a Cauchy (resp. convergent) sequence with respect to $|||\cdot|||$ Thus the completeness of $\left(C^1[a,b],\|\cdot\|\right)$ is equivalent to the completeness of $\left(C^1[a,b],|||\cdot|||\right)$, and convergence with respect to $\|\cdot\|$ or $|||\cdot|||$ is each equivalent to uniform convergence of both the function sequence (x_n) as well as its sequence (x_n') of derivatives. Thus if (x_n) is a $\|\cdot\|$- (resp. $|||\cdot|||$-) Cauchy sequence, then (x_n) and (x_n') are $\|\cdot\|_\infty$-Cauchy sequences, and so by the completeness of $C[a,b]$ each has a limit, say x und y. A well-known theorem from analysis now gives that x is differentable with $x'=y$. Thus x is indeed in $C^1[a,b]$ and $\|x_n-x\|\to 0$.

In summary:

- $C^1[a,b]$ is a Banach space with repect to $\|\cdot\|$ and $|||\cdot|||$ norms but not with repect to the supremum norm

Similarly

$$C^r[a,b] := \{x\in C[a,b] \colon x \text{ is } r\text{-times continuously differentiable}\}\;.$$

- $C^r[a,b]$, endowed with the norm $|||x||| = \sum_{i=0}^r \left\|x^{(i)}\right\|_\infty$, is a Banach space.

We shall also discuss a more general space of functions. For this, multi-index notation is used. Let $\Omega\subset\mathbb{R}^n$ be open and $\varphi\in C^r(\Omega)$, i.e., $\varphi\colon\Omega\to\mathbb{K}$ is r-times continuously differentiable. Thus, for $\alpha=(\alpha_1,\ldots\alpha_n)\in\mathbb{N}_0^n$ with $|\alpha|{:=}\alpha_1+\ldots+\alpha_n\le r$, the partial derivative

$$D^\alpha\varphi \;=\; \frac{\partial^{\alpha_1}\cdots\partial^{\alpha_n}}{\partial t_1^{\alpha_1}\cdots\partial t_n^{\alpha_n}}\,\varphi$$

of order $|\alpha|$ exists and is continuous; thereby the order of differentiation is irrelevant.

Now let $\Omega \subset \mathbb{R}^n$ be open and *bounded*. Set

$$C^r\left(\overline{\Omega}\right) = \left\{ \varphi\colon \Omega \to \mathbb{K}\colon \quad \begin{array}{l} \varphi \text{ is } r\text{-times continuously differentiable,} \\ \text{and for each multi-index } \alpha \text{ with } |\alpha| \le r \\ D^\alpha \varphi \text{ extends continuously to } \overline{\Omega} \end{array} \right\} .$$

In particular, for each $|\alpha| \le r$, the derivative $D^\alpha \varphi$ is bounded and thus

$$|||\varphi|||:= \sum_{|\alpha|\le r} \|D^\alpha \varphi\|_\infty$$

is finite. As in one dimension one shows:

- $C^r\left(\overline{\Omega}\right)$, endowed with the norm $|||\cdot|||$, is a Banach space.

Example 5. The space of holomorphic (= anlaytic) functions.
Let $\mathbb{D} = \{z \in \mathbb{C}\colon\ |z| < 1\}$ be the open unit disc in $\mathbb{C}$. Then H^∞ denotes the vector space of all bounded holomorpic functions from $\mathbb{D}$ to $\mathbb{C}$. One can endow H^∞ with the supremum norm and realize it as a subspace of $\ell^\infty(\mathbb{D})$ or $C^b(\mathbb{D})$. A theorem from function theory says that the uniform limit of holomorpic functions is still holomorphic; thus, H^∞ is closed.

A related space is the so-called Disk Algebra

$$A(\mathbb{D}) := \left\{ f \in C\left(\overline{D}\right) : f|_{\mathbb{D}} \text{ is holomorphic} \right\} .$$

Also here one see that $A(\mathbb{D})$ is $\|\cdot\|_\infty$-closed in $C\left(\overline{\mathbb{D}}\right)$. So one obtains

- H^∞ and $A(\mathbb{D})$, endowed with the supremum norm, are Banach spaces.

This result stands in stark contract to Example 4, where we saw that the space C^1 of real-valued differentiable functions, endowed with the supremum norm, is *not* complete.

Example 6. The sequence spaces: d, c_0, c, ℓ^∞.
We consider the vector spaces

$$\begin{aligned} d &= \{(t_n) : t_n \in \mathbb{K},\ t_n \ne 0 \text{ for atmost finitely many } n\} \\ c_0 &= \{(t_n) : t_n \in \mathbb{K},\ \lim_{n\to\infty} t_n = 0\} \\ c &= \{(t_n) : t_n \in \mathbb{K},\ (t_n) \text{ converges}\} \\ \ell_\infty &= \ell_\infty(\mathbb{N}) = \{(t_n) : t_n \in \mathbb{K},\ (t_n) \text{ is bounded}\} \end{aligned}$$

and, on each space, put the supremum norm

$$\|(t_n)\|_\infty = \sup_{n\in\mathbb{N}} |t_n| .$$

(The space d of truncated sequences can be regarded as the discrete analog of the to be discussed space $\mathscr{D}$ of *test functions*.) Note that

$$d \subset c_0 \subset c \subset \ell^\infty .$$

As we shall see, c_0 and c are closed in ℓ_∞ but d is not.

In order to prove this, we must consider a sequence of sequences; the usage of double indices is inescapable. Now let (x_n) be a sequence from c such that $\|x_n - x\|_\infty \to 0$ for an $x \in \ell^\infty$. We must show that $x \in c$ and use an $\frac{\varepsilon}{3}$-argument as on p. 5. We write

$$x_n = \left(t_m^{(n)}\right)_{m\in\mathbb{N}} \ , \ x = (t_m)_{m\in\mathbb{N}} \ , \ t_\infty^{(n)} = \lim_{m\to\infty} t_m^{(n)} \ .$$

Since $|\lim_{m\to\infty} s_m| \le \|(s_m)\|_\infty$ for each $(s_m) \in c$, we have that $\left(t_\infty^{(n)}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{K}$ (because (x_n) is a Cauchy sequence in c). Hence $t_\infty := \lim_{n\to\infty} t_\infty^{(n)}$ exists. To show that $x \in c$, it is enough to show that $\lim_{m\to\infty} t_m = t_\infty$.

To this end, for an $\varepsilon > 0$, choose a natural number N with

$$\|x_N - x\|_\infty \le \frac{\varepsilon}{3} \ , \ \left|t_\infty^{(N)} - t_\infty\right| \le \frac{\varepsilon}{3} \ .$$

Then fix $m_0 \in \mathbb{N}$ such that

$$m \ge m_0 \ \Rightarrow \ \left|t_m^{(N)} - t_\infty^{(N)}\right|\Big\|_\infty \le \frac{\varepsilon}{3} \ .$$

It follows that for each $m \ge m_0$

$$\begin{aligned} |t_m - t_\infty| \ &\le \left|t_m - t_m^{(N)}\right| + \left|t_m^{(N)} - t_\infty^{(N)}\right| + \left|t_\infty^{(N)} - t_\infty\right| \\ &\le \|x_N - x\|_\infty + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \le \varepsilon \ . \end{aligned}$$

Now for the completeness of c_0. Let (x_n) be a sequence from c_0 such that $\|x_n - x\|_\infty \to 0$ for an $x \in \ell^\infty$. From the already proven, $x \in c$ (i.e., in the above notation, $t_\infty = \lim_{m\to\infty} t_m$ exists) and we now need to show that $t_\infty = 0$. But this was already done in the above proof, because $t_\infty = \lim_{n\to\infty} t_\infty^{(n)} = 0$ since $x_n \in c_0$.

To finish, consider d. To see that d is not closed, set, for $n \in \mathbb{N}$,

$$x_n = \left(1, \frac{1}{2}, \dots, \frac{1}{n}, 0, 0, \dots\right) \ , \ x = \left(1, \frac{1}{2}, \dots, \frac{1}{n}, \frac{1}{n+1}, \dots\right) = \left(\frac{1}{n}\right)_{n\in\mathbb{N}} \ .$$

Then $x_n \in d$ and $\|x_n - x\|_\infty = \frac{1}{n+1} \to 0$ but $x \notin d$.

To summarize:

- The sequence spaces c_0, c, and ℓ_∞, endowed with the supremum norm, are Banach spaces but d is not a Banach space.

By the way, c_0 is nothing else than the space $C_0(\mathbb{N})$ from Example 3.

We now treat fundamental norms that are not the supremum norm.

Example 7. The sequence spaces ℓ_p for $1 \le p < \infty$.
Let

$$\ell_p = \left\{ (t_n) : t_n \in \mathbb{K}, \sum_{n=1}^{\infty} |t_n|^p < \infty \right\}$$

as well as, for $x = (t_n) \in \ell_p$,

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |t_n|^p \right)^{\frac{1}{p}} ;$$

where $1 \leq p < \infty$ (this restriction on p is important, as we will show, for $\|\cdot\|_p$ to actually be a norm). We will see that $\left(\ell_p, \|\cdot\|_p\right)$ is a Banach space for $1 \leq p < \infty$.

For starters, it is not bloody obvious that ℓ^p is even a vector space. To show this fact, consider two sequence $x = (s_n)$ and $y = (t_n)$ from ℓ^p. Then

$$\begin{aligned}
\sum_{n=1}^{\infty} |s_n + t_n|^P &\leq \sum_{i=1}^{\infty} (|s_n| + |t_n|)^p \\
&\leq \sum_{i=1}^{\infty} (2 \max\{|s_n|, |t_n|\})^P \\
&= 2^p \sum_{i=1}^{\infty} \max\{|s_n|^p, |t_n|^p\} \\
&\leq 2^p \sum_{i=1}^{\infty} (|s_n|^p + |t_n|^p) \\
&= 2^p \left(\sum_{i=1}^{\infty} |s_n|^p + \sum_{i=1}^{\infty} |t_n|^p \right) ,
\end{aligned}$$

so $x + y \in \ell^p$. It is clear that if $\lambda \in \mathbb{K}$ and $x \in \ell^p$ then $\lambda x \in \ell^p$. Therefore ℓ^p is indeed a vector space.

Next we show that the norm properites for $\|\cdot\|_p$ hold. Here, only the triangle inequality (except for the case $p = 1$) is not obvious. We show first an important inequality. For two sequences $x = (s_n)$ and $y = (t_n)$, set $xy = (s_n t_n)$.

Theorem 1.1.4 (Hölder's Inequality, sequence version).

(a) *If $x \in \ell^1$ and $y \in \ell^\infty$, then $xy \in \ell^1$ and*

$$\|xy\|_1 \leq \|x\|_1 \|y\|_\infty .$$

(b) *Let $1 < p < \infty$ and $q = \frac{p}{p-1}$ (so $\frac{1}{p} + \frac{1}{q} = 1$). If $x \in \ell^p$ and $y \in \ell^q$, then $xy \in \ell^1$ and*

$$\|xy\|_1 \leq \|x\|_p \|y\|_q .$$

One can formulate both parts simultaneously by defining the "conjugate exponent" of $p = 1$ as $q = \infty$. Then the notation of ℓ_∞ for the space of bounded sequences is natural. (Another reason: $\lim_{p \to \infty} \|x\|_p = \|x\|_\infty$, see Problem 1.4.11.)

Proof. Please see the book.

$$(1.2)$$

□

The triangle inequality for $\|\cdot\|_p$ follows as a corollary, which has its own name.

Corollary 1.1.5 (Minkowski's Inequality, sequence version).
If $x, y \in \ell^p$ with $1 \le p < \infty$, then

$$\|x+y\|_p \le \|x\|_p + \|y\|_p \ .$$

Proof. Please see the book. □

That $\|\lambda x\|_p = |\lambda|\,\|x\|_p$ and $\|x\|_p = 0 \Leftrightarrow x = 0$ is clear, so $\left(\ell^p, \|\cdot\|_p\right)$ is a normed space. We now show the completeness of ℓ_p. Similar to the case of ℓ_∞, the completeness of $\mathbb{K}$ is the crucial key.

Let (x_n) be a Cauchy sequence from ℓ^p. We write $x_n = \left(t_m^{(n)}\right)_{m\in\mathbb{N}}$. Then for each $x = (t_m) \in \ell^p$ and for each $m \in \mathbb{N}$, the inequality $|t_m| \le \|x\|_p$ holds. Thus, for each m, the scalar sequence $\left(t_m^{(n)}\right)_{n\in N}$ is Cauchy. Let

$$t_m = \lim_{n\to\infty} t_m^{(n)} \qquad \text{and} \qquad x = (t_m)_{m\in\mathbb{N}} \ .$$

We now want to show that $x \in \ell^P$ and $\|x_n - x\|_p \to 0$. For $\varepsilon > 0$ find $N = N(\varepsilon)$ with

$$\|x_n - x_{n'}\|_p \le \varepsilon \qquad \text{and} \qquad \forall n, n' \ge N \ .$$

In particular, it follws that for each $M \in \mathbb{N}$

$$\left(\sum_{m=1}^{M} \left|t_m^{(n)} - t_m^{(n')}\right|^p\right)^{\frac{1}{p}} \le \|x_n - x_{n'}\|_p \le \varepsilon \qquad \forall n, n' \ge N \ .$$

Now take the limit as $n' \to \infty$ to obtain

$$\left(\sum_{m=1}^{M} \left|t_m^{(n)} - t_m\right|^p\right)^{\frac{1}{p}} \le \varepsilon \qquad \forall M \in \mathbb{N}\ ,\ \forall n \ge N \ .$$

Since M was arbitrary

$$\left(\sum_{m=1}^{\infty} \left|t_m^{(n)} - t_m\right|^p\right)^{\frac{1}{p}} \le \varepsilon \qquad \forall n \ge N$$

and it follows first that $x - x_N \in \ell^p$ and thus $x = (x - x_N) + x_N \in \ell^p$ as well as $\|x_n - x\|_p \to 0$.

Summarizing:

- $\left(\ell_p, \|\cdot\|_p\right)$ is a Banach space for $1 \leq p < \infty$.

One can naturally put the ℓ^p-norm on the finite dimensional space $\mathbb{K}^n$, so normed, $\mathbb{K}^n$ is denoted by $\ell^p(n)$.

Example 8. The Lebesgue spaces L_p for $1 \leq p < \infty$.
In searching for how to define for integrable functions an analog to the ℓ_p sequence Banach spaces, one stumbles upon big hurdles. We will conquer each hurdle!

Consider first on $C[a,b]$ the norm $\|f\|_1 = \int_a^b |f(t)|\, dt$. In this norm $C[a,b]$ is not complete. Indeed, [2] take, for example, $a = 0$ and $b = 2$ and let

$$f_n(t) = \begin{cases} t^n & 0 \leq t \leq 1 \\ 1 & 1 < t \leq 2\,. \end{cases}$$

Then (f_n) is a Cauchy sequence in $C[0,2]$ w.r.t. $\|\cdot\|_1$ since, for $n \geq m$,

$$\|f_n - f_m\|_1 = \int_0^1 (t^m - t^n)\, dt \;\leq\; \frac{1}{m+1}\,.$$

The sequence (f_n) appears to converge to f where

$$f(t) = \begin{cases} 0 & 0 \leq t < 1 \\ 1 & 1 \leq t \leq 2\,. \end{cases}$$

and f is not continuous. (In fact $\int_0^2 |f_n(t) - f(t)|\, dt = \frac{1}{n+1} \to 0$.) In the same way, one consider the (admittedly also discontinuous) funtion

$$\widetilde{f}(t) = \begin{cases} 0 & 0 \leq t \leq 1 \\ 1 & 1 < t \leq 2\,. \end{cases}$$

as a possible limit of (f_n). Can one perhaps still find a limit in $C[0,2]$ if one searches long enough? It does not look likely.

One can also consider $\|\cdot\|_1$ on the space $R[0,2]$ of Riemann integrable functions from $[0,2]$ to $\mathbb{K}$. But there defined, $\|\cdot\|_1$ is only a seminorm, so convergent sequences need not have a uniquely determined limit. (Above we found f and $\widetilde{f}$ as limits of (f_n).) We condsider thus the linear span of $\{f\}$ and $C[0,2]$ and denote this vector subspace of $R[0,2]$ by X. Then $\|\cdot\|_1$ is a norm on X; indeed, consider $g \in C[0,2]$ and $\lambda \in \mathbb{K}$ with $\|g + \lambda f\|_1 = 0$. Then

$$\int_0^1 |g(t)|\, dt = 0 = \int_1^2 |g(t) + \lambda|\, dt$$

From the continuity of g it follows that $g(t) = 0$ for $0 \leq t \leq 1$ and $g(t) = -\lambda$ for $1 \leq t \leq 2$. Thus $\lambda = 0$ and $g = 0$. Now we conclude that $(C[0,2], \|\cdot\|_1)$ is not a closed

[2] Maria: ask Lutz what O.E. stands for

subspace of $(X, \|\cdot\|_1)$ (indeed, the sequence (f_n) in X has a uniquely determined limit value $f \notin C[0,2]$). Thus the following now follows from Lemma 1.1.3(b).

- $C[0,2]$ (and in general $C[a,b]$), endowed with the $\|\cdot\|_1$, is no Banach space.

One could now define formally the completion (cf. p. 2); however, this would furnish little information on the concrete structure of the obtained Banach space. For example, it is not clear whether the appearing elements could be considered as functions, and when so, which functions. Another idea would be to define $\|\cdot\|_1$ on $R[a,b]$, possibly allowing also improper Riemann integrable functions. But we have already established that one obtains only a seminormed space. Worse yet, the desired completeness is not yet there. (The former deficiency seems to be in fact unavoidable in this approach.) One should find an entirely new approach.

The fundamental innovation to use is the Lebesgue integration theory, whose main features are presented in Appendix A.

Consider an interval $I \subset \mathbb{R}$ (which can be open, clopen, closed, bounded or unbounded), the σ-algebra Σ of Borel measurable subsets of I, and the Lebesgue measure λ on I. For $1 \leq p < \infty$, set (keep in mind that when f is measurable, so is $|f|^p$)

$$\mathscr{L}^p(I) = \left\{ f\colon I \to \mathbb{K}\colon f \text{ is measurable }, \int_I |f|^p \, d\lambda < \infty \right\}$$

$$\|f\|_p^* = \left(\int_I |f|^p \, d\lambda \right)^{\frac{1}{p}} \qquad \text{for } f \in \mathscr{L}^p(I) \; .$$

(We will use the notation $\int f \, d\lambda$ as well as the more traditional notation $\int f(t)\, dt$.) We will show that $\mathscr{L}^p(I)$ is a complete *seminormed* space. Just as in the case for the ℓ^p spaces, we first establish that $\mathscr{L}^p(I)$ is a vector space (which is not obvious), which will follow from the Hölder and Minkowski inequalities, and then close with a proof of the completeness of $\mathscr{L}^p(I)$.

So let $f, g \in \mathscr{L}^p(I)$. Then $f+g$, as a sum of measuable functions, is itself measurable. Furthermore

$$\begin{aligned}
\int_I |f+g|^p \, d\lambda &\leq \int_I (|f| + |g|)^p \, d\lambda \\
&\leq \int_I (2 \max\{|f(t)|, |g(t)|\})^p \, dt \\
&= 2^p \int_I \max\{|f(t)|^p, |g(t)|^p) \, dt \\
&\leq 2^p \int_I (|f(t)|^p + |g(t)|^p) \, dt \\
&= 2^p \left(\int_I |f|^p \, d\lambda + \int_I |g|^p \, d\lambda \right) < \infty \, .
\end{aligned}$$

So $f+g \in \mathscr{L}^p(I)$. Lastly, it is trivial to see that $\alpha f \in \mathscr{L}^p(I)$ for each $\alpha \in \mathbb{K}$. Thus $\mathscr{L}^p(I)$ is a vector space.

Next we show Hölder's inequality for function spaces; as in the cas with ℓ^p, we will deduce from it the triangle inequality for $\|\cdot\|_p^*$.

Theorem 1.1.6 (Hölder's Inequaltiy, $\mathscr{L}^p(I)$ version).
Let $1 < p < \infty$ and $q = \frac{p}{p-1}$, so $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in \mathscr{L}^p(I)$ and $g \in \mathscr{L}^q(I)$, then $fg \in \mathscr{L}^1(I)$ and

$$\|fg\|_1^* \leq \|f\|_p^* \, \|g\|_q^* \ .$$

Proof. Please see the book. □

Corollary 1.1.7 (Minkowski's Inequaltiy, $\mathscr{L}^p(I)$ version).
Let $1 \leq p < \infty$ and $f, g \in \mathscr{L}^p(I)$. Then

$$\|f+g\|_p^* \leq \|f\|_p^* + \|g\|_p^* \ .$$

Proof. Please see the book. □

mg finished up to here

Lemma 1.1.8. *For a seminormed space $(X, \|\cdot\|)$, the following are equivalent.*

(i) X is complete.

(ii) Each absolutely convergent series converges. (Specifically: for each sequence (x_n) in X with $\sum_{n=1}^{\infty} \|x_n\| < \infty$, there exists an element $x \in X$ so that $\lim_{N\to\infty} \left\| x - \sum_{n=1}^{N} x_n \right\| = 0$.)

$\mathscr{L}^p(I)$ is a complete seminormed space. Limits in this space are not uniquely determined; they are determined only modulo elements from the kernel N_p of the seminorm $\|\cdot\|_p^*$, namely

$$N_p = \{f \in \mathscr{L}^p(I) : \ f = 0 \text{ almost everywhere}\} \ .$$

This suggests to identify functions that agree almost everywhere. In such a situation, the appropriate mathematical procedure is to consider, instead of an individual function f, its equivalence class $[f]$ in the associated quotient vector space

$$L^p(I) := \mathscr{L}^p(I) / N_p \ .$$

This procedure is illustrated in the following lemma.

Lemma 1.1.9. *Let $(X, \|\cdot\|^*)$ be a seminormed space.*

(a) $N := \{x : \ \|x\|^* = 0)$ is a vector subspace of X.

(b) $\|[x]\| := \|x\|^*$ defines a norm on X/N.

(c) If X is complete, then X/N is a Banach space.

We denote the quotient norm on $L_p(I)$ by $\|\cdot\|_p$ (finally we can get rid of that star) or by $\|\cdot\|_{L^p}$.

- The space $\left(L_p(I), \|\cdot\|_p\right)$ is a Banach space for $p \geq 1$.

Above we have considered the L^p-spaces over an interval in $\mathbb{R}$ with the Lebesgue measure. Everything carries through to a measurable subset $\Omega \subset \mathbb{R}^n$ (with the corresponding Borel σ-algebra and the n-dimensional Lebesgue measure). More generally, one can take an arbitrary measure space (Ω, Σ, μ). The resulting L^p-space is denoted by $L^p(\Omega, \Sigma, \mu)$, or just $L_p(\mu)$ for short.

- The space $\left(L^p(\mu), \|\cdot\|_p\right)$ is a Banach space for $p \geq 1$.

Example 9. The L^∞ space.
For clarity, we introduce first $L^\infty(I)$ for an interval I and then later $L^\infty(\mu)$ for an arbitrary measure.

$$\mathscr{L}^\infty(I) = \left\{ f : I \to \mathbb{K} : \begin{array}{l} f \text{ is measurable} \\ \exists N \in \Sigma,\ \lambda(N) = 0 : f|_{I\ N} \text{ is bounded} \end{array} \right\}$$
$$\|f\|^*_{L^\infty} = \inf_{\substack{N \in \Sigma \\ \lambda(N)=0}} \sup_{t \in I \setminus N} |f(t)| \; = \; \inf_{\substack{N \in \Sigma \\ \lambda(N)=0}} \left\| f|_{I \setminus N} \right\|_\infty .$$

$\mathscr{L}^\infty(I)$ is a vector space and $\|\cdot\|^*_{L^\infty}$ is a seminorm.

- $(\mathscr{L}^\infty(I), \|\cdot\|^*_{L^\infty})$ is a complete seminormed space.

Let N_∞ is the kernel of the seminorm $\|\cdot\|^*_{L^\infty}$ and

$$L^\infty(I) := \mathscr{L}^\infty(I) / N_\infty .$$

Endow $L^\infty(I)$ with its quotient norm, which is denoted by $\|\cdot\|_{L^\infty}$ and is called the *essential supremum* norm.

- The space $L^\infty(I)$, endowed with the essential supremum norm, is a Banach space.

Now let (Ω, Σ, μ) be an arbitrary measure space.

- The space $L^\infty(\mu)$, endowed with the essential supremum norm, is a Banach space.

Theorem 1.1.10 (Hölder's Inequaltiy, general version).
Let $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ (with the convention $\frac{1}{\infty} = 0$). Let (Ω, Σ, μ) be a measure space. If $f \in L^p(\mu)$ and $g \in L^q(\mu)$, then $fg \in L^1(\mu)$ and

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q} .$$

The special case where $p = q = 2$ is known as the *Cauchy Schwartz inequality*.

Example 10. The space of measures.
Let Σ be a σ-algebra on a set T. A mapping $\mu: \Sigma \to \mathbb{R}$ (resp. $\mathbb{C}$) is a *signed* (resp. *complex*) measure provided that for each sequence of pairwise disjoint sets $A_i \in \Sigma$

$$\mu\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu\left(A_i\right) . \tag{1.3}$$

Let $M(T,\Sigma)$ be the collection of all measures on Σ.

Through the rule

$$|\mu|(A) = \sup_{\mathscr{Z}} \sum_{E \in \mathscr{Z}} |\mu(E)| \ ,$$

where the supremum is taken over all decompositions of A into finitely many pairwise disjoint measurable subsets A_i from Σ, a positive measure $|\mu|$ is associated, the so-called *variation* of μ.

Now associate with μ is its *variation* norm

$$\|\mu\| = |\mu|(T) \ .$$

- $M(T,\Sigma)$, endowed with its variation norm, is a Banach space.

1.2 Properties of Normed Spaces

Theorem 1.2.1. *Let X be a normed space.*

(a) *If $x_n \to x$ and $y_n \to y$, then $x_n + y_n \to x + y$.*

(b) *If $\lambda_n \to \lambda$ in $\mathbb{K}$ and $x_n \to x$, then $\lambda_n x_n \to \lambda x$.*

(c) *If $x_n \to x$, then $\|x_n\| \to \|x\|$.*

Corollary 1.2.2. *If U is a vector subspace of a normed space X, then its closure $\overline{U}$ is also a vector subspace of X.*

Definition 1.2.3. Two norms $\|\cdot\|$ and $|||\cdot|||$ on a vector space X are *equivalent* provided there exists numbers $0 < m \leq M$ so that

$$m\|x\| \leq |||x||| \leq M\|x\| \qquad \forall x \in X \ . \tag{1.4}$$

Theorem 1.2.4. *Let $\|\cdot\|$ and $|||\cdot|||$ be two norms on X. Then the following statements are equivalent.*

(i) *$\|\cdot\|$ and $|||\cdot|||$ are equivalent.*

(ii) *A sequence converges with respect to $\|\cdot\|$ if and only if it converges with respect to $|||\cdot|||$; moreover, the limits agree.*

(iii) *A sequence is $\|\cdot\|$-null convergent if and only if it is $|||\cdot|||$-null convergent.*

Theorem 1.2.5. *Each pair of norms on a <u>finite</u> dimensional <u>normed linear</u> space are equivalent.*

Lemma 1.2.6 (Riesz's Lemma). *Let U be a closed subspace of a normed space X with $U \neq X$. Let $0 < \delta < 1$. Then there exists $x_\delta \in X$ such that $\|x_\delta\| = 1$ and*

$$\|x_\delta - u\| \geq 1 - \delta \qquad \forall u \in U \ .$$

Theorem 1.2.7. *For a normed space X, the following are equivalent.*

(i) $\dim X < \infty$.

(ii) *$B_X := \{x \in X : \|x\| \leq 1\}$ is compact.*

(iii) *Each bounded sequence in X has a convergent subsequence.*

Definition 1.2.8. A metric (or topological) space is *separable* provided it contains a countable dense subset.

A subset D of a set T is called *dense* provided $\overline{D} = T$.

Lemma 1.2.9. *For a normed space X, the following are equivalent.*

(i) *X is separable.*

(ii) *There is a countable set A such that $X = \overline{\text{lin}}\, A := \overline{\text{lin}\, A}$.*

Theorem 1.2.10 (Weierstraß's Approximation Theorem).
The subspace $P[a,b]$ of polyonomial functions on an interval $[a,b]$ in $\mathbb{R}$ is a dense subset in $(C[a,b], \|\cdot\|_\infty)$.

(1.5)

(1.6)

(1.7)

Corollary 1.2.11. *$C[a,b]$ is separable.*

Theorem 1.2.12. *$C[a,b]$ is dense in $L^p[a,b]$ if $1 \leq p < \infty$.*

Definition 1.2.13. Let Σ be the Borel [3] σ-algebra of a metric (or topological) space T. A measure μ on Σ is called a *Borel measure*. A positive Borel measure is called *regular* provided

(a) $\mu(C) < \infty$ for each compact subset C

(b) for each $A \in \Sigma$

$$\begin{aligned} \mu(A) &= \sup\{\mu(C) : C \subset A,\ C \text{ compact}\} \\ &= \inf\{\mu(O) : A \subset O,\ O \text{ open}\}\ . \end{aligned}$$

A signed or complex Borel measure is called regular when its variation (cf. Example 10) is regular. $M(T)$ denotes the collection of signed or complex regular Borel measures.

Theorem 1.2.14.

(a) Let T be one of the following.

- *A compact metric space.*
- *A complete separable metric space.*
- *An open subset of $\mathbb{R}^n$.*

Then each finite Borel measure on T is regular. Also, the Lebesgue measure on $\mathbb{R}^n$ is regular.

(b) If T is a compact topological space, then $M(T)$ is a (in general, proper) closed vector subspace of $M(T,\Sigma)$.

Corollary 1.2.15. *$L^p[a,b]$ is separable for $1 \leq p < \infty$.*

Here is a small collection of yet further facts about spaces of the type C or L^p.

- $(C(T), \|\cdot\|_\infty)$ is separable when T is a compact metric space.

[3] Thus Σ is the σ-algebra generated by the open subsets of T.

- For compact topological (Hausdorff) spaces T, the space $C(T)$ is separable precisely when T is metrizable.
- If μ is a finite regular Borel measure on a compact metric (or merely a topological) space T, then $C(T)$ lies densely in $L^p(\mu)$ for $1 \leq p < \infty$.
- If T is a compact metric space or an open subset of $\mathbb{R}^n$ with μ a regular Borel measure on T, then $L^p(\mu)$ is separable for $1 \leq p < \infty$. (This statement is no longer valid for merely compact topological spaces.) In particular, $L^p(\mathbb{R}^n)$ is separable for this range of p's.

1.3 Quotients and Sums of Normed Spaces

Definition 1.3.1. Let A be a subset of a normed space X. The *distance* from $x \in X$ to A is

$$d(x,A) := \inf\{\|x-y\| : y \in A\} \ .$$

Theorem 1.3.2. *Let U be a subspace of a normed space X. For $x \in X$ denote by $[x] = x+U \in X/U$ the corresponding equivalence class.*

(a) $\|[x]\| := d(x,U)$ *defines a seminorm on X/U.*

(b) *If U is closed then $\|\cdot\|$ is a norm.*

(c) *If X is complete and U is closed, then X/U is a Banach space.*

Theorem 1.3.3. *Let X and Y be normed spaces.*

(a) *Let $1 \le p \le \infty$. Then*

$$\|(x,y)\|_p := \begin{cases} (\|x\|^p + \|y\|^p)^{1/p} & \text{if} \quad p < \infty \\ \max\{\|x\|, \|y\|\} & \text{if} \quad p = \infty \end{cases}$$

defines a norm on the direct sum $X \oplus Y$. Endowed with this norm, we denote the direct sum by $X \oplus_p Y$.

(b) *Each of these norms are equivalent and generate the product topology on $X \times Y$,* i.e. $(x_n, y_n) \to (x,y)$ *with respect to $\|\cdot\|_p$ if and only if $x_n \to x$ and $y_n \to y$.*

(c) *If X and Y are complete, then $X \oplus_p Y$ is complete.*

Chapter 2
Functionals and Operators

2.1 Examples and Properties of Continuous Linear Operators

Definition 2.1.1. A continuous linear mapping between normed spaces is called a continuous *operator*. If the range space is the scalar field, then it called a *functional*.

Theorem 2.1.2. *Let X and Y be normed spaces and $T : X \to Y$ be linear. Then the following are equivalent.*

(i) *T is continuous.*

(ii) *T is continuous at* 0*.*

(iii) *There exists $M \geq 0$ such that*

$$\|Tx\|_Y \leq M\|x\|_X \quad \forall x \in X \; .$$

(iv) *T is uniformly continuous.*

Definition 2.1.3. The smallest constant for which (iii) of Theorem 2.1.2 holds is denoted by $\|T\|$; thus,

$$\|T\| := \inf\{M \geq 0 : \; \|Tx\| \leq M\|x\| \;\; \forall x \in X\} \; .$$

$$\|Tx\| \leq \|T\| \; \|x\| \quad \forall x \in X \; . \tag{2.1}$$

By Theorem 2.1.2, for a continuous operator, the image of the unit ball

$$\{Tx \in Y : \; \|x\| \leq 1\}$$

is a bounded set; thus, one can also use the terminology *bounded operator*.

Consider now

$$L(X,Y) := \{T : X \to Y : \; T \text{ is linear and continuous}\} \; .$$

Theorem 2.1.4.

(a) $\|T\| = \sup_{\|x\|\leq 1} \|Tx\|$ *defines a norm on* $L(X,Y)$*, the so called operator norm.*

(b) *If* Y *is complete (and independent of the completeness of* X*), then* $L(X,Y)$ *is complete.*

For a normed space X, let

$$B_X := \{x \in X : \|x\| \leq 1\} \quad \text{(the (closed) unit ball of } X\text{)}$$
$$S_X := \{x \in X : \|x\| = 1\} \quad \text{(the unit sphere of } X\text{)} .$$

Theorem 2.1.5. *Let* D *be a dense subspace of a normed space* X *and* Y *be a Banach space. Let* $T \in L(D,Y)$*. Then there exists exactly one continuous extension* $\widehat{T} \in L(X,Y)$, i.e. *a continous operator with* $\widehat{T}\,|_D = T$*. Furthermore,* $\left\|\widehat{T}\right\| = \|T\|$.

Lemma 2.1.6. *Let* $S \in L(X,Y)$ *and* $T \in L(Y,Z)$*. Then* $TS \in L(X,Z)$ *with*

$$\|TS\| \leq \|T\| \; \|S\| \; .$$

Example (l) . Let $k\colon [0,1]\times[0,1] \to \mathbb{K}$ be continuous and $x \in C[0,1]$. Define T_k by

$$(T_k x)\,(s) := \int_0^1 k(s,t)x(t)\,dt \; .$$

T_k is called a Fredholm Integral Operator with kernel k.

(2.2)

(2.3)

(2.4)

(2.5)

Definition 2.1.7. Let X and Y be normed spaces. A linear mapping $T\colon X \to Y$ is called a *quotient mapping* provided T maps the open ball $\{x \in X\colon \ \|x\| < 1\}$ to the open ball $\{y \in y\colon \ \|y\| < 1\}$; i.e. $T\left(\{x \in X\colon \ \|x\| < 1\}\right) = \{y \in y\colon \ \|y\| < 1\}$.

A quotient map is surjective and continuous with operator norm 1.

Theorem 2.1.8. *Let U be a closed subspace of a normed space X. Then the natural mapping $\omega\colon X \to X/U$, given by $x \mapsto [x]$, is a quotient mapping.*

Definition 2.1.9. A continuous linear operator $T\colon X \to Y$ is called an *isomorphism* provided T is bijective and T^{-1} is continuous. A linear map $T\colon X \to Y$ satisfying $\|Tx\| = \|x\|$ for each $x \in X$ called an *isometry*. Normed spaces for which there exists an isomorphism (resp. isometric isomorphism) between them are said to be isomorphic (resp. isometrically isomorphism) and we write $X \simeq Y$ (resp. $X \cong Y$).

Theorem 2.1.8.5 Let X and Y be normed linear spaces and $T \in L(X,Y)$.
Let U be a closed subspace of X such that $U \subset \ker T$.
Let $q\colon X \to X/U$ be the natural mapping given by $x \mapsto [x]$.

(a) Then there is a unique $\tilde{T} \in L(X/U,Y)$ such that $T = \tilde{T}q \in L(X,Y)$ (i.e. such that the following diagram commutes).

Also, $\|T\| = \left\|\tilde{T}\right\|$ and $\operatorname{range} T = \operatorname{range}\tilde{T}$.

(b) If futhermore $U = \ker T$, then $\tilde{T}$ is injective.

<u>Preview</u> We will show the Open Mapping Theorem. One corollary of the OMT is that a *bijective* bounded linear operator between *Banach spaces* is an isomorphism. What will this give when applied to the above theorem?

Theorem 2.1.8.7 Let X and Y be normed linear spaces and $T \in L(X,Y)$.
If T is a quotient mapping (i.e. $T(B_X^0) = B_Y^0$) then the following 3 conditions hold.

(1) $X/(\ker T) \cong Y$ (i.e. $X/(\ker T)$ and Y are isometric).

(2) $\|T\| = 1$

(3) T is surjective.

Theorem 2.1.10. *(a)* $c \simeq c_0$

(b) If D is a closed subset of $[0,1]$ and $J_D := \{x \in C[0,1]\colon x|_D = 0\}$ then $C(D) \cong C[0,1]/J_D$

A close analysis of the proof of part (b) leads to the following.

- If K is a compact metric (or topological) space and D is a closed subset of K, then the restriction mapping $C(K) \to C(D)$ given by $x \mapsto x|_D$ is a quoient mapping.
- If $T\colon X \to Y$ is a quotient mapping, then $X/\ker(T) \cong Y$.

Topology Excursion

Definition A Hausdorff (i.e. T_2) is *normal* provided each pair of disjoint closed sets have disjoint neighborhoods.

Basic Exercise from a Topology Class. A metric space is normal. A compact Hausdorff space is normal.

Corollary B.1.6 Let A be a closed subset of a metric space T. Let $f\colon A \to \mathbb{K}$ be a bounded continuous function. Then there exists a bounded continuous extension $F\colon T \to K$ of f with $\|F\|_\infty = \|f\|_\infty$.

Corollary B.2.6 Let A be a closed subset of a normal topological space T. Let $f\colon A \to \mathbb{K}$ be a bounded continuous function. Then there exists a bounded continuous extension $F\colon T \to K$ of f with $\|F\|_\infty = \|f\|_\infty$.

Tietze's Extension Theorem Let X be a Hausdorff topological space. The following two properties are equivalent.

(1) X is normal.

(2) For each closed $A \subset X$ and each continuous $f\colon A \to \mathbb{R}$ has a continuous extension $F\colon X \to \mathbb{R}$. Furthermore, if $|f(a)| < c$ on A, then F can be chosen so that $|F(x)| < c$ on X.

Back to Functional Analysis

Theorem 2.1.11. *Let X be a normed space and $T \in L(X)$.*

(a) If

$$\sum_{n=0}^{\infty} T^n \ \text{ converges in } \ L(X) \tag{(*)}$$

then $Id - T$ is invertible with

$$(Id - T)^{-1} = \sum_{n=0}^{\infty} T^n \ .$$

(b) If X is a Banach space and $\|T\| < 1$, then ((*)) holds and

$$\left\|(Id - T)^{-1}\right\| \le (1 - \|T\|)^{-1} \ .$$

Remark. If $|r| < 1$, then $(1-r)^{-1} = \sum_{n=0}^{\infty} r^n$.

2.2 Dual spaces and their Representations

Definition 2.2.1. The space $L(X,\mathbb{K})$ of continuous linear functionals on a normed space X is called the *dual space* of X and is denoted by X'.

Corollary 2.2.2. *The dual space of a normed space, endowed with the norm*

$$\left\|x'\right\| = \sup_{\|x\|\leq 1} \left|x'(x)\right| \ ,$$

is always a Banach space.

If $1 \leq p,q \leq \infty$ and $\frac{1}{p}+\frac{1}{q}=1$ (with the convension $\frac{1}{\infty}=0$), then the pair p and q are called *conjugate exponents*; also, q is said to be the conjugate exponent of p.

Theorem 2.2.3.

(a) *Let $1 \leq p < \infty$ and $\frac{1}{p}+\frac{1}{q}=1$. Then the mapping*

$$T : \ell^q \to (\ell^p)' \quad , \quad (Tx)(y) = \sum_{n=1}^{\infty} s_n t_n$$

(where $x=(s_n)\in \ell^q$, $y=(t_n)\in \ell^p$) is an isometric isomorphism.

(b) *Similarly, the above mapping defines an isometric isomorphism from ℓ_1 to $(c_0)'$.*

However, Theorem 3.1.11 will show that $(\ell^\infty)'$ is strictly larger than ℓ^1. So

$$\begin{aligned}
(\ell^p)' &\cong \ell^q \qquad 1 \leq p < \infty \qquad (\text{so } 1 < q \leq \infty)\\
(\ell^\infty)' &\ncong \ell^1\\
(c_0)' &\cong \ell^1 \ .
\end{aligned}$$

The above proof also works for n-dimensional ℓ^p spaces $\ell^p(n)$. In this context, $(\ell^p(n))' \cong \ell^q(n)$ for $1 \leq p \leq \infty$.

Theorem 2.2.4. *Let $1 \leq p < \infty$ and (Ω,Σ,μ) be a σ-finite measure space. Again let $\frac{1}{p}+\frac{1}{q}=1$. Then the mapping*

$$T : L^q(\mu) \to (L^p(\mu))' \quad , \quad (Tg)(f) = \int_\Omega fg\,d\mu \ ,$$

is an isometric isomorphism.

In Theorem 2.2.4, if $p \neq 1$ then the σ-finite assumption can be dropped. Also,

$$\begin{aligned}
(L^p)' &\cong L^q \qquad 1 \leq p < \infty \qquad (\text{so } 1 < q \leq \infty)\\
(L^\infty)' &\ncong L^1\\
(\text{any normed linear space})' &\ncong L^1 \ .
\end{aligned}$$

Section 1.1: Example 10/Beispiel j: The space of measures.
Let Σ be a σ-algebra on a set T. A mapping $\mu\colon \Sigma \to \mathbb{R}$ (resp. $\mathbb{C}$) is a *signed* (resp. *complex*) measure provided that for each sequence of pairwise disjoint sets $A_i \in \Sigma$

$$\mu\left(\cup_{i=1}^{\infty} A_i\right) \;=\; \sum_{i=1}^{\infty} \mu\left(A_i\right) \;. \tag{1.3}$$

Let $M(T,\Sigma)$ be the collection of all measures on Σ. Through the rule

$$|\mu|(A) = \sup_{\mathscr{Z}} \sum_{E\in\mathscr{Z}} |\mu(E)| \;,$$

where the supremum is taken over all decompositions of A into finitely many pairwise disjoint measurable subsets A_i from Σ, a positive measure $|\mu|$ is associated, the so-called *variation* of μ. Now associate with a mass μ its *variation* norm

$$\|\mu\| = |\mu|(T) \;.$$

- $M(T,\Sigma)$, endowed with its variation norm, is a Banach space.

Recall from Definition 1.2.13 that $M(T)$ is the collection of all *regular Borel* measures, endowed with the variation norm.

Theorem 2.2.5 (Riesz's Representation Theorem). *Let K be a compact metric (or topological) space. Then $[C(K)]' \cong M(K)$ under the mapping*

$$T\colon M(K) \to [C(K)]' \quad , \quad (T\mu)(x) = \int_K x\, d\mu \;.$$

Theorem 2.2.6. *Let X be a n-dimensional normed space. Then there exists a basis $\{b_1,\ldots,b_n\}$ for X and a basis $\{b_1',\ldots,b_n'\}$ for X' such that*

$$b_j'(b_i) = \delta_{ij} \qquad \textit{and} \qquad \|b_i\| = \left\|b_j'\right\| = 1$$

for all $i,j = 1,\ldots n$.

A basis as in Theorem 2.2.6 is called a Auerbach basis.

Corollary 2.2.7. *Let $\{b_1,\ldots,b_n\}$ be an Auerbach basis of a n-dimensional space X. Then*

$$\frac{1}{n}\sum_{i=1}^{n}|\alpha_i| \le \left\|\sum_{i=1}^{n}\alpha_i b_i\right\| \le \sum_{i=1}^{n}|\alpha_i| \quad \forall\ (\alpha_1,\ldots\alpha_n) \in \mathbb{K}^n \;. \tag{2.8}$$

Thus the operator $T\colon \ell^1(n) \to X$, given by $(\alpha_i) \mapsto \sum_i \alpha_i b_i$ is an isomorphism with

$$\|T\| \le 1 \;,\; \left\|T^{-1}\right\| \le n \;. \tag{2.9}$$

2.3 Compact Operators

Notation[Aufgabe I.4.1, p. 35] Let (X,d) be a metric space, $x \in X$, and $\varepsilon > 0$.

$$\begin{aligned} U(x,\varepsilon) &:= \{y \in X : d(x,y) < \varepsilon\} \;=\; N_\varepsilon(x) \\ B(x,\varepsilon) &:= \{y \in X : d(x,y) \le \varepsilon\} \quad \text{and} \quad B_X{:=}B(0,1) \\ S(x,\varepsilon) &:= \{y \in X : d(x,y) = \varepsilon\} \quad \text{and} \quad S_X{:=}S(0,1)\ . \end{aligned}$$

Topology Excursion

Definition Let K be a subset of a topological space X. K is *sequentially compact* provided each sequence in K contains a subsequence converging to a point of K. K is *relatively compact* provided $\overline{K}$ is compact.
Definition A subset K of a metric space X is *totally bounded* provided for each $\varepsilon > 0$ there are finitely many elements $k_1, \ldots k_{N_\varepsilon} \in K$ such that $K \subset \cup_{i=1}^{N_\varepsilon} N_\varepsilon(k_i)$.
Theorem B.1.7[from Appendix=Anhang] For a metric space (X,d), TFAE.

(i) X is compact.
(ii) X is sequentially compact.
(iii) X is complete and totally bounded.

Theorem 2.3.0 Let K be a subset of a metric space X. Then TFAE.

(i) $\overline{K}$ is compact.
(ii) Each sequence in K contains a subsequence converging to a point in X.
(iii) $\overline{K}$ is complete and K is totally bounded.

Thus a compact metric space is complete and separable.

Back to Functional Analysis

Theorem 1.2.7[recall] For a normed space X, the following are equivalent.

(i) $\dim X < \infty$.
(ii) $B_X{:=}\{x \in X : \|x\| \le 1\}$ is compact.
(iii) Each bounded sequence in X has a convergent subsequence.

Definition 2.3.1. A linear mapping T between normed spaces X and Y is called *compact* when $T(B_x)$ is relatively compact (i.e., when $\overline{T(B_x)}$ is compact). The collection of compact operators is denoted by $K(X,Y)$; again, we let $K(X) = K(X,X)$.

From p. 66 Obviously, for a linear operator $T : X \to Y$, the TFAE.

(1) T is compact.
(2) T maps bounded sets to relatively compact sets.
(3) For each bounded sequence (x_n) in X, the sequence (Tx_n) in Y has a convergent subsequence.

Theorem 2.3.2.

(a) *Let X and Y be Banach spaces. Then $K(X,Y)$ is a closed subspace of $L(X,Y)$. In particular, $K(X,Y)$ is itself a Banach space.*

(b) *Let Z be another Banach space. Let $T \in L(X,Y)$ and $S \in L(Y,Z)$. If T or S is compact, the ST is compact.*

Example (p. 67)

(a) If X is finite dimensional, then each linear operator $T : X \to Y$ is compact. Such a T is necessarily continuous (Example II.1(b)) and hence maps the compact set B_X to a compact set.

(b) If $T \in L(X,Y)$ and its image space ran (T) is finite dimensional, then T is compact because $T(B_x)$ is bounded and bounded subsets of a finite dimensional space are relatively compact.

Corollary 2.3.3. *Let X and Y be Banach spaces and $T \in L(X,Y)$. If (T_n) is a sequence from $L(X,Y)$ such that $\|T_n - T\| \to 0$ and each T_n has finite dimensional range, then $T \in K(X,Y)$.*

Definition Let (X,d_X) and (Y,d_Y) be metric spaces and $\mathscr{F}$ be a family of functions from X to Y.

(1) $\mathscr{F}$ is called *equicontinuous at a point* $x_0 \in X$ provided

$$\forall \varepsilon > 0 \exists \delta > 0 \left[x \in X \ \wedge \ d_X(x,x_0) < \delta \Rightarrow \sup_{f \in \mathscr{F}} d_Y(f(x), f(x_0)) < \varepsilon \right].$$

(2) $\mathscr{F}$ is called *equicontinuous* if it is equicontinuous at each point of X.

(3) $\mathscr{F}$ is called *uniformly equicontinuous* provided

$$\forall \varepsilon > 0 \exists \delta > 0 \left[x_i \in X \ \wedge \ d_X(x_1,x_2) < \delta \Rightarrow \sup_{f \in \mathscr{F}} d_Y(f(x_1), f(x_2)) < \varepsilon \right].$$

It is easy to show that (uniform) equicontinuity passes to closures.

Theorem 2.3.4 (Arzelá-Ascoli Theorem - beefed up). *Let (K,d) be a* ***compact*** *metric space and (as usual) endow $C(K)$ with the supremum norm. Let $M \subset C(K)$.*

(1) TFAE.

(a) M is compact.

(b) M is closed, bounded, and equicontinuous.

(c) M is closed, bounded, and uniformly equicontinuous.

(2) TFAE.

(a) M is relatively compact.

(b) M is bounded and equicontinuous.

(c) M is bounded and uniformly equicontinuous.

Let $F(X,Y)$ be the space of continuous linear operators from X to Y with finite dimensional range (F stands for *finite rank*). Corollary 2.3.3 says that

$$\overline{F(X,Y)} \subset K(X,Y)$$

for all Banach spaces X and Y.

Theorem 2.3.5. *Let X be an arbitrary Banach space and Y be a (separable) Banach space with the property that there exists a bounded sequence (S_n) in $F(Y)$ such that*

$$\lim_{n\to\infty} S_n y = y \quad \forall y \in Y \ . \tag{2.12}$$

Then $\overline{F(X,Y)} = K(X,Y)$.

Corollary 2.3.6. *Let X be any Banach space. Let Y be one of the following Banach spaces: c_0, ℓ_p, $C[0,1]$, $L^p[0,1]$ where $1 \leq p < \infty$. Then $\overline{F(X,Y)} = K(X,Y)$.*

2.4 Interpolation of Operators on L^P-spaces

Lemma 2.4.1 (Liapounoff's Inequality).
Let $1 \le p_0, p_1 \le \infty$ *and* $0 \le \theta \le 1$. *Define* p *by*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \ .$$

Then $L^{p_0}(\mu) \cap L^{P_1}(\mu) \subset L^p(\mu)$*; more precisely*

$$\|f\|_{L^p} \le \|f\|_{L^{p_0}}^{1-\theta} \ \|f\|_{L^{p_1}}^{\theta} \quad \forall f \in L^{p_0}(\mu) \cap L^{P_1}(\mu) \ .$$

Furthermore $L^{p_0}(\mu) \cap L^{P_1}(\mu)$ *is dense in* $L^p(\mu)$ *for* $1 \le p < \infty$.

Theorem 2.4.2 (Riesz-Thorin Interpolation Theorem).
Let $1 \le p_0, p_1, q_0, q_1 \le \infty$. *Also, let* $0 < \theta < 1$ *and define* p *and* q *by*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad , \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \ . \tag{1.13}$$

Let μ *and* ν *be* σ*-finite measures. Let* T *be a linear mapping with*

$$T: L^{p_0}(\mu) \to L^{q_0}(\nu) \quad \textit{continuous with norm } M_0 \ ,$$
$$T: L^{p_1}(\mu) \to L^{q_1}(\nu) \quad \textit{continuous with norm } M_1 \ .$$

Then

$$\|Tf\|_{L^q} \le M_0^{1-\theta} M_1^{\theta} \ \|f\|_{L^p} \quad \forall f \in L^{p_0}(\mu) \cap L^{p_1}(\mu) \tag{2.14}$$

for $\mathbb{K} = \mathbb{C}$ *and*

$$\|Tf\|_{L^q} \le 2M_0^{1-\theta} M_1^{\theta} \ \|f\|_{L^p} \quad \forall f \in L^{p_0}(\mu) \cap L^{p_1}(\mu) \tag{2.15}$$

for $\mathbb{K} = \mathbb{R}$. *The operator extends to a continuous linear mapping*

$$T: L^p(\mu) \to L^q(\nu) \quad \textit{with norm } \le cM_0^{1-\theta} M_1^{\theta}$$

($c = 1$ *for* $\mathbb{K} = \mathbb{C}$ *and* $c = 2$ *for* $\mathbb{K} = \mathbb{R}$*).*

Let

$$S = \{z \in \mathbb{C} : 0 \le Re\ z \le 1\} \ . \tag{2.16}$$

Theorem 2.4.3 (Hadamard's Three Line Theorem).
Let $F : S \to \mathbb{C}$ *be a bounded continuous function that is analytic on the interior of* S. *For* $0 \le \theta \le 1$ *set*

$$M_\theta = \sup_{y \in \mathbb{R}} |F(\theta + iy)| \ .$$

Then

$$M_\theta \le M_0^{1-\theta} M_1^{\theta} \ .$$

Let $\mathbb{T} = \{z \in \mathbb{C}\colon\ |z| = 1\} = \{e^{it}\colon 0 \le t \le 2\pi\}$. Endow $\mathbb{T}$ with the normalized Lebesgue measure $\frac{dt}{2\pi}$. For two complex valued functions $f, g \in L^1(\mathbb{T})$ set

$$(f * g)\left(e^{is}\right) \ = \ \int_0^{2\pi} f(e^{it})\, g(e^{i(s-t)})\, \frac{dt}{2\pi}\ .$$

The function $f * g$ is called the *convolution* of f and g.

Theorem 2.4.4 (Young's Inequality).
Let $1 \le p, q \le \infty$ and

$$\frac{1}{r} := \frac{1}{p} + \frac{1}{q} - 1 \ge 0\ . \qquad (+)$$

If $f \in L^p(\mathbb{T})$ and $g \in L^q(\mathbb{T})$, then $f * g \in L^r(\mathbb{T})$ and

$$\|f * g\|_{L^r} \le \|f\|_{L^p}\ \|g\|_{L^q}\ .$$

Note that (+) is equivalent to

$$\frac{1}{r'} := \frac{1}{p'} + \frac{1}{q'}$$

where the prime indicates the conjugate exponent.

Theorem 2.4.5. *Let the assumptions and notation from the Riesz-Thorin theorem be. Additionally, let $T\colon L^{p_0}(\mu) \to L^{q_0}(\nu)$ be compact. Then $T\colon L^p(\mu) \to L^q(\nu)$ is compact.*

2.5 Problems

2.6 Remarks and Overviews

Chapter 3
The Hahn Banach Theorem and its Consequences

3.1 Extension of Functionals

In the last chapter, the representation of continuous linear functional on several normed spaces were given. However it is still not clear whether, in general, each normed space has a continuous linear functional (attempt, e.g., the qoutient space ℓ^∞/c_0). The existence of functionals with prescribed properties is the theme of this chapter.

The basic existence theorem (Theorem 3.1.2) belongs to the area of linear algebra. We formulate it first for *real* vector spaces. The following definition, though, applies to both a real or a complex vector space.

Definition 3.1.1. Let X be a vector space over $\mathbb{K}$. A mapping $p\colon X \to \mathbb{R}$ is called *sublinear* provided

(a) $p(\lambda x) = \lambda p(x)$ for each $\lambda \geq 0$ and $x \in X$,
(b) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$.

Notice that the definition is purely algebraic (X need not be a normed space) and that sublinear mappings can take on negative values. Note that, even for $\mathbb{C}$ vector spaces, a sublinear map is $\mathbb{R}$ valued.

Examples

(a) Each seminorm is sublinear.
(b) Each linear mapping on a real vector space is sublinear.
(c) $(t_n) \mapsto \limsup t_n$ is sublinear on ℓ^∞ over $\mathbb{R}$.
$(t_n) \mapsto \limsup \mathrm{Re}\, t_n$ is sublinear on ℓ^∞ over $\mathbb{C}$.

Another example (Minkowski functional) will be introduced in the next section.

Theorem 3.1.2 (Hahn-Banach Theorem: $\mathbb{R}$-linear algebra version).
Let U be a vector subspace of a real vector space X. Also, let

$$\ell\colon U \to \mathbb{R} \qquad \textit{be linear}$$
$$p\colon X \to \mathbb{R} \qquad \textit{be sublinear}$$

with

$$\ell(x) \le p(x) \quad \forall x \in U \ .$$

Then there exists a linear extension $L\colon X \to \mathbb{R}$, $L\,|_U = \ell$, with

$$L(x) \le p(x) \quad \forall x \in X \ .$$

Remark. The extension L need not be unique.

Recall: the following definitions used in Zorn's Lemma.

1. A *partial ordering* $\le$ on a set A is a binary relation that satisifies the following 3 properties:
   (1) $a \le a$ (reflexivity),
   (2) ff $a \le b$ and $b \le a$ then $a = b$ (antisymmetry),
   (3) if $a \le b$ and $b \le c$ then $a \le c$ (transitivity),

   for each $a, b, c \in A$. In which case, $(A, \le)$ is called a *partially ordered set*.
2. The word *partially* emphasizes that A may contain elements a and b for which neither $a \le b$ nor $b \le a$ holds (in which case, a and b are *incomparable*). Two elements of A are *comparable* provided they are not incomparable.
3. A subset C of A is a *chain* provided it is totally ordered (i.e., every two elements are comparable).
4. An *upper bound* of a subset B of A is an element $a \in A$ such that $b \le a$ for each $b \in B$.
5. A *maximal element* of A is an $m \in A$ such that if $m \le a \in A$ then $m = a$.

Zorn's Lemma
Let $(A, \le)$ be a partially ordered nonempty set such that each chain in A has an upper bound. Then A contains at least one maximal element.

Next we formulate Theorem 3.1.2 for complex vector spaces. For a $\mathbb{C}$-valued mapping $\ell\colon X \to \mathbb{C}$, though, "$\ell(x) \le p(x)$" no longer makes sense. A vector space X over $\mathbb{C}$ is naturally also a vector space over $\mathbb{R}$; note, that if $x \neq 0$ then x and ix are linearly independent over $\mathbb{R}$. Hence, Theorem 3.1.2 can be applied to X for $\mathbb{R}$ linear functionals. There is a close relationship between $\mathbb{R}$ linear and $\mathbb{C}$ linear functionals, which we first describe in detail.

Lemma 3.1.3. *Let X be a $\mathbb{C}$ vector space.*

(a) *Let $\ell\colon X \to \mathbb{R}$ be a $\mathbb{R}$ linear functional, i.e.,*

$$\ell(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 \ell(x_1) + \lambda_2 \ell(x_2) \quad \forall \lambda_i \in \mathbb{R},\ x_i \in X\ .$$

Set

$$\tilde{\ell}(x) := \ell(x) - i\ell(ix)\ .$$

Then $\tilde{\ell}\colon X \to \mathbb{C}$ is a $\mathbb{C}$ linear functional and $\ell = Re\ \tilde{\ell}$.

(b) *Let $h\colon X \to \mathbb{C}$ be a $\mathbb{C}$ linear functional. Set $\ell{:=}Re\, h$ and define $\tilde{\ell}$ as in (a). Then ℓ is $\mathbb{R}$ linear and $\tilde{\ell} = h$.*

(c) *Let*

$$\begin{aligned} p\colon X \to \mathbb{R} &\qquad \textit{be a seminorm} \\ \ell\colon X \to \mathbb{C} &\qquad \textit{be } \mathbb{C} \textit{ linear}\ . \end{aligned}$$

Then the following are equivalent.

$$|\ell(x)| \le p(x) \quad \forall x \in X \qquad \Leftrightarrow \qquad |Re\,\ell(x)| \le p(x) \quad \forall x \in X\ .$$

(d) *If X is a normed space and $\ell\colon X \to \mathbb{C}$ is $\mathbb{C}$ linear and continuous, then $\|\ell\| = \|Re\,\ell\|$.*

In other words, the mapping

$$\ell \mapsto \mathrm{Re}\,\ell$$
$$\big(\text{space of } \mathbb{C}\text{-linear } \mathbb{C}\text{-valued functionals}\big) \mapsto \big(\text{space of } \mathbb{R}\text{-linear } \mathbb{R}\text{-valued functionals}\big)$$

ia a $\mathbb{R}$-linear bijective mapping. In the normed case, it is also an isometry.

Remark

(i) If X is a $\mathbb{K}$-vector space and the mapping $\ell\colon X \to \mathbb{K}$ is $\mathbb{K}$-linear, then $p(x){:=}|\ell(x)|$ is a seminorm on $(X,\mathbb{K})$.

(ii) If X is a $\mathbb{C}$-vector space and the mapping $\ell\colon X \to \mathbb{C}$ is $\mathbb{C}$-linear, then $q(x){:=}|\mathrm{Re}\,\ell(x)|$ is a seminorm on $(X,\mathbb{R})$ but needs not be a seminorm on $(X,\mathbb{C})$. This is because $\mathrm{Re}\,\ell$ is $\mathbb{R}$-linear but needs not be $\mathbb{C}$-linear. Note that, in 3.1.3(c), the function p is a seminorm on $(X,\mathbb{C})$.

Theorem 3.1.4 (Hahn-Banach Theorem: $\mathbb{C}$-linear algebra version).
Let U be a vector subspace of a complex vector space X. Also, let

$$\ell : U \to \mathbb{C} \qquad \text{be linear}$$
$$p : X \to \mathbb{R} \qquad \text{be sublinear}$$

with

$$Re\,\ell(x) \leq p(x) \quad \forall x \in U \ .$$

Then there exists a linear extension $L : X \to \mathbb{C}$, $L\,|_U = \ell$, with

$$Re\,L(x) \leq p(x) \quad \forall x \in X \ .$$

We now extend the algebraic Hahn-Banach theorems to normed spaces.

Theorem 3.1.5 (Hahn-Banach Theorem: extension version).
Let U be a vector subspace of a normed space X.
For each continuous linear functional

$$u' : U \to \mathbb{K}$$

there exists a continuous linear functional

$$x' : X \to \mathbb{K}$$

with

$$x'\,|_U = u' \quad , \quad \left\|x'\right\| = \left\|u'\right\| \ .$$

In short, each continuous functional has a norm preserving continuous extension.

We should observe that such an extension x', in general, needs not be unique and that the analogue of Theorem 3.1.5 for operators, in general, is false. We will see, in the remarks after Theorem 4.6.5, that there exists *no* continuous operator $T : \ell^\infty \to c_0$ and thus, the identity $Id : c_0 \to c_0$ has not continuous extension to the whole of ℓ^∞. For a positive result in this direction, see Aufgabe 3.6.22(a); also, compare Theorem 2.1.5 for the case the U is dense in X.

The following corollaries indicate that the dual space of normed spaces X is adequately comprehensive (broad ... all telling) to be able to encode properties of X. Thereby, problems about vectors ultimately reduce to problems about numbers. The $x'(x)$, where x' runs through the dual space of X, can be regarded as the "coordinates" of x.

Corollary 3.1.6. *Let X be a normed space and $0 \neq x \in X$. Then there exists a functional $x' \in X'$ such that*

$$\left\|x'\right\| = 1 \quad \text{and} \quad x'(x) = \|x\| \ .$$

In particular, X' separates points of X; i.e. *if $x_1, x_2 \in X$ and $x_1 \neq x_2$, then there exists $x' \in X'$ such that $x'(x_1) \neq x'(x_2)$.*

Corollary 3.1.7. *For each normed space X,*

$$\|x\| \;=\; \sup_{x' \in B_{X'}} \left|x'(x)\right| \qquad \forall x \in X \;. \tag{3.3}$$

Notice the symmetry in formula (3.3) and the definition

$$\left\|x'\right\| \;=\; \sup_{x \in B_X} \left|x'(x)\right| \qquad \forall x' \in X' \;. \tag{3.3'}$$

In contrast: the supremum in (3.3) is always obtained but the supremum in (3.3') need not be obtained.

Corollary 3.1.8. *Let U be a <u>closed</u> subspace of a normed space X and $x \in X \setminus U$. Then there exists an $x' \in X'$ such that*

$$x' \mid_U = 0 \qquad \textit{and} \qquad x'(x) \neq 0 \;.$$

The next corollary follows directly from Corollary 3.1.8 and Theorem 2.1.5.

Corollary 3.1.9. *Let U be a vector subspace of a normed space X. The following are equivalent.*

(i) *U is dense in X.*

(ii) *If $x' \in X'$ and $x' \mid_U = 0$, then $x' = 0$.*

There are still more applications. First some notation. Let X be a normed space and consider subsets:

$$\begin{aligned} U &\subset X \\ V &\subset X' . \end{aligned}$$

Then

$$U^{\perp} := \left\{ x' \in X' : \quad x'(x) = 0 \ \ \forall x \in U \right\} \quad \subset \quad X' \tag{3.4}$$

$$V_{\perp} := \left\{ x \in X \ \ : \quad x'(x) = 0 \ \ \forall x' \in V \right\} \quad \subset \quad X \ . \tag{3.5}$$

$U^{\perp}$ (resp. $V_{\perp}$) is a closed subspace of X' (resp. X).
$U^{\perp}$ is called the *annihilator* of U in X'.
$V_{\perp}$ is called the *annihilator* of V in X.

Theorem 3.1.10. *Let U be a closed subspace of a normed space X. Then there are the following canonical isometric isomorphisms.*

$$(X/U)' \ \cong \ U^{\perp} \tag{3.6}$$

$$U' \ \cong \ X'/U^{\perp} \tag{3.7}$$

The above isometries are realized by the following mappings.

$$\begin{aligned} T\colon \ (X/U)' &\to U^{\perp} \\ \ell &\to \ell \circ q \end{aligned} \tag{3.6*}$$

where $q\colon X \to X/U$ is the natural quotient mapping.

$$\begin{aligned} S\colon X'/U^{\perp} &\to U' \\ x' + U^{\perp} &\to x' \,|_U \end{aligned} \tag{3.7*}$$

Theorem 3.1.11. *The mapping* $T : \ell^1 \to (\ell^\infty)'$*, given by*

$$(Tx)(y) := \sum_{n=1}^{\infty} s_n t_n \quad \textit{for} \quad x = (s_n) \in \ell^1 \ , \ y = (t_n) \in \ell^\infty \ ,$$

is an isometry but is not surjective.

That there is actually no isomorphism between ℓ^1 and $(\ell^\infty)'$ is shown in the following theorem. (Remember: ℓ^1 is separable but ℓ^∞ is not; Example 1.2(a) and (c)).

Theorem 3.1.12. *A normed space* X *with a separable dual space is itself separable.*

Theorem 3.1.13. *Let* X *be a normed space and* I *be an indexing set. Let* $x_i \in X$ *and* $c_i \in \mathbb{K}$ *for each* $i \in I$*. Then the following conditions are equivalent.*

(i) *There exists* $x' \in X'$ *with* $x'(x_i) = c_i$ *for each* $i \in I$*.*

(ii) *There exists* $M \geq 0$ *such that, for each finite subset* $F \subset I$*,*

$$\left| \sum_{i \in F} \lambda_i c_i \right| \ \leq \ M \ \left\| \sum_{i \in F} \lambda_i x_i \right\| \tag{3.8}$$

for each choice of $(\lambda_i)_{i \in F}$ *from* $\mathbb{K}$*.*

3.2 Separation of Convex Sets

In this section, the geometric versions of the Hahn-Banach Theorems will be presented. The goal is the separation of convex sets in normed spaces by continuous linear functionals.

⊛⊛⊛⊛⊛ see niffty picture top of page 101 ⊛⊛⊛⊛⊛

The separation problem is as following.

Separation Problem Let U and V be convex subsets of a normed space X. Does there exist $0 \neq x' \in X'$ with the following property?

$$\sup_{x\in U} x'(x) \;\le\; \inf_{x\in V} x'(x) \qquad (\mathbb{K}=\mathbb{R})$$
$$\sup_{x\in U} \operatorname{Re} x'(x) \;\le\; \inf_{x\in V} \operatorname{Re} x'(x) \qquad (\mathbb{K}=\mathbb{C})$$

The following definition is purely algebraic, as is the convexity notion.

Definition 3.2.1. Let A be a subset of a vector space X. The *Minkowski functional*

$$p_A\colon X \to [0,\infty]$$

is defined by

$$p_A(x) := \inf\left\{\lambda > 0\colon \tfrac{x}{\lambda} \in A\right\} .$$

The set A is *absorbing* provided $p_A(x) < \infty$ for each $x \in X$.

As an example, if A is the open unit ball of a normed space, then $p_A(x) = \|x\|$.

mg notation In the setting of Def 3.2.1

$$M_A(x) := \left\{\lambda > 0\colon \tfrac{x}{\lambda} \in A\right\} \;=\; \{\lambda > 0\colon x \in \lambda A\} .$$

so

$$p_A(x) \;=\; \inf\, M_A(x) .$$

Lemma 3.2.2. *Let U be a convex subset of a normed space X such that $0 \in int\, U$.*

(a) U is absorbing. More precisely: if $N_\varepsilon(0) \subset U$ then $p_U(x) \leq \frac{1}{\varepsilon}\, \|x\|\ \forall x \in X$.

(b) p_U is sublinear.

(c) If U is open, then $U = p_U^{-1}([0,1))$.

Notation We will be using (e.g. in Proof of Lemma 3.2.3) the suggestive notation

$$A \pm B := \{a \pm b\colon\ a \in A\ ,\ b \in B\}$$

for subsets $A, B \subset X$. If A and B are convex, then so is $A \pm B$, as follows directly for the definitions.

The following lemma is the basis for the Hahn-Banach separation theorems.

Lemma 3.2.3. *Let V be a subset of a normed space X such that*

(1) V is convex

(2) V is open

(3) $0 \notin V$.

Then there exists $x' \in X'$ with

$$Re\, x'(x) < 0 \quad \forall x \in V\ .$$

In Lemma 3.2.3, the openness of V cannot be omitted, in general. For example, consider the normed space $(d, \|\cdot\|_\infty)$ over $\mathbb{R}$. Let

$$V = \{(s_n\} \in d \setminus \{0\} : s_N > 0 \text{ for } N{:=}\max\{i\colon\ s_i \neq 0\}\}\ .$$

It is easy to see that V is convex and that $0 \notin V$. Nevertheless, there does not exist $x' \in d'$ with $x' \mid_V < 0$. One can identify $x' \in d'$ with a sequence $(t_n) \in \ell^1$ (Theorems 2.1.5 and 2.2.3). There are two seperate cases. If $t_k \geq 0$ for a k, then $e_k \in V$ but $x'(e_k) = t_k \geq 0$. If $t_n < 0$ for each n, then consider $x = -\frac{t_2}{t_1} e_1 + e_2 \in V$; in this case, $x'(x) = -\frac{t_2}{t_1} t_1 + t_2 = 0$.

Theorem 3.2.4 (Hahn-Banach Theorem; Separation Version I). *(beefed-up)*
Let V_1 and V_2 be subsets of a normed space X such that

(1) $V_1 \cap V_2 = \emptyset$
(2) V_1 *and* V_2 *are convex*
(3) V_1 *is open.*

Then there exists $x' \in X'$ and $\gamma \in \mathbb{R}$ with

$$Re\, x'(v_1) < \gamma \le Re\, x'(v_2) \quad \forall v_1 \in V_1 \ , \ v_2 \in V_2 \ .$$

Thus

$$\begin{aligned} V_1 \ &\subset \ \left\{ x \in X : Re\, x'(x) < \gamma \right\} \\ V_2 \ &\subset \ \left\{ x \in X : Re\, x'(x) \ge \gamma \right\} \ . \end{aligned}$$

Definition In Theorem 3.2.4, the *real hyperplane*

$$H := \left\{ x \in X : \operatorname{Re} x'(x) = \gamma \right\}$$

separates V_1 and V_2.

Theorem 3.2.5 (Hahn-Banach Theorem; Separation Version II). *(beefed-up)*
Let V_1 and V_2 be subsets of a normed space X such that

(1) $V_1 \cap V_2 = \emptyset$
(2) V_1 *and* V_2 *are convex*
(3) V_1 *is closed and* V_2 *is compact.*

Then there exists $x' \in X'$ and $\gamma \in \mathbb{R}$ with

$$Re\, x'(v_1) < \gamma < Re\, x'(v_2) \quad \forall v_1 \in V_1 \ , \ v_2 \in V_2 \ .$$

Thus

$$\begin{aligned} V_1 \ &\subset \ \left\{ x \in X : Re\, x'(x) < \gamma \right\} \\ V_2 \ &\subset \ \left\{ x \in X : Re\, x'(x) > \gamma \right\} \ . \end{aligned}$$

Here we say the the real hyperplane

$$H := \left\{ x \in X : \operatorname{Re} x'(x) = \gamma \right\}$$

strictly separates V_1 and V_2.
Actually, we will show that there exists $\gamma_i \in \mathbb{R}$ such that

$$\operatorname{Re} x'(v_1) < \gamma_1 < \gamma_2 < \operatorname{Re} x'(v_2) \quad \forall v_1 \in V_1 \ , \ v_2 \in V_2 \ .$$

3.3 Weak Convergence and Reflexivity

mg Notation Let X be a normed space. For $x \in X$ and $x' \in X'$, denote

$$x'(x) := \; < x, x' > \; .$$

Let X be a normed space, X' be the dual space of X, and $X'' := (X')'$ be the dual space of X'. The space X'' is called the *bidual* or *second dual* space of X.

Step 1: Fix $x \in X$. Define the corresponding mapping

$$\hat{x} \colon X' \to \mathbb{K}$$

in a canonical way by

$$\hat{x}(x') := x'(x) \; ,$$

i.e., in other notation

$$< x', \hat{x} > \; := \; < x, x' > \; ;$$

thus, in the expression $x'(x)$, one considers x' as the variable and the x as fixed. (Loosely speaking $\hat{x}$ is point evaluation of x' at the fixed point x.) It is clear that $\hat{x}$ is a linear mapping on X'. Also the continuity of $\hat{x}$ is clear since

$$\left|\hat{x}(x')\right| \; = \; \left|x'(x)\right| \; \le \; \|x\| \; \left\|x'\right\| \; .$$

Thus $\hat{x} \in X''$ and $\|\hat{x}\|_{X''} \le \|x\|_X$. The Hahn-Banach Theorem provides the much sharper statement

$$\|\hat{x}\|_{X''} = \|x\|_X \; ;$$

indeed Corollary 3.1.7 gives that

$$\|x\|_X \overset{\text{H-B}}{=} \sup_{x' \in B_{X'}} \left|x'(x)\right| \overset{\text{def}}{=} \|\hat{x}\|_{X''} \; .$$

Step 2: Define a mapping

$$i \colon X \to X'' \qquad \text{by} \qquad i(x) := \hat{x} \qquad \text{so} \qquad i(X) := \hat{X} \subset X''$$

and

$$\begin{aligned} (\, i(x)\,)\,\left(\, x' \,\right) \; &= \; x' \; (\, x \,) \qquad && \forall x' \in X' \qquad (i_X) \\ < \; x' \; , \; i(x) \; > \; &= \; < \; x \; , \; x' \; > && \forall x' \in X' \end{aligned}$$

Clearly i is linear. So we have shown the following theorem.

Theorem 3.3.1. *The mapping* $i\colon X \to X''$ *given in* (i_X) *is an isometric embedding. It need not be surjective.*

We call i the *canonical mapping* of a normed space X into its bidual space; to indicate the dependence on X one occassionally writes i_X. In this way X can be identified as a subspace of X''. If X is complete then so is $i(X)$; thus, a Banach space X can be identified as a closed subspace of X''. For *any* normed space X, the subspace $\overline{i(X)}$ of X'' is closed and thus is complete. So, for *any* normed space X,

$$X \cong \hat{X} \qquad \text{and} \qquad \overline{\hat{X}}^{X''} \quad \text{is a Banach space .}$$

This gives the following corollary, which provides an elegant construction of the completion of *any* normed space.

Corollary 3.3.2. *Each normed space is isometrically isomorphic to a dense subspace of some Banach space.*

Examples

(a) Let $X = c_0$. By Theorem 2.2.3, $X' = \ell^1$ (i.e., X' is, can be identified with, ℓ^1) and $X'' = \ell^\infty$. Let $x = (s_n) \in c_0$. What is $\hat{x} \in \ell^\infty \cong \left(\ell^1\right)^*$? Well, for the k^{th}-unit vector e_k of ℓ^1

$$< e_k \,,\, \hat{x} > \stackrel{\text{def}}{=} < x \,,\, e_k > = < (s_n) \,,\, e_k > = s_k \,.$$

So $\hat{x} = (s_n) \in \ell^\infty$.

(b) By Theorem 3.1.11, i_{ℓ^1} is also not surjective.

(c) As in (a), one can see that, for $1 < p < \infty$, the canonical embedding i_{ℓ^p} agrees with the identity operator $Id\colon \ell^p \to \ell^p$ and thus is surjective. The same ideas hold for $L^p(\mu)$.

Definition 3.3.3. A Banach space X is called *reflexive* provided i_X is surjective.

(Naturally an incomplete space has no chance of being reflexive.) For reflexive spaces it follows from the definition that $X \cong X''$; but this condition is *not* sufficient: Aufgabe 1.4.8 describes the James space J which has the property $J \cong J''$ but i_J is not surjective. This phenomenon was found in 1950 by R.C. James, for the proof see Lindenstrauss/Tzafriri [1977], p. 25.

The above examples give the following.

- ℓ^P and $L^p(\mu)$ are reflexive for $1 < p < \infty$.
- c_0 and ℓ^1 are not reflexive.
- Furthermore, by Example II.1 (b), each finite dimensional space X is, in a trivial way, reflexive and dim X = dim X' = dim X''.

Theorem 3.3.4.

(a) *Closed subspaces of reflexive spaces are reflexive.*

(b) *A Banach space X is reflexive if and only if X' is reflexive.*

It follows from Theorem 3.3.4 (Aufgabe 3.6.15) that the following spaces are not reflexive: ℓ^∞, $L^1[0,1]$, $L^\infty[0,1]$, and $C[0,1]$.

Note another immediate consequence of Theorem 3.1.12. [1]

Corollary 3.3.5. *A reflexive space is separable if and only if its dual is separable.*

Next the concept of weak convergence of a sequence will be introduced and subsequently studied, in particular, in reflexive spaces.

Definition 3.3.6. A sequence (x_n) in a normed space X is *weakly convergent* to $x \in X$ provided

$$\lim_{n\to\infty} x'(x_n) \ = \ x'(x) \qquad \forall x' \in X' \ .$$

Since X' separates points in X (Corollary 3.1.6), the weak limit, if it happens to exist, is unique. If (x_n) converges weakly to x, then we write [2]

$$x_n \stackrel{\sigma}{\to} x \quad \text{or} \quad \sigma - \lim_{n\to\infty} x_n = x \ .$$

Naturally, (norm) convergent sequences are weakly convergent. [3] The converse is false: consider the sequence (e_n) of unit vectors in ℓ^p for $1 < p < \infty$ or c_0. Then $weak - \lim_{n\to\infty} e_n = 0$ but $\|e_n\| = 1$. In Corollary 4.2.3 we will prove the not obvious fact that weakly convergent sequences are necessarily bounded.

Example For a bounded sequnce (x_n) in $C[0,1]$ the following statements are equivalent.

(i) (x_n) converges weakly to 0.

(ii) (x_n) converges pointwise to 0, i.e. $\lim_{n\to\infty} x_n(t) = 0$ for each $t \in [0,1]$.

In the next theorem a form of *weak compactness* will be shown. (Recall: precisely in finite dimensional spaces is the closed unit ball compact; Theorem 1.2.7).

Theorem 3.3.7. *In a reflexive space X, each bounded sequence has a weakly convergent subsequence.*

Thm 1.2.7 + a Fact-to-come Let X be a Banach space with closed unit ball B_X.

- X is finite dimensional $\iff$ B_X is (norm) compact.
- X is reflexive $\iff$ B_X is weakly compact.

[1] A normed space with separable dual is separable.

[2] or the σ can be replaced with $\sigma(X,X')$ or with *wk* or with *weak*.

[3] with the norm and weak limit points agreeing

We have already observed that the unit vectors (e_n) in ℓ_2 tend weakly $\sigma(\ell^2, \ell^2)$ to the zero vector. The e_n's lie in the closed unit sphere S_{ℓ^2} but their weak limit is not! (Norm) Closed sets need not be *weakly closed*. However, for convex sets the following theorem holds.

Theorem 3.3.8. *Let V be a closed convex subset of a normed space X. If a sequence (x_n) from V converges weakly to $x \in X$, then $x \in V$.*

Corollary 3.3.9. *If $x_n \to x$ weakly $\sigma(X, X')$, then there is a sequence (y_n) of convex combinations*

$$y_n = \sum_{i=1}^{N^{(n)}} \lambda_i^{(n)} x_i \qquad \left(\lambda_i^{(n)} \geq 0 \, , \, \sum_i \lambda_i^{(n)} = 1\right)$$

such that $\|y_n - x\| \to 0$.

Proof. Apply Theorem 3.3.8 to $V := \overline{\text{co}}\ (x_i)_i$.

The analysis of weak convergence (as well as the general weak topologies) will resume in Chapter 8.

3.4 Adjoint Operators

Similar to how a normed space was canonically related to its bidual, a continuous linear operator shall now be associated with another operator.

Definition 3.4.1. Let X and Y be normed spaces and $T \in L(X,Y)$.
The *adjoint operator* $T' : Y' \to X'$ is defined by

$$\left(T'y'\right)(x) \; = \; y'(Tx)$$

i.e.

$$< \, x \, , \, T'y' \, > := \, < \, Tx \, , \, y' \, >$$

for $y' \in Y'$ and $x \in X$. [4]

One can directly verify [5] that $T'y'$ is indeed in X' and that $T' \in L(Y', X')$.

Examples

(a) Let $1 \leq p < \infty$ and $X = Y = \ell^p$. Consider the *shift operator* (more precisely, the left shift)

$$T: \; (s_1, s_2, \ldots) \mapsto (s_2, s_3, \ldots) \; .$$

What is T'? We identify X' and Y' with ℓ^q in accordance with Theorem 2.2.3 so T' is an operator on ℓ^q. With the identification $y' = (t_n) \in \ell^q$, namely $y'((s_n)) = \sum_n s_n t_n$ for $(s_n) \in \ell^p$, we can now write

$$y'(T(s_n)) = \sum_{n=1}^{\infty} s_{n+1} t_n = \sum_{n=2}^{\infty} s_n \tilde{t}_n$$

where $\tilde{t}_n = t_{n-1}$ for $n \geq 2$. Thus T' is the right shift

$$T': \; (t_1, t_2, \ldots) \mapsto (0, t_1, t_2, \ldots) \; .$$

In the case $p = 2$, observe that $TT' = Id$ but $T'T \neq Id$. [6]

---

[4] Loosely speaking, $T'y' \in X'$ is a point evaluation , namely, $T'y'$ evaluates y' at the point (Tx)

[5] See class notes.

[6] Thus T and T' need not compute.

(b) Let $1 \le p < \infty$ and $X = Y = L^p[0,1]$. For $h \in L^\infty[0,1]$, we consider the multiplication operator $f \mapsto fh$ on $L^p[0,1]$ and denote this operator by $T_{(p)}$, in order to indicate to dependence on p. Furthermore we identify X' with the function space $L^q[0,1]$ where $\frac{1}{p}+\frac{1}{q}=1$ (Theorem 2.2.4). Then $T'_{(p)} = T_{(q)}$; indeed,

$$\begin{aligned}\left(T'_{(p)}g\right)(f) \;&=\; \int_0^1 \left(T_{(p)}f\right)(t)\, g(t)\, dt \;=\; \int_0^1 f(t)\, g(t)\, h(t)\, dt \\ &=\; \int_0^1 f(t)\, \left(T_{(q)}g\right)(t)\, dt \;=\; \left(T_{(q)}g\right)(f)\end{aligned}$$

for $f \in L^p[0,1]$ and $g \in L^q[0,1]$.

(c) Consider the integral operator T with L^2 kernel k on $L^2[0,1]$:

$$(Tf)(s) := \int_0^1 k(s,t)\, f(t)\, dt \quad \text{(almost everywhere)}\ .$$

As in (b), one sees that T' is an integral operator with kernel $k'(s,t) = k(t,s)$.

(d) For a normed space X, consider the canonical embedding

$$i\colon X \to X'' \qquad \text{where} \qquad <\, x'\,,\, i(x)\, > \;:=\; <\, x\,,\, x'\, >\ .$$

What is i'? From the definition it follows that

$$i'\colon X''' \to X' \qquad \text{where} \qquad <\, x\,,\, i'(x''')\, > \;=\; <\, i(x)\,,\, x'''\, >\,,$$

i.e. i' is the norm-one operator $x''' \mapsto x'''\,|_{i(X)}$.

In the American literature, the notation T^* is often used instead of T'. However, we will denote by T^* the Hilbert space adjoint, which is discussed in Chapter 5.

Theorem 3.4.2.

(a) The mapping

$$L(X,Y) \ni T \quad \mapsto T' \in L(Y',X')$$

is linear and isometric, i.e. $\|T\| = \|T'\|$. *It need not be surjective.*

(b) $(ST)' = T'S'$ *for* $T \in L(X,Y)$ *and* $S \in L(Y,Z)$.

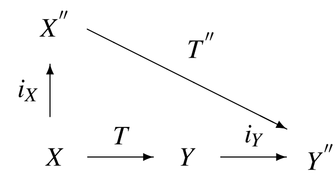
Lemma 3.4.3. *Let* $T \in L(X,Y)$. *Then*

$$T'' \circ i_X \;=\; i_Y \circ T\ .$$

Lemma 3.4.3 basically says that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\;T\;} & Y \\ {\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle i_Y} \\ X'' & \xrightarrow{\;T''\;} & Y'' \end{array}$$

One can consider X (resp. Y) as a subspace of X'' (resp. Y'') and view $T'' \in L(X'', Y'')$ as an extension of $T \in L(X,Y)$, admittedly with values in a bigger space Y''.



Here is another outcome of Lemma 3.4.3 (again identifying $X \subset X''$ and $Y \subset Y''$).

- $S \in L(Y', X')$ is an adjoint operator if and only if $S'(X) \subset Y$.

With this observation one can easily find a counterexample to surjectivity in Theorem 3.4.2(a). Let $X = Y = c_0$, thus (Theorem 2.2.3) $X' = Y' = \ell^1$ and $X'' = Y'' = \ell^\infty$. Define $S \in L(\ell^1)$ by

$$\ell^1 \ni (t_n) \overset{S}{\mapsto} \Big(\sum_{n=1}^{\infty} t_n\ ,\ 0\ ,\ 0\ ,\ \ldots\Big) \in \ell^1\ .$$

One can (and should) verify that, for $(u_n) \in \ell^\infty$,

$$S'((u_n)) = (u_1, u_1, u_1, \ldots)\ .$$

Thus $S'(c_0) \not\subset c_0$, and so S can not be an adjoint operator.

Basic Lemma: Let X be a normed space.
Let K be a bounded subset of X and F be a bounded subset of X'. Then

$$M := \{f\,|_K : K \to \mathbb{K} \quad | \quad f \in F\}$$

is a bounded equicontinuous subset of $C(K)$.

Recall Arzelá-Ascoli Theorem (Thm 2.3.4) for a compact metric space K.
A subset M of $C(K)$ is relatively compact $\Leftrightarrow$ M is bounded and equicontinuous.

Theorem 3.4.4 (Schauder's Theorem).
Let $T : X \to Y$ be a continous linear operators between Banach spaces X and Y. Then T is compact if and only if T' is compact.

Now we use adjoint operators to discuss the solution to operator equalities.

For an operator $T : X \to Y$ between normed linear spaces X and Y, we denote by

$\ker T := \{x \in X : Tx = 0\}$	its kernel	… always closed in X
$\operatorname{ran} T := \{Tx \in Y : x \in X\}$	its range	… need not be closed in Y .

By the linearity of T, the $\ker T$ and $\operatorname{ran} T$ are always vector subspaces; by the continuity of T, the $\ker T = T^{-1}(\{0\})$ is always closed. However, the ran T need not be closed; consider for example the formal identity operator from $C[0,1]$ to $L^1[0,1]$. Yet the following theorem does holds.

Theorem 3.4.5. *Let $T \in L(X,Y)$ for normed spaces X and Y. Then*

$$\overline{\mathit{ran}\ T} \;=\; \left(\mathit{ker}\ T'\right)_\perp \ .$$

Recall [Definition of annihilators, Section 3.1] Let X be a normed linear space and

$$\begin{array}{lll} U & \text{be a subset of} & X \\ V & \text{be a subset of} & X' \ . \end{array}$$

Then

$$U^\perp := \left\{x' \in X' : \ x'(x) = 0 \ \ \forall x \in U\right\} \quad \text{is a closed subspace of} \quad X' \tag{3.4'}$$

$$V_\perp := \left\{x \in X \ : \ x'(x) = 0 \ \ \forall x' \in V\right\} \quad \text{is a closed subspace of} \quad X \ . \tag{3.5'}$$

Also

$$U \subset \operatorname{lin} U \subset \left(U^\perp\right)_\perp \qquad \text{and} \qquad V \subset \operatorname{lin} V \subset (V_\perp)^\perp \ .$$

Furthermore

$$\left(U^\perp\right)_\perp = \overline{[\ \operatorname{lin} U\ ]}^{\text{ norm topology}} \qquad \text{and} \qquad (V_\perp)^\perp = \overline{[\ \operatorname{lin} V\ ]}^{\text{weak}^*\text{-topology}}$$

Theorem Let $T \in L(X,Y)$ for normed spaces X and Y. Then

$$\ker T' = [\operatorname{ran} T]^\perp \qquad \text{and} \qquad \ker T = \left[\operatorname{ran} T'\right]_\perp$$

Proof. In each column, each statement is equivalent.

$y' \in \ker T'$	$x \in \ker T$
$T'y' = 0$	$Tx = 0$
$< x, T'y' >= 0 \ \forall x \in X$	$< Tx, y' >= 0 \ \forall y' \in Y'$
$< Tx, y' >= 0 \ \forall x \in X$	$< x, T'y' >= 0 \ \forall y' \in Y'$
$y' \in [\operatorname{ran} T]^\perp$	$x \in \left[\operatorname{ran} T'\right]_\perp$.

Proof (Theorem 3.4.5). $\quad (\ker T')_\perp \overset{\text{Thm}}{=} \left(\ [\ \operatorname{ran} T\ ]^\perp\ \right)_\perp \overset{\text{Recall}}{=} \overline{\operatorname{ran} T}$.

Corollary 3.4.6. *Let $T \in L(X,Y)$ for normed spaces X and Y.*
Let T have closed range and fix $y_0 \in Y$.
Then the operator equality

$$Tx = y_0$$

has a solution $x \in X$ if and only if the implication

$$T'y' = 0 \quad \Rightarrow \quad y'(y_0) = 0$$

holds.

Proof. Let the givens be given.

$$\begin{aligned}
&Tx = y_0 \text{ has a solution } x \in X \\
&\iff y_0 \in \operatorname{ran} T \overset{\text{assumption}}{=} \overline{\operatorname{ran} T} \overset{\text{thm 3.4.5}}{=} \left(\ker T'\right)_\perp \\
&\quad \vdots \quad T' : Y' \to X' \quad \textit{so} \quad \ker T' \subset Y' \\
&\iff \left[\, y' \in \ker T' \quad \Rightarrow \quad y'(y_0) = 0 \,\right] \\
&\iff \left[\, T'y' = 0 \quad \Rightarrow \quad y'(y_0) = 0 \,\right] .
\end{aligned}$$

In Chapter 6 it will be shown that operators of the form $T = Id - S$ with $S \in K(X)$ have closed range (Theorem 6.2.1).

The virtue of Corollary 3.4.6 is that the existence of a solution is guaranteed by a condition on the kernel of T'. Keep in mind, the kernel of an operator is often relatively easy to compute. In particular, the above implication is always fulfilled when T' is injective.

3.5 Differentation of nonlinear mappings

3.6 Problems

3.7 Remarks and Overviews

The next page will be a blank page between chapters.

Chapter 4
The Fundamental Theorems for Operators on Banach Spaces

TOPOLOGY EXCURSION

(X, τ) is a topological space
$A \subset X$

CATEGORY DEFINITIONS

The below terminology is due to Baire and unfortunately is somewhat nondescriptive; but, it is entrenched in the literature. [1]

Definition 4.1.2 Let M be a subset of a topological space (X, τ).

(a) M is *nowhere dense* provided its closure, $\overline{M}$, contains no interior points.
(b) M is of 1^{st} *category* $\Leftrightarrow \exists$ nowhere dense sets (M_n) with $M = \cup_{n\in\mathbb{N}} M_n$.
(c) M is of 2^{nd} *category* provided M is not of 1^{st} category.

An easy example: $\mathbb{Q}$ is of 1^{st} category in $\mathbb{R}$. [2]

NWD = NOWHERE DENSE

Note that

$$A^\circ \text{ is empty} \iff A^C \text{ is dense .} \qquad (*)$$

Proof of (*) by bi-counterpositive.

$$x \in A^\circ \iff \exists U \in \tau \text{ s.t. } x \in U \subset A \iff \exists U \in \tau \text{ s.t. } x \in U \;\&\; U \cap \left(A^C\right) = \emptyset \iff x \notin \overline{A^C} .$$

Thus

$$A \text{ is nowhere dense} \overset{\text{def}}{\Leftrightarrow} (\overline{A})^\circ = \emptyset \Leftrightarrow \overline{A} \text{ is nowhere dense} \overset{(*)}{\Leftrightarrow} (\overline{A})^C \text{ is dense in } X .$$

It also follows from (*) that

$$A \text{ is closed and nowhere dense} \iff A^C \text{ is open and dense .} \qquad (**)$$

G_δ and F_σ

One calls a countable intersection of open sets a G_δ set (whereby coming from G for *Gebiet*[3] and δ for *Durchschnitt*[4]). One calls a countable union of closed sets a F_σ set (whereby coming from F for *Ferme*[5] and σ for union).

[1] A polite way of saying ”cope”.
[2] Indeed, $\mathbb{Q}$ is a countable union of one point sets.
[3] region
[4] intersection
[5] closed

FIP = FINITE INTERSECTION PROPERTY

Reference: *Topology, a first course* by James R. Munkres

LOCALLY COMPACT HAUSDORFF SPACES

Reference: *Topology* by James Dugundji

4.1 Preparation: Baire Category Theorem

In Section 4.1: (T,τ) **is a complete metric space or locally compact** T_2 **space.**

This chapter will present results about operators on Banach spaces. These results rest upon principles of completeness in metric spaces that R. Baire (in the case of $\mathbb{R}^n$) had discovered in 1899. The Baire Category Principle is introduced first, now.

It is easy to see that in each metric space the intersection of two dense [6] open [7] sets is a dense set. Baire showed that, for $\mathbb{R}^n$, more holds.

Theorem 4.1.1 (Baire's Theorem).
If $(O_n)_{n\in\mathbb{N}}$ be a sequence of dense open subsets of T, then $\cap_{n\in\mathbb{N}}O_n$ is dense.

The assumption of completeness is essential in Theorem 4.1.1, as one see from the following example. Let $T=\mathbb{Q}$, with the usual metric, and consider an enumeration $\{x_1,x_2,\dots\}$ of $\mathbb{Q}$ along with the open dense subsets $O_n=\mathbb{Q}\setminus\{x_n\}$. Then $\cap_{n\in\mathbb{N}}\,O_n=\emptyset$.

Note that the openness of $\cap_{n\in\mathbb{N}}O_n$ was not asserted in Theorem 4.1.1; it is also even not true. (Example?) Here is a reformulation of Theorem 4.1.1.

- A countable intersections of dense G_δ sets from T is a dense G_δ set.

Another redrafting is also often useful. Towards this:

Definition 4.1.2. already did: nowhere dense, first & second category

By taking complements, one obtains the following from Theorem 4.1.1. Indeed, $[\cup_{n\in\mathbb{N}}M_n]^C=\cap_{n\in\mathbb{N}}M_n^C\supset\cap_{n\in\mathbb{N}}\left(\overline{M_n}\right)^C$ and if M_n is nowhere dense then $\left(\overline{M_n}\right)^C$ is dense and open.

Corollary 4.1.3 (Baire's Category Thm.).
If M is of 1^{st} category in T , then M^C is dense (in T). [8]

Frequently the following weaker form is all that will be needed.

Corollary 4.1.4. *A (nonempty) complete metric space (or locally compact T_2 space) is of 2^{nd} category in itself.*

mgRemark All this Baire Stuff expressed vaguely. nwd = nowhere dense.

$$\bigcap_{n\in\mathbb{N}}(\text{ dense open sets }) = \text{dense in } T \qquad \text{(Thm 4.1.1)}$$

$$\left[\bigcup_{n\in\mathbb{N}}(\text{ nwd sets })\right]^C = \text{dense in } T \qquad \text{(Cor 4.1.3)}$$

$$\bigcup_{n\in\mathbb{N}}(\text{ nwd sets }) \neq T \qquad \text{(Cor 4.1.4)}$$

[6] Recall: a subset D of a topological space (T,τ) is dense $\Leftrightarrow \forall\emptyset\neq U\in\tau,\ U\cap D\neq\emptyset$

[7] Open is important; for example, $\mathbb{Q}\cap\mathbb{Q}^C$ is not dense in $\mathbb{R}$.

[8] So sets of first category are small in a certain sense.

Often the Baire Category Theorem gives relatively easy (but nonconstuctive) proofs for existance statements. Here is the archetype.

- Sought after is an object with a desired Property $\circledast$.
- Show then that the collection of explored objects form a complete metric space, where in the objects without Property $\circledast$ is a subset of 1^{st} category.
- Consequently, there are objects with Property $\circledast$, actually, the set of objects with Property $\circledast$ is dense!

The proof of the next theorems illustrates this method.

Theorem 4.1.5. *There is a nowhere differentiable continuous function on* $[0,1]$.

Definition. Let X be a Banach space. A subset A of X is an *algebraic basis* of X provided

$$\operatorname{lin} A := \left\{ \sum_{n=1}^{N} \lambda_n a_n \colon a_n \in A \ , \ \lambda_n \in \mathbb{K} \ , \ N \in \mathbb{N} \right\} = X \ .$$

Theorem. In a Banach space, the cardinality (that is, the number of the elements) of an algebraic basis is either finite or uncountable.

4.2 Uniform Boundedness Principle, called by some folks Principle of Uniform Boundedness (PUB)

In this and the following sections, we apply the Baire category theorem in a functional analysis context. The first result in this direction is the *principle of uniform boundedness* (PUB), which is also known as the *Banach Steinhaus Theorem*.

Theorem 4.2.1 (Banach Steinhaus Theorem - PUB).
Let X be a Banach space and Y be a normed space.
Consider $T_i \in L(X,Y)$ for i in some indexing set I. If

$$\sup_{i\in I} \|T_i x\|_Y < \infty \quad \forall x \in X$$

then

$$\sup_{i\in I} \|T_i\|_{L(X,Y)} < \infty .$$

A rewording of the above. If for each $x \in X$ (suffices to just consider $x \in S_X$) there exists a $M_x \in \mathbb{R}$ such that

$$\sup_{i\in I} \|T_i x\|_Y \le M_x$$

then there exists $M \in \mathbb{R}$ such that

$$\sup_{i\in I} \|T_i\|_{L(X,Y)} \overset{\text{note}}{=} \sup_{i\in I} \sup_{x\in B(X)} \|T_i(x)\|_Y \le M.$$

Note that the proof gives no information on how big $\sup_i \|T_i\|$ can be, it only shows that it is some finite number.

The completeness of X is essential for the validity of Theorem 4.2.1, as seen by the following example. Consider the normed space $(d, \|\cdot\|_\infty)$ of finitely supported sequences and $T_n \colon d \to \mathbb{K}$ given by $(s_m)_{m\in\mathbb{N}} \mapsto ns_n$. Since only a finite number of the s_m can differ from zero, (T_n) fulfills the assumptions of the Banach-Steinhaus Theorem; but $\|T_n\| = n$.

Corollary 4.2.2. *For a subset M of a normed space X, T.F.A.E. .*

(i) *M is (norm) bounded.*
(ii) *For each $x' \in X'$, the set $x'(M) := \{x'(x) \colon x \in M\} \subset \mathbb{K}$ is bounded.*

In particular we now can show a fact mentioned just after Definition 3.3.6 of weak convergence.

Corollary 4.2.3. *A weakly convergent sequence is (norm) bounded.*

Corollary 4.2.4. *Let X be a Banach space and $M \subset X'$. The T.F.A.E. .*

(i) *M is (norm) bounded.*
(ii) *For each $x \in X$, the set $\{x'(x) \colon x' \in M\} \subset \mathbb{K}$ is bounded.*

In Corollary 4.2.4, which is the dual version of Corollary 4.2.2, completeness must be required.

Now we cover the pointwise convergence of sequences of operators. It is well-known that, for a sequence of continuous functions, pointwise convergence is not sufficient to guarantee that the limit function is continuous. Therefore the following result is remarkable/striking/notable.

Corollary 4.2.5. *Let X be a Banach space and Y be a normed space. Consider $T_n \in L(X,Y)$ for $n \in \mathbb{N}$. If*

$$\forall x \in X \qquad \textit{the limit} \qquad Tx := \lim_{n\to\infty} T_n x \qquad \textit{exists}$$

then $T \in L(X,Y)$.

In certain cases, the pointwise convergence of an operator sequence follows from relatively weak assumptions.

Definition Let $C_{\mathbb{R}}[0,1]$ be the (real) Banach space $C[0,1]$ over $\mathbb{R}$, so

$$C_{\mathbb{R}}[0,1] := \{ f\colon [0,1] \to \mathbb{R} \quad | \quad f \text{ is continuous} \} \ .$$

An operator $T \in L(C_{\mathbb{R}}[0,1])$ is called *positive* provided

$$x \in C_{\mathbb{R}}[0,1] \quad \text{and} \quad x \geq 0 \qquad \Longrightarrow \qquad Tx \geq 0$$

or, more specifically,

$$x \in C_{\mathbb{R}}[0,1] \quad \text{and} \quad x(t) \geq 0 \quad \forall t \in [0,1] \qquad \Longrightarrow \qquad (Tx)(t) \geq 0 \quad \forall t \in [0,1] \ .$$

Note that if T is a positive operator then

$$f,g \in C_{\mathbb{R}}[0,1] \quad \text{and} \quad f(t) \leq g(t) \quad \forall t \in [0,1] \quad \Longrightarrow \quad (Tf)(t) \leq (Tg)(t) \quad \forall t \in [0,1] \ .$$

The next theorem handles sequences (T_n) of positive operators on $C_{\mathbb{R}}[0,1]$ and shows the pointwise convergence $T_n \to Id$ for each $x \in C_{\mathbb{R}}[0,1]$ provided one has the pointwise convergence for just 3 (specially choosen) points in $C_{\mathbb{R}}[0,1]$.

Theorem 4.2.6 (First Korovkin Theorem).
*Let (T_n) be a sequence of positive operators in $L(C_{\mathbb{R}}[0,1])$.
For $i \in \{0,1,2\}$, define $x_i \in C_{\mathbb{R}}[0,1]$ by*

$$x_i(t) := t^i \qquad \forall t \in [0,1] \ .$$

If, for each $i \in \{0,1,2\}$,

$$\lim_{n\to\infty} T_n x_i \ = \ x_i \qquad\quad \text{i.e.,} \quad \lim_{n\to\infty} \|x_i - T_n x_i\|_{C_{\mathbb{R}}[0,1]} \ = \ 0$$

then, for each $x \in C_{\mathbb{R}}[0,1]$,

$$\lim_{n\to\infty} T_n x \;=\; x \qquad \text{i.e.,} \quad \lim_{n\to\infty} \|x - T_n x\|_{C_{\mathbb{R}}[0,1]} \;=\; 0\;.$$

The Weierstraß Approximation Theorem [9] follows obviously from Korovkin's Theorem when one sets $T_n x$ to be the n^{th} Bernstein polynomial of x (cf. the calculations on page 30). Having said that, the proof of Weierstraß's Approximation Theorem in this textbook is nothing else than a copy of the above proof.

We still want the trigometric version of Korovkin's Theorem. Consider the real Banach space (with the $\|\cdot\|_\infty$ norm) of 2π periodic continuous function on $\mathbb{R}$. This space will be identified with

$$C_{2\pi} \;=\; \{x \in C[-\pi,\pi] : x(-\pi) = x(\pi)\}\;.$$

Theorem 4.2.7 (Second Korovkin Theorem).
Let (T_n) be a sequence of positive operators in $L(C_{2\pi})$.
For $i \in \{0,1,2\}$, define $x_i \in C_{\mathbb{R}}[0,1]$ by

$$x_0(t) := 1 \quad , \quad x_1(t) := \cos t \quad , \quad x_2(t) := \sin t \qquad \forall t \in [0,1]\;.$$

If, for each $i \in \{0,1,2\}$,

$$\lim_{n\to\infty} T_n x_i \;=\; x_i \qquad \text{i.e.,} \quad \lim_{n\to\infty} \|x_i - T_n x_i\|_{C_{2\pi}} \;=\; 0$$

then, for each $x \in C_{2\pi}$,

$$\lim_{n\to\infty} T_n x \;=\; x \qquad \text{i.e.,} \quad \lim_{n\to\infty} \|x - T_n x\|_{C_{2\pi}} \;=\; 0\;.$$

---

[9] Theorem 1.2.10: The subspace $P[a,b]$ of polyonomial functions on an interval $[a,b]$ in $\mathbb{R}$ is a dense subset in $(C[a,b], \|\cdot\|_\infty)$.

4.3 Open Mapping Theorem

We begin with a definition.

Definition 4.3.1. A mapping between metric spaces is called *open* provided it maps open sets onto open sets.

In contrast to the analogue for continuous functions, one can not here replace open sets with closed sets; in other words, a open map needs not map closed sets onto closed sets. Here is an example: the mapping $p\colon \mathbb{R}^2 \to \mathbb{R}$ defined via $(s,t) \mapsto s$ is open but maps the closed set $\{(s,t) : s \geq 0\ ,\ st \geq 1\}$ to the set $(0,\infty)$.

The above definition is customized to analyse the continuity of an inverse map, for it is clear that a bijective map is open if and only if its inverse is continuous.

We are interested in open linear maps between normed spaces. Therefore the following criterion is helpful.

Lemma 4.3.2. *For a linear map $T\colon X \to Y$ between normed spaces X and Y, T.F.A.E. .*

(i) *T is open.*

(ii) *T maps the open balls around 0 to 0-neighborhoods, in other words, with $U_r := \{x \in X\colon \|x\| < r\}$ and $V_\varepsilon := \{y \in Y\colon \|y\| < \varepsilon\}$,*

$$\forall r > 0\ \ \exists \varepsilon > 0\ \ V_\varepsilon \subset T(U_r)\ .$$

(iii)

$$\exists \varepsilon > 0 \quad V_\varepsilon \subset T(U_1)\ .$$

Examples (a) Each quotient map (Definiton 2.1.7) [10] is open.
(b) The map $T\colon \ell^\infty \to c_0$, defined by $(t_n) \mapsto (\frac{t_n}{n})$ is not open since the set $T(U_1) = \big\{(t_n) \in c_0\colon\ |t_n| < \frac{1}{n}\ \forall n \in \mathbb{N}\big)$ and this set does not contain a neighborhood about 0.

Clearly [11]

an open linear map is surjective. (*)

The following theorem of Banach, which says that the converse of (*) holds in complete spaces, is one of the most important theorems in Functional Analysis, as it numereous corollaries clearly indicate.

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[10] A linear mapping $T\colon X \to Y$ between norm spaces is a *quotient mapping* provided T maps the open ball $\{x \in X\colon\ \|x\| < 1\}$ to the open ball $\{y \in y\colon\ \|y\| < 1\}$.

[11] Consider condition (iii) in Lemma 4.3.2.

Theorem 4.3.3 (Open Mapping Theorem).
Let X and Y be Banach spaces.
Let $T \in L(X,Y)$ be surjective.
Then T is open.

The above remark about inverse mappings gives quickly an important consequence.
Corollary 4.3.4.
Let X and Y be Banach spaces.
Let $T \in L(X,Y)$ be bijective.
Then the inverse operator T^{-1} is continuous.

Corollary 4.3.5. *Let $\|\cdot\|$ and $|||\cdot|||$ be two norms on a vector space X. If*

1. $(X, \|\cdot\|)$ *is a Banach space*
2. $(X, |||\cdot|||)$ *is a Banach space*
3. *there exists $M > 0$ such that $\|x\| \leq M\,|||x|||\ \ \forall x \in X$*

then $\|\cdot\|$ and $|||\cdot|||$ are equivalent.
Corollary 4.3.6.
Let X and Y be Banach spaces.
Let $T \in L(X,Y)$ be injective.
Then T^{-1}, as an operator from ran T to X is continuous if and only if ran T is closed.

The results in the following section can also be viewed as corollaries to the Open Mapping Theorem.

4.4 Closed Graph Theorem

This section discusses operators possessing various properties which produce a continuous operator without the operator itself being continuous.

Definition 4.4.1. Let X and Y be normed spaces and D be a vector subspace of X. A linear map $T : D \to Y$ is called *closed* provided the following holds.

If a sequence $\{x_n\}$ from D converges to $x \in X$ and $\{Tx_n\}$ converges to some $y \in Y$, then $x \in D$ and $Tx = y$.

If T is defined on D then one writes $\text{dom}(T) = D$, or alternatively

$$T : X \supset \text{dom}(T) \to Y \ .$$

Observe how the closeness and continuity of T differ: for the special case that dom $(T) = X$ consider the statements:

(a) $x_n \to x$

(b) (Tx_n) covergences, say $Tx_n \to y$

(c) $Tx = y$.

Then the following holds.

- T is continuous if (a) $\Longrightarrow$ [(b) and (c)] .
- T is closed if [(a) and (b)] $\Longrightarrow$ (c) .

A remark to note. Definition 4.4.1 should not be thought of as the anologue of Definition 4.3.1. In the sense of Definiton 4.4.1, a closed operator does not necessarily map closed sets to closed sets. More accurate is the term *closed graph*, as Lemma 4.4.2 shows.

For a linear mapping $T : X \supset D \to Y$, where D is a vector subset of X, the *graph* of T is defined by

$$\text{gr } T := \{(x, Tx) : x \in D\} \subset X \times Y \ .$$

Lemma 4.4.2. *Let X, Y, D, and T be as in Definition 4.4.1.*

(a) gr (T) is a vector subspace of $X \times Y$.

(b) T is closed if and only if gr (T) is closed in $X \oplus_1 Y$.

Examples Many differential operators are closed. We examine the easy case, the differentation operator for functions of a real variable. We denote the derivative here with $\dot{x}$ instead of x'.

(a) Let $X = Y = C[-1,1]$ and $D = C^1[-1,1]$. Define the operator $T : X \supset D \to Y$ by $Tx = \dot{x}$. A well-known theorem from analysis about the interchange of differentiation and limits [12] shows that T is closed.

[12] Let $J \subset \mathbb{R}$ be a bounded interval and let (f_n) be a sequence of functions from J to $\mathbb{R}$. Suppose that there exists $x_0 \in J$ such that $f_n(x_0)$ converges and that the sequence (f_n') of derivatives exists on J and converge uniformly on J to a function g. Then the sequence (f_n) converges unifromly on J to a function f that has a derivative at each point of J and $f' = g$.

(b) Now let $X = Y = L^2[-1,1]$ and D and T be as above. Then T is not closed. Consider namely $x_n(t) = (t^2 + \frac{1}{n})^{1/2}$ and $x(t) = |t|$ and

$$y(t) := \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0\,. \end{cases}$$

Then (x_n) converges uniformly to X, and even more in $L^2[-1,1]$. Also, (Tx_n) converges to y in $L^2[-1,1]$ (but not uniformly), however, $x \notin D$.
(c) That the T in Example (b) is not closed lies, more or less, in the domain of definition. Instead of $D = C^1$, consider the domain of definition to be

$$D_0 := \left\{ x \in L^2[-1,1] \colon x \text{ is absolutely continuous and } \dot{x} \in L^2[-1,1] \right\} \,.$$

For the definition of absolutely continuous, see Definition A.1.9. [13] The main theorem for differential and integral calculations in Lebesgue spaces, Theorem A.1.10, gives that, for an absolutely continuous function x, the derivative $\dot{x}$ exists almost everywhere and can be viewed as an element in $L^1[-1,1]$.

We will show that $T : X \supset D_0 \to Y$ is closed. To this end, let $x_n \in D_0$ and $x, y \in L^2[-1,1]$ with $x_n \to x$ and $\dot{x}_n \to y$ in $L^2[-1,1]$. For each $t \in [-1,1]$

$$x_n(t) \;=\; x_n(-1) \;+\; \int_{-1}^{t} \dot{x}_n(s)\,ds$$

(cf. Theorem A.1.10). [14] From this we will conclude that (x_n) is uniformly convergent. Indeed, for $t \in [-1,1]$ (the third line uses Hölder's inequality)

$$\begin{aligned} \left| \int_{-1}^{t} \dot{x}_n(s)\,ds \;-\; \int_{-1}^{t} y(s)\,ds \right| \;&\le\; \int_{-1}^{t} |\dot{x}_n(s) - y(s)|\;ds \\ &\le\; \int_{-1}^{1} |\dot{x}_n(s) - y(s)|\;ds \\ &\le\; \left(\int_{-1}^{1} |\dot{x}_n(s) - y(s)|^2\;ds \right)^{\frac{1}{2}} \left(\int_{-1}^{1} 1^2\,ds \right)^{\frac{1}{2}} \\ &=\; \sqrt{2}\;\|Tx_n - y\|_{L^2} \\ &\to\; 0\,. \end{aligned}$$

---

[13] A function $f\colon J \to \mathbb{K}$ is *absolutely continuous* if $\forall \varepsilon > 0\ \exists \delta > 0$ such that if $\{(a_k, b_k)\}_{k=1}^{n}$ are pairwise disjoint intervals in J and $n \in \mathbb{N}$ and $\sum_{k=1}^{n}(b_k - a_k) < \delta$ then $\sum_{k=1}^{n} |f(b_k) - f(a_k)| \;<\; \varepsilon$.

[14] Thm A.1.10. Let J be an interval in $\mathbb{R}$.
(a) If $f\colon J \to \mathbb{K}$ is absolutely continuous, then the derivative f' exists almost everywhere, f' is integrable over each compact subinterval of J, and $f(t) - f(s) = \int_s^t f'\,d\lambda\ \forall s,t \in J$. (Thereby $\int_s^t = -\int_t^s$ for $s > t$).
(b) If $g\colon J \to \mathbb{K}$ is integrable over each compact subinterval of J and $a \in J$ is arbitrary and $f(t) := \int_a^t g\,d\lambda$, then f is absolutely continuous, f' exists almost everywhere and $f' = g$ almost everywhere.

Thus

$$\int_{-1}^{t} \dot{x}_n(s)\,ds \;\to\; \int_{-1}^{t} y(s)\,ds$$

uniformly in t. Furthermore

$$x_n(-1) - x_m(-1) \;=\; x_n(t) - x_m(t) + \int_{-1}^{t} (\dot{x}_m(s) - \dot{x}_n(s))\;ds\;,$$

and so (from above and the triangle inequality for $\|\cdot\|_{L^2}$)

$$\begin{aligned} |x_n(-1) - x_m(-1)| \;&=\; \frac{1}{\sqrt{2}}\left(\int_{-1}^{1} |x_n(-1) - x_m(-1)|^2\,dt\right)^{\frac{1}{2}} \\ &\le\; \frac{1}{\sqrt{2}}\left[\left(\int_{-1}^{1} |x_n(t) - x_m(t)|^2\,dt\right)^{\frac{1}{2}}\right. \\ &\quad \left. + \left(\int_{-1}^{1}\left|\int_{-1}^{t} (\dot{x}_n(s) - \dot{x}_m(s))\;ds\right|^2 dt\right)^{\frac{1}{2}}\right]\;, \end{aligned}$$

so $(x_n(-1))$ is a Cauchy sequence (in $\mathbb{R}$) since the sequence $(\int_{-1}^{t}\dot{x}_n(s)\,ds)$ is, by the first part, a Cauchy sequence, in fact, uniformly in t.

Set $\alpha := \lim_{n\to\infty} x_n(-1)$ and $z(t) = \alpha + \int_{-1}^{t} y(s)\,ds$. The function z is absolutely continuous as the integral of an L^1-function and so $\dot{z}(t) = y(t)$ almost everywhere (cf. Theorem A.1.10). So $z \in D_0$ as well as $Tz = y$. Because however $x_n \to z$ uniformly and $x_n \to x$ in $L^2[-1,1]$ so $x(t) = z(t)$ almost everywhere so $x = z$ in $L^2[-1,1]$. Therefore T is closed.

The connection between continuous and closed operators is made in the following lemma. The thereabout emerging norm is known as the *graph norm*.

Lemma 4.4.3. *Let X and Y be Banach spaces, D be a vector subspace of X, and $T : X \supset D \to Y$ be closed. Then the following hold.*

(a) *D, endowed with the norm $|||x||| := \|x\|_X + \|Tx\|_Y$, is a Banach space.*

(b) *T, as a map from $(D, |||\cdot|||)$ to $(Y, \|\cdot\|_Y)$, is continuous.*

The following theorem is the analog to the theorem for open maps to closed operators. It shows, in particular, the continuity of the inverse operator of a closed map between Banach spaces. [15]

Theorem 4.4.4.
Let X and Y be Banach spaces and D be a vector subspace of X.
Let $T: X \supset D \to Y$ be closed.

(1) *If T is surjective, then $T: (D, |||\cdot|||_{gr(T)}) \to (Y, \|\cdot\|_Y)$ is open.*
(2) *If T is bijective, then $T^{-1}: (Y, \|\cdot\|_Y) \to (D, \|\cdot\|_X)$ is continuous.*

Now we show the main theorem of this section.

Theorem 4.4.5 (Closed Graph Theorem).
Let X and Y be Banach spaces and $T: X \to Y$ be a linear closed map.
Then T is continuous.

A short formulation of the closed graph theorem is that a closed operator between Banach spaces is automatically continuous. This theorem and Corollary 4.2.5 illustrate why it is practically impossible to explicity write down a noncontinuous linear operator on a Banach space.

Now, as an application, we sholl show a theorem of Toeplitz about summability methods. Given is an infinite matrix $(a_{ij})_{i\in\mathbb{N}, j\in\mathbb{N}}$. To a sequence $(s_i)_{i\in\mathbb{N}}$ assign the sequence $(\sigma_i)_{i\in\mathbb{N}}$ where

$$\sigma_i := \sum_{j=1}^{\infty} a_{ij} s_j \tag{IV.9}$$

provided the series in (IV.() converges. This well-known example is the Cesäro-Method, where $a_{ij} = \frac{1}{i}$ for $j \leq i$ and $a_{ij} = 0$ for $j > i$. (This summability method was already encountered in Theorem 4.2.11.) Toeplitz showed in 1911 the following theorem, for which Banach later gave a functional analytic proof. A *regular matrix summability method* is a matrix transformation of a convergent sequence which preserves the limit.

Theorem 4.4.6. *An infinite matrix $(a_{i,j})_{i,j\in\mathbb{N}}$ with complex valued entries defines a regular summability method if and only if it satisfies all of the following properties.*

(a) $\lim_{i\to\infty} a_{ij} = 0 \quad \forall j \in \mathbb{N}$ *(any column sequence converges to 0)*
(b) $\sup_i \sum_{j=1}^{\infty} |a_{ij}| < \infty$ *(the absolute row sums are bounded)*
(c) $\lim_{i\to\infty} \sum_{j=1}^{\infty} a_{ij} = 1$ *(the row sums converge to 1).*

For a proof, read the book and/or see
http://planetmath.org/encyclopedia/SilvermanToeplitzTheorem.html.

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[15] Compare with OMT:
4.3.3 (a surjective blop btw B-sp's is open) and
4.3.4 (a bijective blop btw B-sp's has cont. inverse)

4.5 Closed Range Theorem

Corollary 3.4.6 gave a criterion for the solution of the operator equality $Tx = y$ for continuous linear operators with closed range. The following criterion is useful in checking the closed range assumption.

Theorem 4.5.1 (Closed Range Theorem).
Let X and Y be Banach spaces and $T \in L(X,Y)$. T.F.A.E. .

- *(i)* *ran (T) is closed.*
- *(ii)* *ran $(T) \; = \; [ker \; (T')]_{\perp}$.*
- *(iii)* *ran (T') is closed.*
- *(iv)* *ran $(T') \; = \; [ker \; (T)]^{\perp}$.*

The notations $U^{\perp}$ and $V_{\perp}$ were introduced in (3.4) and (3.5). The proof will require two lemmas, each of which is interesting in its own right.

Lemma 4.5.2 (Banach's Homeomorphism Theorem).
Let X and Y be Banach spaces and $T \in L(X,Y)$ have closed range.
Then there exists $K \geq 0$ such that

$$\forall y \in ran\ (T) \;\; \exists x \in X \;\; : \;\; Tx = y \;\; and \;\; \|x\| \leq K\|y\| \; .$$ [16]

Lemma 4.5.3.
Let X and Y be Banach spaces and $T \in L(X,Y)$.
Let $T' \in L(Y',X')$ satisfy, for some $c > 0$: $\; c\|y'\| \leq \|T'y'\| \;\; \forall y' \in Y'$.
Then T is open and thus surjective.

MG REMARKS

Let's look closer at the proof [17] of Lemma 4.5.3 to see what we really have.

Lemma 4.5.3$^+$
Let X and Y be Banach spaces and $T \in L(X,Y)$.
Let $U{:=}\{x \in X : \; \|x\| < 1\}$ and $V{:=}\{y \in Y : \; \|y\| < 1\}$. Let $\delta > 0$.
Consider the following statements.

- (a) $\|T'y'\| \; \geq \; \delta\,\|y'\| \;\; \forall y' \in Y'$.
- (b) $\overline{T\,(U)} \; \supset \; \delta V$.
- (c) $T\,(U) \; \supset \; \delta V$.
- (d) $T(X) = Y$.

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d).
Furthermore, if (d) holds then (a) holds for some $\delta > 0$.

---

[16] In particular, wlog $K > 0$, so $\forall y \in$ ran $(T) \;\; \exists x \in X \;\; : \;\; Tx = y$ and $\frac{1}{K}\|x\| \leq \|Tx\| \leq \|T\|\,\|x\|$.
[17] English Reference: *Functional Analysis*, Walter Rudin, 2nd ed., Thm. 4.13

Proof of Lemma 4.5.3$^+$ The proof in class of Lemma 4.5.3 showed that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d). So now assume (d), ie. assume T is onto. Then T is open (by OMT Thm 4.3.3) and so $\exists\delta > 0$ so that $T(U) \supset \delta V$ (by Lemma 4.3.2). Hence

$$\begin{aligned} \|T'y'\|_{X'} &= \sup\{|< x, T'y' >| : x \in U\} \\ &= \sup\{|< Tx, y' >| : x \in U\} \\ &\geq \sup\{|< y, y' >| : y \in \delta V\} \\ &= \delta \|y'\|_{Y'} \end{aligned}$$

for each $y' \in Y'$. Thus (a) holds. □

To tie things together, let's introduce a definition and some notation.

Definition. Let $T: X \to Y$ be a linear map between normed linear spaces X and Y. Then T is an *isomorphic embedding* provided T is continuous and

$$\exists m > 0 \quad \text{so that} \quad m\|x\| \leq \|Tx\| \quad \forall x \in X .$$

Clearly, an isomorphic embedding is injective. Also, for $T \in L(X,Y)$, note that T is an isomorphic embedding **IFF** the corresponding map $T_0: X \to (\text{ran}T)$, obtained by restricting the range of T, is an isomorphism [18], in which case

$$\|T_0^{-1}\|^{-1} \|x\| \leq \|Tx\| \leq \|T\| \|x\| \qquad \forall x \in X .$$

Notation. Let $T: X \to Y$ be a linear map between normed linear spaces X and Y.

$$\begin{aligned} &T: X \twoheadrightarrow Y && \text{denotes that } T \text{ is surjective ,} \\ &T: X \rightarrowtail Y && \text{denotes that } T \text{ is injective ,} \\ &T: X \hookrightarrow Y && \text{denotes that } T \text{ is an isomorphic embedding .} \end{aligned}$$

Observation. Let $T \in L(X,Y)$ for <u>normed</u> linear spaces X and Y. Then

$$T \text{ has dense range} \quad \Leftrightarrow \quad T' \text{ is injective .}$$

Indeed: $\overline{\text{ran}T} = Y \overset{\text{Thm 3.4.5}}{\Longleftrightarrow} (\ker T')_\perp = Y \overset{\text{easy}}{\Longleftrightarrow} \ker T' = \{0\}$.

Observation. Let $T \in L(X,Y)$ for <u>Banach</u> spaces X and Y. Then

$$\begin{aligned} \overline{T(X)} = Y \quad &\Leftrightarrow \quad T': Y' \rightarrowtail X' && \text{(Obs. 4.5-1)} \\ \Uparrow \qquad & \qquad\qquad \Uparrow && \\ T: X \twoheadrightarrow Y \quad &\Leftrightarrow \quad T': Y' \hookrightarrow X' && \text{(Obs. 4.5-2)} \\ T: X \hookrightarrow Y \quad &\Leftrightarrow \quad T': Y' \twoheadrightarrow X' . && \text{(Obs. 4.5-3)} \end{aligned}$$

Indeed: the $\Uparrow$'s are clear, (Obs. 4.5-1) is the above Observation, and (Obs. 4.5-2) follows from Lemma 4.5.3$^+$. If $T: X \hookrightarrow Y$ then, since X is a Banach space, T has closed range and so $\text{ran}T' \overset{\text{Thm. 4.5.1}}{=} [\ker T]^\perp = \{0\}^\perp = X'$ and so $T': Y' \twoheadrightarrow X'$ If $T': Y' \twoheadrightarrow X'$, then $T'': X'' \hookrightarrow Y''$ by (Obs. 4.5-2), and so $T: X \hookrightarrow Y$ by Lemma 3.4.3.

[18] See Definition 2.1.9: T_0 isomorphism **IFF** T_0 is bijective with T_0 and T_0^{-1} continuous.

4.6 Projections on Banach Spaces

A REVIEW and SOME MOTIVATION

Unfortunately the terminology varies greatly so let's fix our verbiage and notation. For sanity reasons, let's just fix our underlying field $\mathbb{K}$ now.

RECALL

(Hamel) Basis Extension. [19] Let V be a vector space.
A (Hamel) basis B of V is a linearly independent subset of V that spans (or generates) V. In V, between any linearly independent set and any generating set there is a basis. More formally: if L is a linearly independent set in V and G is a generating set of V containing L, then there exists a basis of V that contains L and is contained in G. In particular (taking $G = V$), any linearly independent set L can be "extended" to form a basis of V. These extensions are not unique.

Isomorphism. Roughly speaking, an isomorphism is a bijective map between two spaces that preserves the structure of the spaces. Thus, $j\colon X \to Y$ is: [20]

X, Y vector spaces:	an algebraic isomorphism	$\Leftrightarrow$	j bijective linear
X, Y topological spaces:	a topological isomorphism	$\Leftrightarrow$	j bijective with j & j^{-1} continuous
X, Y normed spaces:	a normed space isomorphism	$\Leftrightarrow$	j bijective linear with j & j^{-1} continuous
X, Y Banach spaces:	a Banach space isomorphism	$\Leftrightarrow$	j bijective linear with j & j^{-1} continuous

Recall Theorem 1.3.3. Let X and Y be normed spaces. Let $1 \le p \le \infty$.
For $(x,y) \in X \times Y$, define

$$\|(x,y)\|_p := \begin{cases} \left(\|x\|_X^p + \|y\|_Y^p\right)^{1/p} & \text{if} \quad 1 \le p < \infty \\ \max\{\|x\|_X, \|y\|_Y\} & \text{if} \quad p = \infty \end{cases}$$

(a) Then $\|(\cdot,\cdot)\|_P$ defines a norm on the direct sum $X \oplus Y$. Endowed with this norm, we denote the direct sum by $X \oplus_p Y$.

(b) All of these norms are equivalent and generate the product topology on $X \times Y$, i.e. $[(x_n, y_n) \overset{\|\cdot\|_p}{\to} (x,y)]$ **IFF** $[x_n \overset{\|\cdot\|_X}{\to} x$ and $y_n \overset{\|\cdot\|_Y}{\to} y]$.

(c) If X and Y are complete, then $X \oplus_p Y$ is complete.

---

[19] Recall from Linear Algebra. Taken from: http://en.wikipedia.org/wiki/Basis_(linear_algebra).

[20] $X \simeq Y$ denotes the existence of such a j. Dirk's book uses $\simeq$ for the normed space setting (Def. 2.1.9).

DIRECT SUM OF VECTOR SPACES

Definition. Let V_1 and V_2 be vector spaces.
The algebraic (external) *direct sum* of V_1 and V_2 is the vector space

$$V_1 \oplus V_2 := \{(v_1, v_2) \colon v_1 \in V_1 \text{ and } v_2 \in V_2\} \ ,$$

endowed with the inherited coordinatewise vector space operations:

$$\alpha(v_1, v_2) + \beta(\tilde{v}_1, \tilde{v}_2) = (\alpha v_1 + \beta \tilde{v}_1, \alpha v_2 + \beta \tilde{v}_2) \ .$$

Thoughts. Now let V_1 and V_2 be vector subspaces of <u>a vector space V</u>.
Clearly, the (Minkowski) *sum*

$$V_1 + V_2 := \{v_1 + v_2 \colon v_1 \in V_1 \ , \ v_2 \in V_2\}$$

of V_1 and V_2 is a vector subspace of V. Thus

$$V_1 \oplus V_2 \ \subset \ V \times V \qquad \text{while} \qquad V_1 + V_2 \ \subset \ V \ .$$

How do the two sums compare? Let's consider the *natural mapping*

$$j \colon V_1 \oplus V_2 \ \to \ V \qquad \text{given by} \qquad (v_1, v_2) \overset{j}{\to} v_1 + v_2 \ . \tag{4.$\tilde{6}$.1}$$

Clearly, j is a well-defined linear map. Clearly,

j is surjective **IFF** $V = V_1 + V_2$
j is injective **IFF** $V_1 \cap V_2 = \{0\}$. [21]

All these thoughts motivate the next two definition.
Definition. A vector space V is algebraically isomorphic (via the natural map) to the algebraic external direct sum of V_1 and V_2, denoted[22] by

$$V \overset{\text{alg}}{\underset{n}{\approx}} V_1 \oplus V_2 \ ,$$

provided V_1 and V_2 are vector subspaces of V and the natural map j in (4.$\tilde{6}$.1) is an algebraic isomorphism.
Definition. Let V_1 and V_2 be vector subspaces of <u>a vector space V</u>.
V is the algebraic (internal) *direct sum* of V_1 and V_2 provided

$$V = V_1 + V_2 \qquad \text{and} \qquad V_1 \cap V_2 = \{0\} \ ,$$

or equiv., provided $\forall\, v \in V \ \exists! v_1 \in V_1 \ \exists! v_2 \in V_2$ such that $v = v_1 + v_2$.
Thus: V is the algebraic internal direct sum of V_1 and V_2 **IFF** V is algebraically isomorphic (via the natural map) to the algebraic external direct sum of V_1 and V_2.
For this reason, the adjectives internal/external are often omitted when confusion seems unlikely.

---

[21] Note that: $V_1 \cap V_2 = \{0\}$ **IFF** $\forall\, v \in V_1 + V_2 \ \exists! v_1 \in V_1 \ \exists! v_2 \in V_2$ such that $v = v_1 + v_2$.
[22] just here in these notes, very nonstandard notation, see the small n for natural?

Definition. V_1 is (algebraically) *complemented* in a vector space V provided V_1 is a vector subspace of V and $\exists$ a vector subspace V_2 of V such that $V \overset{\text{alg}}{\cong} V_1 \oplus V_2$.
In this case, we say that:

- V_1 and V_2 are algebraically *complementary subspaces* of V
- V_2 is algebraically *complementary* to V_1 in V
- V_2 is an algebraic *complement* of V_1 in V.

Remark. A complement need not be unique. Indeed, $\{(0,y) \colon y \in \mathbb{R}\}$ and $\{(y,y) \colon y \in \mathbb{R}\}$ are each a complement of $\{(x,0) \colon x \in \mathbb{R}\}$ in $V = \mathbb{R}^2$.

Definition. A mapping $P \colon V \to V$ is a *projection* provided $P^2 = P$. More generally, a projection P is a mapping from V to a subset of V satisfying $Pv = P^2 v \ \forall v \in V$.

Recall. Let $P \colon V \to V$ be a linear projection on a vector space V. Define $Q \colon V \to V$ by $P + Q = \mathrm{Id}_V$.

(1) Q is a linear projection.

(2) $P \circ Q = 0 = Q \circ P$.

(3) $\mathrm{ran}P = \ker Q = \{v \in V \colon Pv = v\}$.

(4) $\mathrm{ran}P$ and $\ker P$ are vector subspaces of V.

The following fundamental linear algebra results are straightforward to verify.

Motivation 4.$\tilde{6}$.1. Let X be a vector space.

(1) If $P \colon X \to X$ is a linear projection, then $X \overset{\text{alg}}{\cong} \mathrm{ran}(P) \oplus \ker(P)$.
Furthermore,

$$\begin{aligned} U &:= \mathrm{ran}(P) = \ker(\mathrm{Id}_X - P) && \text{is a vector subspace of } X \\ V &:= \ker(P) = \mathrm{ran}(\mathrm{Id}_X - P) && \text{is a vector subspace of } X \ . \end{aligned} \tag{4.$\tilde{6}$.2}$$

Also, j, as in (4.$\tilde{6}$.1), is an algebraic isomorphism where

$$\begin{aligned} j \colon U \oplus V &\to X && \text{with} && (u,v) \overset{j}{\to} u+v \\ j^{-1} \colon X &\to U \oplus V && \text{with} && x \overset{j^{-1}}{\to} \left(P(x), (\mathrm{Id}_X - P)(x)\right) \ . \end{aligned} \tag{4.$\tilde{6}$.3}$$

(2) If $X \overset{\text{alg}}{\cong} U \oplus V$, then the mapping $P \colon X \to X$ given by

$$x = u + v \in X \text{ with } u \in U \text{ and } v \in V \quad \Rightarrow \quad P(x) = u \tag{4.$\tilde{6}$.4}$$

is a well-defined linear projection satisfying (4.$\tilde{6}$.2) and (4.$\tilde{6}$.3).

(3) If U is a vector subspace of X, then there is a linear projection $P \colon X \to X$ with $\mathrm{ran}(P) = U$. In other words, each vector subspace of X is algebraically complemented in X. [23]

[23] Indeed, a simple Hamel basis extension argument gives this.

DIRECT SUM OF NORMED LINEAR SPACES

The normed space version of Motivation 4.$\tilde{6}$.1 should preserve norm structure (not only the norm itself but also the topology generated by the norm).

A Discussion. Motivation 4.$\tilde{6}$.1 part (2) for normed linear spaces.
Let U and V be (not necessarily closed) subspaces of a normed linear space X with

$$X \overset{\text{alg}}{\cong} U \oplus V \ .$$

We have just discussed that the *natural mapping*

$$j\colon U \oplus_1 V \ \to \ X \qquad \text{given by} \qquad (u,v) \ \overset{j}{\to} \ u+v \tag{4.$\tilde{6}$.5}$$

is an *algebraic* isomorphism. But is j a *normed space* isomorphism? Note that

$$\|j((u,v))\|_X \ = \ \|u+v\|_X \ \le \ \|u\|_X \ + \ \|v\|_X \ = \ \|(u,v)\|_{U\oplus_1 V} \ ,$$

so j is continuous. Thus j is a normed space isomorphism **IFF** j^{-1} is continuous. Note that if j^{-1} is continuous, then U and V are (norm) closed subspaces of X. [24]

Remark. Let U and V be vector spaces of a normed space X. Let $X \overset{\text{alg}}{\cong} U \oplus V$ (i.e. let $X = U + V$ and $U \cap V = \{0\}$). For the natural map j in (4.$\tilde{6}$.5), T.F.A.E. .

(i) j is a normed space isomorphism.
(ii) j is a topological isomorphism.
(iii) the inverse map of j is continuous.

If (ii) holds, then we say that X is topologically isomorphic to $U \oplus_1 V$ via the natural map j and we write[25]

$$X \overset{\text{top}}{\cong} U \oplus_1 V \ .$$

Also, if (ii) holds then U and V must be (norm) closed subspaces of X.

A Discussion. Motivation 4.$\tilde{6}$.1 part (1) for normed linear spaces.
Let $P\colon X \to X$ be a linear projection on a normed space X. Set U:=ran(P) and V:=ker$P$. We have just discussed natural map j in (4.$\tilde{6}$.5) is a continuous *algebraic* isomorphism. Again, the question is whether j is a *normed space* isomorphism. Note

$$\left\|j^{-1}(x)\right\|_{U\oplus_1 V} \ = \ \|P(x)\|_X \ + \ \|(\mathrm{Id}_X - P)(x)\|_X \qquad \forall x \in X \ .$$

Thus, j^{-1} is continuous **IFF** P is continuous. [26]
Also, if P is continuous, then U and V must be (norm) closed subspaces of X. [27]

---

[24] Indeed, if $U \ni u_n \overset{\|\cdot\|_X}{\to} x \in X$ and j^{-1} is continuous, then $j^{-1}(u_n) = (u_n, 0) \overset{\|\cdot\|_{U\oplus_1 V}}{\to} j^{-1}(x) \overset{\text{so}}{=} (x,0)$.
[25] again, just here in these notes, very nonstandard notation, see the small n for natural?
[26] Indeed, with $\|P\| \ \le \ \left\|j^{-1}\right\| \ \le \ \|P\| + \|\mathrm{Id}_X - P\|$.
[27] Follows from above or the fact that U is the kernel of $\mathrm{Id}_X - P$ and V is the kernel of P.

We now have a norm space version of Motivation 4.$\tilde{6}$.1. .

Motivation 4.$\tilde{6}$.2. Let X be a normed linear space.

(1) If $P\colon X \to X$ is a *continuous* linear projection, then $X \overset{\text{top}}{\cong} \operatorname{ran}(P) \oplus_1 \ker(P)$. Furthermore,
$$\begin{aligned} U &:= \operatorname{ran}(P) \;=\; \ker(\mathrm{Id}_X - P) \text{ is a (norm) closed subspace of } X \\ V &:= \ker(P) \;=\; \operatorname{ran}(\mathrm{Id}_X - P) \text{ is a (norm) closed subspace of } X\ . \end{aligned} \tag{4.$\tilde{6}$.6}$$
Also, the map j in (4.$\tilde{6}$.5) is a normed space isomorphism where
$$\begin{aligned} j\colon U \oplus_1 V &\to X \qquad \text{with} \qquad (u,v) \overset{j}{\to} u+v \\ j^{-1}\colon X &\to U \oplus_1 V \qquad \text{with} \qquad x \overset{j^{-1}}{\to} (P(x), (\mathrm{Id}_X - P)(x))\ . \end{aligned} \tag{4.$\tilde{6}$.7}$$
Also, $\widetilde{Q} \in L(X/U, V)$ is bijective where $\widetilde{Q}^{-1}(v) := v + U$. [28]

(2) If $X \overset{\text{top}}{\cong} U \oplus_1 V$, then the mapping $P\colon X \to X$ given by
$$x = u + v \in X \text{ with } u \in U \text{ and } v \in V \qquad \Rightarrow \qquad P(x) = u \tag{4.$\tilde{6}$.8}$$
is a well-defined continuous linear projection satisfying (4.$\tilde{6}$.6) and (4.$\tilde{6}$.7).

(3) There is a normed space Y with (norm) closed subspaces V and W such that $Y \overset{\text{alg}}{\cong} V \oplus W$ but $Y \overset{\text{top}}{\ncong} V \oplus_1 W$. [29]

DIRECT SUM OF BANACH SPACES

A Discussion: Banach spaces. Open Mapping Theorem (OMT) to the rescue. Now let U and V be closed [30]subspaces of a Banach space X such that $X \overset{\text{alg}}{\cong} U \oplus V$. Then the natural mapping j in (4.$\tilde{6}$.5) is a bijective continuous linear map between Banach spaces and so j^{-1} is continuous by the **OMT** (Cor. 4.3.4). **Wunderbar**:

Theorem 4.$\tilde{6}$.3. Let U and V be closed subspaces of a Banach space X. Then
$$X \overset{\text{alg}}{\cong} U \oplus V \qquad \Longleftrightarrow \qquad X \overset{\text{top}}{\cong} U \oplus_1 V\ .$$
Thus, T.F.A.E. .

(1) $U + V = X$ and $U \cap V = \{0\}$.

(1′) V is an algebraic complement of U in X.

(2) $X \overset{\text{alg}}{\cong} U \oplus V$.

(2′) $\exists$ a linear projection $P\colon X \to X$ with $\operatorname{ran}P = U$ and $\ker P = V$.

(3) $X \overset{\text{top}}{\cong} U \oplus_1 V$.

(3′) $\exists$ a continuous linear projection $P\colon X \to X$ with $\operatorname{ran}P = U$ and $\ker P = V$.

Definition. A subspace U of a Banach space X is *complemented* in X provided U is closed and $\exists$ a continuous linear projection $P\colon X \to X$ with $\operatorname{ran}P = U$. [31]

---

[28] Apply Thm 2.1.8.5 (p.23) to $T = \mathrm{Id}_X - P$. Note: X is a Banach space $\overset{\text{OMT}}{\Rightarrow}$ $\widetilde{Q}$ is a (B-sp) isomorphism.

[29] See Problem 4.8.23 and take $Y = \{v + w\colon v \in V\ ,\ w \in W\}$.

[30] Thus U and V are Banach spaces in their own right. This is good since, when working in a category of spaces (e.g.: vector, topological, normed, Banach), one usually wants to stay within that category.

[31] Thus a closed subspace U of X is complemented in X **iff** there exists a closed subspace V of X satisfying one (and thus all) of the equivalent conditions in Theorem 4.$\tilde{6}$.3.

BACK TO DIRK'S BOOK

A *projection* on a vector space is a map satisfying $P^2 = P$. In linear algebra it is shown that each vector subspace U of a vector space X has a complementary subspace V such that X is algebraically isomorphic to the direct sum $U \oplus V$. The associated projection from X onto U is then linear. (All this follows from the basis extension theorem.)

If the vector space X is, furthermore, normed then one is interested in the existence of *continuous* linear projections. Also, one wants to know whether the norm space X is not only algebraically, but also topologically, isomorphic to $U \oplus V$, which is naturally endowed with the norm from Theorem 1.3.3, in other words [32], whether $(u_n + v_n)$ converges if and only if (u_n) and (v_n) converge. One speaks then of a *topological direct decomposition*.

First an easy observation.

Lemma 4.6.1. *Let P be a continuous linear projection on a normed space X.*
- *(a) Either $P = 0$ or $\|P\| \geq 1$.*
- *(b) The kernel $ker(P)$ and the image $ran(P)$ are (norm) closed.*
- *(c) $X \simeq ran(P) \oplus_1 ker(P)$ via the normed space isomorphism $x \mapsto (P(x), (Id_X - P)(x))$.*

Examples (a) On $L^p(\mathbb{R})$, for $1 \leq p \leq \infty$, the mapping $f \mapsto f\chi_{[0,1]}$ defines a continuous linear projection with $\|P\| = 1$. The image of P is isometrically isomorphic to $L^p[0,1]$.

(b) The proof of Corollary 2.3.6 defines (conditional expectation) operators on $L^p[0,1]$ that are likewise contractive linear projections.

(c) In Theorem 4.6.5 it is shown that there does not exist a continuous linear projection from ℓ^∞ onto c_0.

Theorem 4.6.2. *Let U be a finite dimensional subspace of a normed space X. Then there exists a continuous linear projection P from X onto U with $\|P\| \leq dim\, U$.*

The next theorem gives a converse of Lemma 4.6.1 in the case of Banach spaces.

Theorem 4.6.3. *Let U be a <u>closed</u> subspace of a Banach space X. Also assume there exists a <u>closed</u> (algebraically) complementary subspace V to U; thus, X is algebraically isomorphic to the direct sum $U \oplus V$. Then:*

- *(a) X is topologically* (i.e., *as Banach spaces) isomorphic to $U \oplus_1 V$.*
- *(b) There is a continuous linear projection from X onto U.*
- *(c) $V \simeq X/U$. Here $\simeq$ indicates Banach space isomorphic.*

More specifically.
- *(a) The map $U \oplus_1 V \ni (u,v) \mapsto u + v \in X$ is a Banach space isomorphism.*
- *(c) The map $V \ni v \mapsto [v] \in X/U$ is a Banach space isomorphism.*
- *(•) $X \simeq U \oplus_1 V$, $V \simeq X/U$, $X \simeq U \oplus_1 X/U$.* [33]

[32] since the norm on $U \oplus_1 V$ generates the product topology on $U \times V$

[33] Here $\simeq$ indicates Banach space isomorphic. Each isomorphism is realized by the *natural* map.

In Chapter 5 it is shown that closed subspaces of L^2 or ℓ^2 always have complementary subspaces. In closing, we now show that this need not hold for arbitrary Banach spaces. To formulate the counterexample we introduce new terminology.

Definition 4.6.4. A subspace U of a Banach space X is called *complemented* (in X) provided U is (norm) closed in X and there exists a continuous linear projection from X onto U.

Theorem 4.6.5. *The subspace c_0 of ℓ^∞ is not complemented in ℓ^∞.*

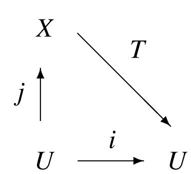
For its proof, we use a combinatorial lemma.

Lemma 4.6.6. *There exists an uncountable family of infinite subsets of $\mathbb{N}$ such that any two members of the family have finite intersection.*

Proof (Theorem 4.6.5). As noted on page 194, the proof of Thm 4.6.5 is due to Phillips (*Trans. Math. Soc.* **48** (1940) 516–541). The proof presented in Dirk's book is due to Whitley (*Amer. Math. Monthly* **73** (1966) 285–286). You should read a proof of this theorem; either: in Dirk's book (so in German) or the above mentioned paper by Whitley (so in English).

The statement of Theorem 4.6.5 can also be interpreted as: the identity operator $\mathrm{Id}\colon c_0 \to c_0$ cannot be extended to a continuous operator from ℓ^∞ to c_0. Thus (unfortuately) there is no hope for a general Hahn-Banach-type extension theorem [34] for operators between Banach spaces instead of functionals (i.e. operators from a Banach space into the scalars).

mgRemark: Complements and extensions.
Let U be a vector subspace of a vector space X.
Consider the following (not necessarily commuting) diagram.



where i is the identity map, j is the inclusion map, and T is some linear map. It is easy to check that the diagram commutes **IFF** T is a projection onto U.

[34] cf. Thm. 3.1.5

4.7 Fixed Point Theorem

4.8 Problems

Problem 4.8.2. Let $\mathscr{P}$ be the vector space of all polynomials on $\mathbb{R}$ and $\|\cdot\|$ be a norm on $\mathscr{P}$. Then $(\mathscr{P}, \|\cdot\|)$ is no Banach space.
(Hint: Baire Category Theorem!)

Problem 4.8.6.

(a) If $1 \le p < q \le \infty$, then ℓ^p is of first category in ℓ^q.

(b) $\cup_{1\le p<q} \ell^p$ is a proper subset of ℓ^q.

Problem 3.6.16. [35] Let X and Y be Banach spaces.

(a) $x_n \xrightarrow{\sigma} x\,,\, T \in L(X,Y) \;\;\Rightarrow\;\; Tx_n \xrightarrow{\sigma} Tx\,.$

(b) $x_n \xrightarrow{\sigma} x\,,\, T \in K(X,Y) \;\;\Rightarrow\;\; Tx_n \to Tx\,.$
(Hint: Use here a fact, to be shown in Chapter 4[36], that (x_n) is bounded.)

(c) Let X be reflexive and $T \in L(X,Y)$ satisfy

$$x_n \xrightarrow{\sigma} x \;\;\Rightarrow\;\; Tx_n \to Tx\,.$$

Then T is compact.

Problem 4.8.8.

(a) Let T be a linear map between normed spaces X and Y. Let T satisfy the condition

$$x_n \to 0 \text{ weakly} \quad \Rightarrow \quad Tx_n \to 0 \text{ weakly}\,.$$

Then T is continuous.
(Together with Problem 3.6.16(a), this statement roughly says that sequential norm continuity and sequential weak continuity of operators are equivalent. The concept of weak continuity will be presented in Chapter 8.

(b) Find counterexamples to show that the completeness of X in Corollary 4.2.4 and 4.2.5 is essential.

Problem 4.8.9. Give necessary and sufficient conditions for weak convergence of a sequence in ℓ^p, with $1 < p < \infty$, or c_0.

---

[35] Weak convergence (denoted by σ) of sequences was introduced in Section 3.3.

[36] Cor. 4.2.3

Problem 4.8.10.

(a) Let $1 \le p < \infty$. For a sequence (x_n) in $L^p[0,1]$, T.F.A.E. .

  (i) $x_n \to 0$ weakly.

  (ii) $\sup_n \|x_n\|_{L^p} < \infty$ and $\int_A x_n(t)\,dt \to 0$ for all Borel sets A.

(b) Are these equivalent also for $p = \infty$?
(Hint: contemplate $x_n(t) = t^n$.)

Problem 4.8.11. A sequence (x_n) in a normed space X is called *weakly Cauchy* provided for all $x' \in X'$ the scalar sequence $(x'(x_n))$ is Cauchy.

(a) Give examples in c_0 and $C[0,1]$ of weakly Cauchy sequences that are not weakly convergent! (One can prove, by the way, that each weakly Cauchy sequence in $L^1[0,1]$ is weakly convergent.)

(b) A weakly Cauchy sequence is bounded.

(c) In a reflexive Banach space a weakly Cauchy sequence is weakly convergent.

Problem 4.8.12. Let X and Y be Banach spaces and $B\colon X \times Y \to \mathbb{K}$ be bilinear[37]. Let B be continuous in each variable separately[38]. Then B is continuous.
(Tip: Banach-Steinhaus Theorem!)

Problem 4.8.13. Let X and Y be Banach spaces. If $T: X \supset \mathrm{dom}(T) \to Y$ is closed, then its kernel (i.e. $\{x \in \mathrm{dom}(T)\colon\, Tx = 0\}$) is a closed subspace of X.

Problem 4.8.14.

(a) Let $X = Y = \ell^2$. Consider $T: X \supset D \to Y$, defined by $T((s_n)) = (ns_n)$, where

  (i) $D = \left\{(s_n) \in \ell^2\colon\, (ns_n) \in \ell^2\right\}$,

  (ii) $D = d$[39] .

Determine whether T is closed.

(b) Let X and Y be normed spaces, D be a vector subspace of X, and $T: X \supset D \to Y$ be given by $Tx = 0$. Is T closed?

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[37] i.e. linear in each variable separately

[38] i.e. $\forall\, x \in X$ the map $y \mapsto B(x,y)$ is continuous and $\forall\, y \in Y$ the map $x \mapsto B(x,y)$ is continuous

[39] See Example 6 from Section 1.1.

Problem 4.8.15.

(a) Let U,V,W be Banach spaces, D be a vector subspace of V,
and $T\colon V \supset D \to W$ be linear. Show the following.

(1) If T is injective and closed, then $T^{-1}\colon W \supset \operatorname{ran}(T) \to V$ is closed.

(2) If T is closed and $S \in L(U,V)$ with $\operatorname{ran}(S) \subset D$, then $TS \in L(U,W)$.

(3) Let T be injective and closed.
If $T^{-1}\colon W \supset \operatorname{ran}(T) \to V$ is continuous then the $\operatorname{ran}(T)$ is closed.

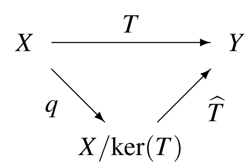
(4) If T is closed and $S \in L(V,W)$, then $S + T\colon V \supset D \to W$ is closed.

(b) Let X,Y,Z be Banach spaces, $K \in K(X,Z)$, and $T \in L(Y,Z)$
with $\operatorname{ran}(T) \subset \operatorname{ran}(K)$. Then T is compact.
(Hint: apply Problem 2.5.18(c) and part (a).)

(c) Show Problem 2.5.18(c). Below the whole of Problem 2.5.18 is translated.
In class we showed parts: (a), (b), and (d).

Problem 2.5.18. Let X and Y be normed spaces and $T \in L(X,Y)$.

(a) There exists a well-defined operator $\widehat{T}$ such that the following diagram commutes (i.e. $T = \widehat{T} \circ q$).



Here q is the canonical map $x \mapsto [x]$ from X onto $X/\ker(T)$.

(b) $\|T\| = \left\|\widehat{T}\right\|$ and $\widehat{T}$ is injective.

(c) T is compact if and only if $\widehat{T}$ is compact.

(d) If T is a quotient map, then $\widehat{T}$ is an isometry.
So in this case, $Y \cong X/\ker(T)$.

Problem 4.8.16. Let X,Y, and Z be Banach spaces. Let:

(1) $T\colon X \to Y$ be linear

(2) $J\colon Y \to Z$ be linear, injective and continuous

(3) $JT\colon X \to Z$ is continuous.

Show that T is continuous.

Problem 4.8.17. Let $\Omega \subset \mathbb{R}^N$ be open and bounded. [40]
Consider the Banach space (!)

$$C^{1,b}(\Omega) = \left\{ f \in C^1(\Omega) : \begin{array}{l} f \text{ and all the partial derivatives} \\ D_i f = \frac{df}{dx_i} \text{ are bounded} \end{array} \right\}$$

under the norm

$$\|f\| := \|f\|_\infty + \sum_{i=1}^{N} \|D_i f\|_\infty .$$

Now let $N \geq 3$. Let $g\colon \Omega \times \Omega \to \mathbb{R}$ be a function such that

- $|g(x,y)| \leq \text{const.}\, |x-y|^{N-2}$ if $x \neq y$

and, for each $f \in C^b(\Omega)$,

- $\int_\Omega g(x,y)\, f(y)\, dy$ exists for each $x \in \Omega$
- the map $\Omega \ni x \mapsto \int_\Omega g(x,y)\, f(y)\, dy \in \mathbb{R}$ is in $C^{1,b}(\Omega)$.

Then there exists a constant A such that, for each $i = 1, \ldots, N$

$$\left| \frac{\partial}{\partial x_i} \int_\Omega g(x,y)\, f(y)\, dy \right| \leq A\ \|f\|_\infty \qquad \forall f \in C^b(\Omega)\ ,\ x \in \Omega\ .$$

(Hint: use Problem 4.8.16.)

[40] Recall from Example 1.1.4:

$$\begin{aligned} C(\Omega) &= \{\varphi\colon \Omega \to \mathbb{R} : \quad \varphi \text{ continuous } \} \\ C^b(\Omega) &= \{\varphi\colon \Omega \to \mathbb{R} : \quad \varphi \text{ is continuous and bounded } \} \\ C^r(\Omega) &= \{\varphi\colon \Omega \to \mathbb{R} : \quad \varphi \text{ is } r\text{-times continuously differentiable } \}\ . \end{aligned}$$

Here $r \in \mathbb{N}$. Recall that $\varphi\colon \Omega \to \mathbb{R}$ is r-times continuously differentiable provided, for each multi-index $\alpha = (\alpha_1, \ldots \alpha_N) \in \mathbb{N}_0^N$ with $|\alpha| := \alpha_1 + \ldots + \alpha_N \leq r$, the partial derivative

$$D^\alpha \varphi = \frac{\partial^{\alpha_1} \cdots \partial^{\alpha_N}}{\partial t_1^{\alpha_1} \cdots \partial t_N^{\alpha_N}}\ \varphi$$

of order $|\alpha|$ exists and is continuous; thereby the order of differentiation is irrelevant.

***Problem 4.8.18.** Let X be a Banach space. [41]
A sequence (x_n) in X is called a *Schauder basis* provided that each $x \in X$ can be respresented *uniquely* as a convergent series $x = \sum_{n=1}^{\infty} c_n x_n$ where $c_n \in \mathbb{K}$[42].

(a) The (standard) unit vectors (e_n) build a Schauder basis in ℓ^p for $1 \le p < \infty$ and in c_0[43] but not in ℓ^∞.

(b) If X possesses a Schauder basis, then X is separable.[44]

(c) Let $n \in \mathbb{N}$. Define the map $x_n^* \colon X \to \mathbb{K}$ by: if $x = \sum_{k=1}^{\infty} c_k x_k$ then $x_n^*(x) = c_n$. Then x_n^* is well-defined and linear.

(e) For $x \in X$ let

$$|||x||| := \sup_{N \in \mathbb{N}} \left\| \sum_{n=1}^{N} x_n^*(x) x_n \right\| .$$

Show $||| \cdot |||$ is indeed a norm on X. Show that $(X, ||| \cdot |||)$ is complete. Show $||| \cdot |||$ and $\|\cdot\|$ are equivalent norms on X (oh yeh, OMT to the rescue).

(d) It always holds that $x_n^* \in X'$. Prove this theorem of Banach by showing that $x_n^* \in (X, ||| \cdot |||)'$ and using part (e).

Definition. x_n^* is called the n^{th} *coefficient functional*.

***Problem 4.8.19-1.** Let X be a Banach space. [45]

(a) Let X have a Schauder basis (x_n) with coefficient functional (x_n^*). Consider the operators $P_n \colon X \to X$ defined by $P_n(x) = \sum_{k=1}^{n} x_k^*(x) x_k$. Then the following conditions hold.
  (1) $P_n \in L(X)$ and $\dim \operatorname{ran}(P_n) = n$ for each $n \in \mathbb{N}$.
  (2) $P_n P_m = P_m P_n = P_{\min(n,m)}$ for all $m, n \in \mathbb{N}$ (in particular, the P_n's are projections).
  (3) $P_n(x) \to x$, i.e. $\lim_{n\to\infty} \|x - P_n(x)\|_X = 0$, for each $x \in X$.

(b) Conversely, if (P_n) is a sequence of projections satisfying conditions (1), (2), and (3), then there exists a Schauder basis to which the P_n's are associated under (a).

Definition. P_n is called the n^{th} *canonical projection* associated with the basis (x_n).

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[41] In the Banach space world, basis usually refers to a Schauder basis.

[42] in other words: provided that for each $x \in X$ there exists a unique sequence (c_n) from $\mathbb{K}$ such that $\lim_{N\to\infty} \left\| x - \sum_{n=1}^{N} c_n x_n \right\|_X = 0$

[43] I think this is obvious ... what about you?

[44] The converse is not true, as P. Enflo showed in 1973 by constructing a separable reflexive Banach space that lacks the Approximation Property (A.P.). See Problem 4.8.19-3. On November 6, 1936, Mazur had entered this converse in the famous *Scottish book* of open problems kept at the Scottish Coffee House in Lvów, Poland, by Banach, Mazur, Stanislaw Ulam, and other mathematicians in their circle. Mazur offered a live goose as the prize for a solution. About a year after solving the problem, Enflo traveled to Warsaw to give a lecture on his solution, after which he was awarded the goose.

[45] Some needed definitions are in Problem 4.8.18.

Problem 4.8.19-2. Schauder bases[46] in $C[0,1]$.

(c) Let $(t_j)_{j=1}^{\infty}$ be a dense sequence of distinct points in $[0,1]$ such that $t_1 = 0$ and $t_2 = 1$. Define projections $P_n \colon C[0,1] \to C[0,1]$ as follows. Let $P_1(f) = f(0)$. Let $P_n(f)$ be the piecewise linear function with nodes at $t_1, t_2, \ldots, t_n$ and $[P_n(f)](t_j) = f(t_j)$ for $j = 1, \ldots, n$. Explain why (P_n) fulfills conditions (1)–(3) of Problem 4.8.19-1(a). For the special (usual) case that (t_n) is the sequence $(0, 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \ldots)$, sketch the first several corresponding (normalized, i.e. of norm one) basis elements.[47]

(d) On the other hand, the functions $f_n \colon t \mapsto t^n$, for $n \geq 0$, do not form a Schauder basis for $C[0,1]$.
(Tip: Is $\sum_{k=0}^{n} a_k f_k \to a_1$ continuous?)

(e) Is there a rearrangement $(f_{\pi(n)})$, where π is a bijection on $\{0, 1, 2, \ldots\}$, that does form a Schauder basis?

---

[46] See Problems 4.8.18 and 4.8.19-1. BTW: the plural of basis is bases.

[47] This basis is called the *Faber-Schauder* basis for $C[0,1]$. Schauder, himself, constructed it!

***Problem 4.8.19-3.** Some basic facts about bases. Again, X is a Banach space.

(f) Let (x_n) be a sequence in X. Then (x_n) is a Schauder basis of X if and only if the follow three conditions hold.
  (1) $x_n \neq 0$ for each $n \in \mathbb{N}$
  (2) There is a constant K so that, for each choice of scalars (c_n) and integers $n < m$

$$\left\| \sum_{k=1}^{n} c_k x_k \right\|_X \leq K \left\| \sum_{k=1}^{m} c_k x_k \right\|_X .$$

  (3) The closed linear span of (x_n) is all of X.

In this case, the smallest constant K for which the inequality in (2) of (f) holds is called the *basis constant* of (x_n) and is denoted by $\mathrm{bc}(x_n)$. Show

$$\mathrm{bc}(x_n) = \sup_{n\in\mathbb{N}} \|P_n\| \qquad \text{and} \qquad \|x_n^*\| \leq \frac{2\,\mathrm{bc}(x_n)}{\|x_n\|} .$$

(g) Let (x_n) be a Schauder basis of X and set $\widetilde{x}_n = \frac{x_n}{\|x_n\|}$.
Then $(\widetilde{x}_n)$ is a normalized[48] Schauder basis of X.

(h) Let X have a Schauder basis (x_n) with coefficient functional (x_n^*).
Then $K \subset X$ is relatively compact if and only if K is bounded and [49]

$$\lim_{N\to\infty} \sup_{x\in K} \sum_{k=N}^{\infty} x_k^*(x) x_n = 0 .$$

(i) If X has a Schauder basis, then X has the Approximation Property.[50]

Problem 4.8.23. Let V and W be closed subspaces of a Banach space X. Show by the following counterexample that $V+W$ need not be closed.

$$X = \ell^1 \oplus_{17} \ell^2 \quad \text{and} \quad V = \ell^1 \oplus_{17} \{0\} \quad \text{and} \quad W = \mathrm{gr}(T)$$

where $T\colon \ell^1 \to \ell^2$ is given by $T(x) = x$.

4.9 Remarks and Overviews

[48] I.e. each vector has norm one

[49] The below condition can also be phrased as: $\lim_{N\to\infty} \sum_{k=N}^{\infty} x_k^*(x) x_n = 0$ uniformly in K.

[50] Corollory 2.3.3 says that $\overline{F(Y,X)} \subset K(Y,X)$ for all Banach spaces X and Y. A Banach space X is said to have the *Approximation Property* (A.P.) provided $\overline{F(Y,X)} = K(Y,X)$ for each Banach space Y (cf. p. 88.). Corollary 2.3.6 says that some of your favorite Banach spaces have the A.P.. Theorem 2.3.5 gives a sufficient condition (oops, this is a big hint) for the A.P..

Chapter 5
Hilbert Spaces

5.1 Defintions and Examples

Hilbert spaces are Banach spaces that, as $\mathbb{K}^n$, have the additional structure of a scalar product. They are among the most important spaces in analysis.

Definition 5.1.1. Let X be a $\mathbb{K}$-vector space. A map $\langle\cdot,\cdot\rangle\colon X\times X\to\mathbb{K}$ is called a *scalar product* (or *inner product*) provided

(a) $\langle x_1+x_2,y\rangle=\langle x_1,y\rangle+\langle x_2,y\rangle$

(b) $\langle\lambda x,y\rangle=\lambda\langle x,y\rangle$

(c) $\langle x,y\rangle=\overline{\langle y,x\rangle}$

(d) $\langle x,x\rangle\geq 0$

(e) $\langle x,x\rangle=0 \;\Leftrightarrow\; x=0$

for each $x,y\in X$ and $\lambda\in\mathbb{K}$. An immediate consequence from (a), (b), and (c) is

(a′) $\langle x,y_1+y_2\rangle=\langle x,y_1\rangle+\langle x,y_2\rangle$

(b′) $\langle x,\lambda y\rangle=\overline{\lambda}\langle x,y\rangle$

for each $x,y,y_1,y_2\in X$ and $\lambda\in\mathbb{K}$.

Thus for $\mathbb{K}=\mathbb{R}$, $\langle\cdot,\cdot\rangle$ is bilinear. However, for $\mathbb{K}=\mathbb{C}$, the scalar in the second factor pulls out as the complex conjugate; one says that $\langle\cdot,\cdot\rangle$ is sesquilinear (sesqui = $1\frac{1}{2}$). The conditions (d) and (e) together is refered to as the *positive definiteness* of scalar products. From (c) follows that $\langle x,x\rangle\in\mathbb{R}$.

As in linear algebra (or analysis), one shows the following important inequality.

Theorem 5.1.2 (Cauchy-Schwarz Inequality).
Let X be a vector space with scalar product $\langle\cdot,\cdot\rangle$. Then

$$|\langle x,y\rangle|^2 \;\leq\; \langle x,x\rangle\,\cdot\,\langle y,y\rangle \qquad \forall x,y\in X\;.$$

Equality holds if and only if x and y are linearly dependent.

For short, by setting

$$\|x\| \;=\; \langle x,x\rangle^{\frac{1}{2}}\ ,$$

Theorem 5.1.2 becomes

$$|\langle x,y\rangle| \;\le\; \|x\|\cdot\|y\|\ .$$

That the notation $\|x\|$ is justified is shown in the next lemma.

Lemma 5.1.3. *$x \mapsto \|x\| := \langle x,x\rangle^{\frac{1}{2}}$ defines a norm.*

Definition 5.1.4. A normed space $(X,\|\cdot\|)$ is a *pre-Hilbert space* provided there is a scalar product $\langle\cdot,\cdot\rangle$ on $X\times X$ such that $\langle x,x\rangle^{\frac{1}{2}} = \|x\|$ for each $x\in X$. A complete pre-Hilbert space is a *Hilbert space*.

We will always denote by $\langle\cdot,\cdot\rangle$ the scalar product that generates the norm on a pre-Hilbert space. $\|\cdot\|$ will always be the norm given in Lemma 5.1.3. (If $(X,\|\cdot\|)$ is a pre-Hilbert space and $|||\cdot|||$ is an equivalent norm on X, then $(X,|||\cdot|||)$ need not be a pre-Hilbert space!)

The Cauchy-Schwarz inequality implies that the maps $x\mapsto\langle x,y\rangle$ and $y\mapsto\langle x,y\rangle$ on a pre-Hilbert space are continuous. As another consequence we present a lemma that later (Corollary 5.3.5) will be generalized further.

Lemma 5.1.5. *Let U be a dense subspace of a pre-Hilbert space X.*
If $x\in X$ satisfies that $\langle x,u\rangle = 0$ for each $u\in U$, then $x=0$.

In a pre-Hilbert space, one can express not only the norm in terms of the scalar product[1] but also the scalar product in terms of the norm. Namely, an easy calculation shows that for $\mathbb{K}=\mathbb{R}$

$$\langle x,y\rangle \;=\; \tfrac{1}{4}\left(\|x+y\|^2-\|x-y\|^2\right) \tag{5.2}$$

and for $\mathbb{K}=\mathbb{C}$

$$\langle x,y\rangle \;=\; \tfrac{1}{4}\left[\left(\|x+y\|^2-\|x-y\|^2\right) \;+\; i\left(\|x+iy\|^2-\|x-iy\|^2\right)\right]\ . \tag{5.3}$$

Lemma 5.1.6. *The scalar product of a pre-Hilbert space X is a continuous map from $X\times X$ to $\mathbb{K}$.* [2]

Theorem 5.1.7 (Parallelogram Equality).
A normed space X is a pre-Hilbert space if and only if

$$\|x+y\|^2 \;+\; \|x-y\|^2 \;=\; 2\,\|x\|^2 \;+\; 2\,\|y\|^2 \qquad \forall x,y\in X\ . \tag{5.4}$$

---

[1] Lemma 5.1.3

[2] mgRmk: In the paragraph before Lemma 5.1.5, it was noted that the scalar product is continuous in each variable separately. Lemma 5.1.6 says that the scalar product is jointly continuous. Compare with Problem 4.8.12.

Theorem 5.1.8.

(a) *A normed space is a pre-Hilbert space if and only if each two dimensional subspace is a pre-Hilbert space (*i.e. $\cong \ell^2(2)$*).*

(b) *Subspaces of pre-Hilbert spaces are pre-Hilbert spaces.*

(c) *The completion of a pre-Hilbert space is a Hilbert space.*

Examples (a) From linear algebra it is well-known that $\mathbb{C}^n$, with the scalar product

$$\langle (s_i),(t_i)\rangle \;=\; \sum_{i=1}^{n} s_i\overline{t_i}\ ,$$

is a Hilbert space.

(b) ℓ^2 is a Hilbert space whose norm is induced by the scalar product

$$\langle (s_i),(t_i)\rangle \;=\; \sum_{i=1}^{\infty} s_i\overline{t_i}\ .$$

(The convergence of the series follows from Hölder's inequality.)

(c) Let $\Omega \subset \mathbb{R}$ be an interval or $\Omega \subset \mathbb{R}^n$ be open (or, more generally, measurable). Then $L^2(\Omega)$ is a Hilbert space, where by the scalar product is defined by

$$\langle f,g\rangle \;=\; \int_{\Omega} f\overline{g}\,d\lambda\ ;$$

here λ denotes Lebesgue measure. (That $f\overline{g}$ is integerable follows once again from Hölder's inequality). Generally all $L^2(\mu)$ spaces are Hilbert spaces.

(d) In numerical mathematics, measure of the form $\mu = w\cdot\lambda$ frequently appear. Here $w \geq 0$ is a measurable function on an interval I and λ is, as above, Lebesgue measure.

$$\langle f,g\rangle \;=\; \int_{I} f\overline{g}\,w\,d\lambda$$

defines the scalar product on $L^2(\mu)$.

(e) One obtains a special case with the counting measure on an arbitrary index set I; the resulting L^2 space is denoted by $\ell^2(I)$. (For $I = \mathbb{N}$ one obtains ℓ^2.) An equivalent description is given by

$$\ell^2(I) \;=\; \left\{ f\colon I\to\mathbb{K} : \begin{array}{l} f(s)\neq 0 \text{ for atmost countably many } s \\ \text{and } \sum_{s\in I}|f(s)|^2<\infty \end{array} \right\}.$$

Thereby the sum "$\sum_{s\in I}$" is understood in the following sense. Let $\{s_1,s_2,\ldots\}$ be an enumeration of $\{s\in I\colon f(s)\neq 0\}$. One sets

$$\sum_{s\in I}|f(s)|^2 \;=\; \sum_{i=1}^{\infty}|f(s_i)|^2\ ;$$

note that because of absolute convergence the order of summation plays no role. For $f, g \in \ell^2(I)$

$$\langle f, g\rangle = \sum_{s\in I} f(s)\overline{g(s)}$$

is a well-defined scalar product that induces the norm

$$\|f\| = \left(\sum_{s\in I} |f(s)|^2\right)^{1/2}$$

with which $\ell^2(I)$ is a complete space. (This statement can be shown in the same manner as we did for ℓ^2.)

(f) For $\lambda \in \mathbb{R}$ consider the function $f_\lambda : \mathbb{R} \to \mathbb{C}$ given by $f_\lambda(s) = e^{i\lambda s}$. Set $X = \operatorname{lin}\{f_\lambda : \lambda \in \mathbb{R}\}$. Through the ansatz[3]

$$\langle f, g\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(s)\overline{g(s)}\, ds\,, \tag{5.5}$$

a scalar product on X is defined (Problem 5.6.4). The completion of X under the norm $\|x\| = \langle x, x\rangle^{1/2}$ is a Hilbert space that is denoted by $AP^2(\mathbb{R})$. Since (Problem 5.6.4)

$$\|f_\lambda - f_{\lambda'}\| = \sqrt{2} \qquad \forall \lambda \neq \lambda'$$

$AP^2(\mathbb{R})$ is not separable. (cf. the proof of the inseparability of ℓ^∞, p. 29); therein lies the important of this example. From results in Section 5.4 it will follow that $AP^2(\mathbb{R}) \cong \ell^2(\mathbb{R})$. (AP stands for *almost periodic*, see the Remarks and Overviews.)

(g) Very important examples of Hilbert spaces are the Sobolev spaces that now shall be defined.

Definition 5.1.9. Let $\Omega \subset \mathbb{R}^n$ be open. Define

$$\mathscr{D}(\Omega) := \left\{\varphi \in C^\infty(\Omega) : \operatorname{supp}(\varphi) := \overline{\{x : \varphi(x) \neq 0\}}^{\mathbb{R}^n} \subset \Omega \text{ is compact}\right\}.$$

(Here $C^\infty(\Omega) = \cap_{m\in\mathbb{N}} C^m(\Omega)$.) $\operatorname{supp}(\varphi)$ is called the *support* of φ; elements from $\mathscr{D}(\Omega)$ are called C^∞ functions with compact support or *test functions*. Why test functions are so named will be clarified on page 430.[4]

Example. Let $|\cdot|$ be the Euclidean norm on $\mathbb{R}^n$ and

$$\varphi(x) = \begin{cases} c \cdot \exp\left(\left(|x|^2 - 1\right)^{-1}\right) & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1\,. \end{cases}$$

Then $\varphi \in \mathscr{D}(\mathbb{R}^n)$. (Cf. Problem 2.5.6).

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[3] From Wikipedia: in physics and mathematics, an ansatz is an educated guess that is verified later.

[4] just after Definition 8.5.3

Lemma 5.1.10. *$\mathscr{D}(\Omega)$ lies[5] densely in $L_p(\Omega)$ for $1 \le p < \infty$.*

Proof. Remarks on proof. [6]
Let $f \in L_p(\Omega)$. For $m \in \mathbb{N}$, define f_m on Ω by

$$\begin{aligned} f_m(x) &:= \int_{K_m} f(y)\varphi_{1/m}(x-y)\,dy \\ &= \int_{\mathbb{R}^n} (f1_{K_m})(y)\varphi_{1/m}(x-y)\,dy \;=\; \left[(f1_{K_m}) * \varphi_{1/m}\right](x) \;, \end{aligned}$$

where: $K_m = \left(x \in \Omega : \; |x| \le m \;,\; d(x, \partial\Omega) \ge \frac{2}{m}\right)$, φ is as in the above example with c chosen so that $\|\varphi\|_{L_1} = 1$, and $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi(x/\varepsilon)$ for $\varepsilon > 0$. Then

$$f_m \in \mathscr{D}(\Omega) \subset L_p(\Omega) \quad \text{and} \quad \lim_{m\to\infty} \|f - f_m\|_{L_p} = 0 \;.$$

Observe that this constructive proof provides a simultaneous approximation in all L_p-spaces: if $f \in L_p(\Omega) \cap L_q(\Omega)$ with $1 \le p, q < \infty$, then there exists $(f_m)_{m\in\mathbb{N}}$ from $\mathscr{D}(\Omega)$ such that $\lim_{m\to\infty} \|f - f_m\|_{L_p} = 0 = \lim_{m\to\infty} \|f - f_m\|_{L_q}$. □

Next the concept of *weak derivatives* is treated. At first, let $\Omega \subset \mathbb{R}$ be an open interval. For $f \in C^1(\overline{\Omega})$ and $\varphi \in \mathscr{D}(\Omega)$

$$\int_\Omega f'(x)\overline{\varphi(x)}\,dx \;=\; -\int_\Omega f(x)\overline{\varphi'(x)}\,dx$$

by the integration by parts formula. (The boundary terms disappear since the support of φ is a compact subset of Ω.) This equality can be written, with the help of the $L^2(\Omega)$ scalar product, as, for $f \in C^1(\overline{\Omega})$

$$\langle f', \varphi\rangle \;=\; -\langle f, \varphi'\rangle \qquad \forall \varphi \in \mathscr{D}(\Omega) \;.$$

Now let $\Omega = (-1,1)$ and $f(x) = |x|$ and $g(x) = x/|x|$ if $x \ne 0$ and $g(0) = 0$. Although $f \notin C^1(\overline{\Omega})$, a similar equality holds, namely,

$$\langle g, \varphi\rangle \;=\; -\langle f, \varphi'\rangle \qquad \forall \varphi \in \mathscr{D}(\Omega) \;.$$

We regard g as a *weak* or *generalized* derivative of f. Analogously, for open $\Omega \subset \mathbb{R}^n$ and $f \in C^1(\overline{\Omega})$

$$\langle \frac{\partial}{\partial x_i} f, \varphi\rangle \;=\; -\langle f, \frac{\partial}{\partial x_i}\varphi\rangle \qquad \forall \varphi \in \mathscr{D}(\Omega) \;.$$

---

[5] Clearly $\mathscr{D}(\Omega) \subset L_p(\Omega)$ for $1 \le p \le \infty$.

[6] If you feel uncomfortable with the proof in the book, read up on it somewhere else. For example, go to Google Book Search (http://books.google.com/) and find the book *Lebesgue Integration on Euclidean Spaces* by Frank Jones. There is a partial (free) preview. Read in Chapter 7 Sections B through D (page 170-180). Note that minor variations on the proof for $p = 1$ give a proof for $1 < p < \infty$ since C_c, ie. continuous functions with compact support, is dense in the reflexive L_p.

by Gauß's integration theorems (see also Problem 5.6.6); here $\langle f,\varphi\rangle = \int_\Omega f\overline{\varphi}$. Correspondingly one obtains for m-times continuously differentiable f and multi-indexes α, with $|\alpha|\le m$

$$\langle D^\alpha f,\varphi\rangle \;=\; (-1)^{|\alpha|}\langle f, D^\alpha \varphi\rangle \qquad \forall \varphi\in\mathscr{D}(\Omega)\;.$$

(The multi-index was introduced in Example 1.1 (4).)[7]

Definition 5.1.11. Let $\Omega\subset\mathbb{R}^n$ be open, α be a multi-index, and $f\in L^2(\Omega)$. $g\in L^2(\Omega)$ is called a *weak* or *generalized* α^{th} derivative of f provided

$$\langle g,\varphi\rangle \;=\; (-1)^{|\alpha|}\langle f, D^\alpha \varphi\rangle \qquad \forall \varphi\in\mathscr{D}(\Omega)\;.$$

Such an g is uniquely determined: namely, if h happens to be anotherr weak derivative of f, then $\langle g-h,\varphi\rangle=0$ for each $\varphi\in\mathscr{D}(\Omega)$, and so by Lemma 5.1.5 as well as Lemma 5.1.10 it follow $g=h$.[8]

The uniquely determined function g in Definition 5.1.11 is denoted by $D^{(\alpha)}f$. So

$$\langle D^{(\alpha)} f,\varphi\rangle \;=\; (-1)^{|\alpha|}\langle f, D^\alpha \varphi\rangle \qquad \forall \varphi\in\mathscr{D}(\Omega)\;.$$

Definition 5.1.12. Let $\Omega\subset\mathbb{R}^n$ be open.

(a) $W^m(\Omega)=\left\{f\in L^2(\Omega)\colon D^{(\alpha)}f\in L^2(\Omega)\ \text{ exists }\ \forall\,|\alpha|\le m\right\}$.

(b) $\langle f,g\rangle_{W^m}=\sum_{|\alpha|\le m}\langle D^{(\alpha)}f, D^{(\alpha)}g\rangle_{L_2}$.

(c) $H^m\left(\overline{\Omega}\right) \;=\; \overline{C^m\left(\overline{\Omega}\right)\cap W^m(\Omega)}$, where the closure is in the norm induced by $\langle\cdot,\cdot\rangle_{W^m}$.

(d) $H_0^m(\Omega)=\overline{\mathscr{D}(\Omega)}$.

These spaces are known as *Sobolev spaces*. It is clear that they are vector spaces and that $\langle\cdot,\cdot\rangle_{W^m}$ is a scalar product. By the way, for bounded domains with sufficiently smooth boundary $W^m(\Omega)=H^m(\overline{\Omega})$ (Adam [1975], p. 54). On the other hand, $C^m(\Omega)\cap W^m(\Omega)$ lies dense in $W^m(\Omega)$ for all Ω (Adam [1975], p. 52).

Theorem 5.1.13. *$W^m(\Omega)$, $H^m\left(\overline{\Omega}\right)$, and $H_0^m(\Omega)$ are Hilbert spaces.*

It is clear that $W^m(\Omega)\subset W^{m-1}(\Omega)$ and the (formal) identity operator $W^m(\Omega)\to W^{m-1}(\Omega)$ is continuous.

With the help of Sobolev theory, existence problems for solutions of (in particular, elliptic) partial differentiable equations can be handled. The following results are frequently used.

- Lemma of Sobolev. For $f\in W^m(\Omega)$ and $m>k+\frac{n}{2}$ there exists $g\in C^k(\Omega)$ with $f=g$ almost everywhere.
- Theorem of Rellich. For a bounded open set $\Omega\subset\mathbb{R}^n$, the embedding $H_0^m(\Omega)\to H_0^{m-1}(\Omega)$ is compact.

These theorems will be proved in the next section.

---

[7] Or see http://en.wikipedia.org/wiki/Multi-index_notation on Wiki.

[8] as L_2 functions, i.e. almost everywhere.

5.2 Fourier Transform and Sobolev Spaces

The goal of this section, which leads us into an excursion, is to prove the just stated theorems of Sobolov and Rellich. Towards this, the Fourier transform, a technical tool playing an essential role, is needed.

We utilize the following notation. For $x, \xi \in \mathbb{R}^n$ let

$$x\xi = \sum_{j=1}^{n} x_j \xi_j \quad , \quad x^2 = \sum_{j=1}^{n} x_j^2 \quad , \quad |x| = \left(\sum_{j=1}^{n} x_j^2 \right)^{1/2} .$$

In this section we will always consider complex valued functions.

Definition 5.2.1. For $f \in L^1(\mathbb{R}^n)$ set

$$(\mathscr{F} f)(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-ix\xi} \, dx \qquad \forall \xi \in \mathbb{R}^n . \tag{5.7}$$

The function $\mathscr{F} f$ is the *Fourier transform* of f. The map $\mathscr{F}$ is the *Fourier transform*.

Clealy $\mathscr{F} f$ is well-defined and measurable, and the map $\mathscr{F}$ is linear.

FT BASICS[9]

mgRemark: For sanity reasons, let's let

$$c_n := \frac{1}{(2\pi)^{n/2}} .$$

The normalizing constant c_n was chosen here as to have $\|f\|_{L_2} = \|\mathscr{F} f\|_{L_2}$ for sufficiently nice f. Warning: in Jones' book, $c_n = 1$.

Here, f is in $L^1(\mathbb{R}^n)$. Other notation: $\mathscr{F} f = \widehat{f}$.

FT1. If $\xi \in \mathbb{R}^n$, then $|\widehat{f}(\xi)| \leq c_n \|f\|_{L^1}$.

FT2. $\widehat{f}$ is continuous.

Better yet, $\widehat{f}$ is uniformly continuous.

Wow. If f is sufficiently small as $|x| \to \infty$, say e.g. $|\cdot| f(\cdot) \in L_1$, then $\widehat{f}$ is Lipschitz!

Key Idea. The decay of $f(x)$ as $|x| \to \infty$ is reflected in the smoothness of $\widehat{f}$.

FT12. Riemann-Lebesgue Lemma. $\lim_{|\xi| \to \infty} \widehat{f}(\xi) = 0$.

Wow. If f is Lipschitz with compact support, then $\exists C > 0$ s.t. $|\widehat{f}(\xi)| \leq \frac{C}{|\xi|}$ $\forall \xi \in \mathbb{R}^n \setminus \{0\}$.

Key Idea. The smoothness of f is reflected in the decay of $\widehat{f}$.

[9] Taken from Frank Jones's book *Lebesgue Integration on Euclidean Spaces*, of which you should read Chapter 13 Section A.

BACK TO DIRK'S BOOK

We now secure an important property of $\mathscr{F}$; the space $(C_0(\mathbb{R}^n), \|\cdot\|_\infty)$ to now come was defined in Example 1.1-3. [10]

Theorem 5.2.2. *If $f \in L_1(\mathbb{R}^n)$ then $\mathscr{F} f \in C_0(\mathbb{R}^n)$. Furthermore,*

$$\mathscr{F} : L^1(\mathbb{R}^n) \rightarrow C_0(\mathbb{R}^n)$$

is a continuous linear operator with $\|\mathscr{F}\| \leq c_n$.

From the functional analysis viewpoint, it is advantageous to restrict $\mathscr{F}$ to an appropriate subspace of $L_1(\mathbb{R}^n)$ consisting of smooth functions. For this, the Schwartz space is now defined. [11]

Definition 5.2.3. A function $f \colon \mathbb{R}^n \to \mathbb{C}$ is *rapidly decreasing* provided

$$\lim_{|x|\to\infty} x^\alpha f(x) = 0 \qquad \forall \alpha \in \mathbb{N}_0^n ; \tag{5.8}$$

here x^α is defined by $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. [12] The space

$$\mathscr{S}(\mathbb{R}^n) := \left\{ f \in C^\infty(\mathbb{R}^n) \colon D^\beta f \text{ is rapidly decreasing } \forall \beta \in \mathbb{N}_0^n \right\}$$

is called the *Schwartz space*, its elements *Schwartz functions*.

An example[13] of a Schwartz function is $\gamma(x) = e^{-x^2}$. Instead of (5.8) one can require the equivalent conditions

$$\lim_{|x|\to\infty} P(x) f(x) = 0 \qquad \forall \text{ polynomial } P \colon \mathbb{R}^n \to \mathbb{C} \tag{5.9}$$

or

$$\lim_{|x|\to\infty} |x|^m f(x) = 0 \qquad \forall m \in \mathbb{N}_0 . \tag{5.10}$$

(Clearly condition (5.8) implies condition (5.9). Also, condition (5.9) yeilds condition (5.10) since for even m one can consider polynomials $P(x) = (x_1^2 + \ldots + x_n^2)^{m/2}$. Finally, $|x^\alpha| \leq |x|^{|\alpha|}$ implies the missing implication.

A C^∞ function f is a Schwartz function if and only if

$$\sup_{x\in\mathbb{R}^n} (1 + |x|^m) \left| D^\beta f(x) \right| < \infty \qquad \forall m \in \mathbb{N}_0 , \ \beta \in \mathbb{N}_0^n . \tag{5.11}$$

[10] Functions g (defined on the locally compact set $\mathbb{R}^n$) that "vanish at infinity" (i.e. for each $\varepsilon > 0$ the set $\{t \in \mathbb{R}^n \colon |g(t)| \geq \varepsilon\}$ is compact).

[11] Let's look back at our two Key Ideas to see just what this *appropriate subspace* should look like.

[12] If you are not familar such multi-index notation, visit
http://en.wikipedia.org/wiki/Multi-index_notation on Wiki.

[13] important, useful, in every analyist's tool bag

To see this, note that (5.11) implies

$$|x|^m \left|D^\beta f(x)\right| = \frac{\left(|x|^m + |x|^{m+1}\right)\left|D^\beta f(x)\right|}{1+|x|}$$
$$\leq \frac{\left[(1+|x|^m) + \left(1+|x|^{m+1}\right)\right]\left|D^\beta f(x)\right|}{1+|x|} \leq \frac{c}{1+|x|} \overset{|x|\to\infty}{\longrightarrow} 0 .$$

When having to divide[14] by the vorfactor; one usually uses the vorfactor $1+|x|^m$, instead of the vorfactor $|x|^m$; this is so helpful it should be pointed out.

By definition, Schwartz functions and all their derivatives decay, when heading to infinity, faster than the reciprocal of any polynomial. Thus $\mathscr{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ for each $p \geq 1$ since, for $mp-(n-1) > 1$ and $f \in \mathscr{S}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |f(x)|^p \, dx \leq \int_{\mathbb{R}^n} \left(\frac{c}{1+|x|^m}\right)^p dx = c^p \omega_{n-1} \int_0^\infty \left(\frac{1}{1+r^m}\right)^p r^{n-1} \, dr < \infty ,$$

where[15] ω_{n-1} denotes the surface area of the sphere $\{x \in \mathbb{R}^n : \ |x| = 1\}$ and c is an appropriate constant. Obviously $\mathscr{S}(\mathbb{R}^n)$ is a vector space containing $\mathscr{D}(\mathbb{R}^n)$[16] so $\mathscr{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$.

The importance of the Schwartz functions is that the Fourier transforms acts as a bijection (Theorem 5.2.8) from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}(\mathbb{R}^n)$, which is not the case for $L^1(\mathbb{R}^n)$. A few necessary prelimaries are needed for the proof of this theorem. In the following the function $x \mapsto x^\alpha$ will be denoted by the symbol x^α; although this is not totally precise, it does ease the formulation of several statements.

For $f \in \mathscr{S}(\mathbb{R}^n)$, the definition immediates yields that

$$x^\alpha f \in \mathscr{S}(\mathbb{R}^n) , \qquad D^\alpha f \in \mathscr{S}(\mathbb{R}^n) \qquad \forall \alpha \in \mathbb{N}_0^n .$$

First let's examine the interaction between differentiation and the Fourier transform.

Lemma 5.2.4. *Let $f \in \mathscr{S}(\mathbb{R}^n)$ and α be a multi-index. Then* [17]

(a) $\mathscr{F} f \in C^\infty(\mathbb{R}^n)$ *and* $D^\alpha(\mathscr{F} f) = (-i)^{|\alpha|} \, \mathscr{F}(x^\alpha f)$.

(b) $\mathscr{F}(D^\alpha f) = i^{|\alpha|} \xi^\alpha \mathscr{F} f$.

mgRemark The proof uses the following two facts. The first is from the appendix of Dirk's book and follows from Lebesgues' dominated convergence theorem (cf. Jones 6.G). The second is from the homework and follows from Fubini's theorem.

---

[14] We really should not divide by zero.

[15] Let $f \in L_1(\mathbb{R}^n)$ be rotationally symmetric (i.e. $f(x)$ depends only on $|x|$) and nonnegative. So f gives rise to a function $\widetilde{f}$ on $[0,\infty)$ via $f(x) = \widetilde{f}(|x|)$. Then $\int_{\mathbb{R}^n} f(x)\,dx = \omega_{n-1} \int_0^\infty \widetilde{f}(r) r^{n-1}\,dr$ where $\omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$. Reference: Jones' book Section 9.C.

[16] Test functions from Def. 5.1.9: $\mathscr{D}(\mathbb{R}^n) = \{\varphi \in C^\infty(\mathbb{R}^n) : \operatorname{supp}(\varphi) \text{ is compact}\}$.

[17] Without the above (not totally precise) notation, these would (precisely) be written as:
$[D^\alpha(\mathscr{F} f)](\xi) = (-i)^{|\alpha|} \, [\mathscr{F}((\cdot)^\alpha f(\cdot))](\xi)$ and $[\mathscr{F}(D^\alpha f)](\xi) = i^{|\alpha|} \xi^\alpha [\mathscr{F} f](\xi)$.
In short: under the Fourier transform, differentiation corresponds to multiplication by a power of x.

Corollary A.3.3 Interchange of derivative and integral.
(Here, denote a point in $\mathbb{R}^{d+1}$ as (t,x) with $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$.)
Let $f\colon \mathbb{R}^{d+1} \to \mathbb{C}$ be continuously differentiable.
For each $t \in \mathbb{R}$, let the function $\mathbb{R}^d \ni x \mapsto f(t,x) \in \mathbb{C}$ be integrable.
Then

$$\frac{d}{dt} \int_{\mathbb{R}^d} f(t,x)\,dx \;=\; \int_{\mathbb{R}^d} \frac{\partial f}{\partial t}\,(t,x)\,dx \qquad \forall t \in \mathbb{R}$$

PROVIDED for each $t_0 \in \mathbb{R}$ there exists a neighborhood U about t_0 and an integrable function $g\colon \mathbb{R}^d \to \mathbb{R}$ such that

$$\left| \frac{\partial f}{\partial t}\,(t,x) \right| \;\le\; g(x) \qquad \forall t \in U \;,\; x \in \mathbb{R}^d \;.$$

Problem 5.6.6 b Integration by parts.
Let $f,g\colon \mathbb{R}^n \to \mathbb{C}$ be continuously differentiable with one of the functions having compact support. Then, for each $j = 1,\ldots,n$,

$$\int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}\,(x)\,g\,(x)\,dx \;=\; -\int_{\mathbb{R}^n} f\,(x)\;\frac{\partial g}{\partial x_j}\,(x)\,dx$$

and so

$$\int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}\,(x)\,\overline{g\,(x)}\,dx \;=\; -\int_{\mathbb{R}^n} f\,(x)\;\overline{\frac{\partial g}{\partial x_j}\,(x)}\,dx$$

and so in the case that f and g are also in L^2

$$\langle D_j f, g\rangle_{L^2} \;=\; -\langle f, D_j g\rangle_{L^2} \;.$$

Now back to Dirk's book.

Lemma 5.2.4 expresses the important property of the Fourier transform that the Fourier transform converts derivatives into multiplication and thus an analytic operation into an algebraic operation.[18]

Lemma 5.2.5. *If $f \in \mathscr{S}(\mathbb{R}^n)$, then $\mathscr{F} f \in \mathscr{S}(\mathbb{R}^n)$.*

We now calculate the Fourier transform of the function

$$\gamma(x) \;=\; e^{-x^2/2} \qquad \forall x \in \mathbb{R}^n \;.$$

This is (up to a constant) the density of the standard normal distribution from probability theory. Set $\gamma_a(x) = \gamma(ax)$ for $a > 0$. Well-known is

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \gamma(x)\,dx \;=\; 1 \;.$$

[18] This can be nice since multiplication is usually easier than differentation.

Lemma 5.2.6. *Let* $\gamma\colon \mathbb{R}^n \to \mathbb{R}$ *be given by* $\gamma(x) = e^{-x^2/2}$.
Fix $a > 0$ *and set* $\gamma_a(x) = \gamma(ax)$.

$$(\mathscr{F}\gamma)(\xi) \;=\; e^{-\xi^2/2} \quad , \quad (\mathscr{F}\gamma_a)(\xi) \;=\; \frac{1}{a^n}(\mathscr{F}\gamma)\left(\frac{\xi}{a}\right) \,.$$

pre-Lemma If $f, g \in \mathscr{S}(\mathbb{R}^n)$ then

$$\int_{\mathbb{R}^n} (\mathscr{F}f)(x)\, g(x)\, dx \;=\; \int_{\mathbb{R}^n} f(x)\, (\mathscr{F}g)(x)\, dx \,,$$

i.e. in other notation (jumping hats)

$$\int_{\mathbb{R}^n} \widehat{f}(x)\, g(x)\, dx \;=\; \int_{\mathbb{R}^n} f(x)\, \widehat{g}(x)\, dx \,.$$

Lemma 5.2.7. *For* $f \in \mathscr{S}(\mathbb{R}^n)$

$$(\mathscr{F}\mathscr{F}f)(x) \;=\; f(-x) \qquad x \in \mathbb{R}^n \,.$$

Theorem 5.2.8. *The Fourier transform is a bijection from* $\mathscr{S}(\mathbb{R}^n)$ *onto* $\mathscr{S}(\mathbb{R}^n)$*. Its inverse operator is given by*

$$\left(\mathscr{F}^{-1}f\right)(x) \;=\; \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\xi)\, e^{ix\xi}\, d\xi \qquad \forall x \in \mathbb{R}^n \,. \tag{5.14}$$

Furthermore[19]

$$\langle \mathscr{F}f \,,\, \mathscr{F}g \rangle_{L^2} \;=\; \langle f \,,\, g \rangle_{L^2} \qquad \forall f, g \in \mathscr{S}(\mathbb{R}^n) \,.$$

In particular

$$\|\mathscr{F}f\|_{L^2} \;=\; \|f\|_{L^2} \qquad \forall f \in \mathscr{S}(\mathbb{R}^n) \,.$$

Thus, viewing $\mathscr{S}(\mathbb{R}^n)$ as a subspace of $L^2(\mathbb{R}^n)$, the operator

$$\mathscr{F}\colon \left(\mathscr{S}(\mathbb{R}^n), \|\cdot\|_{L_2}\right) \to \left(\mathscr{S}(\mathbb{R}^n), \|\cdot\|_{L_2}\right)$$

is well-defined, linear, bijective and norm preserving: it's an isometry! By Lemma 5.1.10, $\mathscr{S}(\mathbb{R}^n)$ lies dense in $L^2(\mathbb{R}^n)$, thus the Fourier transform $\mathscr{F}$ can be extended to an isometric operator on the whole of [20] $L^2(\mathbb{R}^n)$,

$$\mathscr{F}_2\colon \left(L^2(\mathbb{R}^n), \|\cdot\|_{L_2}\right) \to \left(L_2(\mathbb{R}^n), \|\cdot\|_{L_2}\right) \;;$$

this extension, known as the *Fourier Plancherel Transformation*, shall be, for the moment, denoted by $\mathscr{F}_2$. By Theorem 5.2.8, the map $\mathscr{F}_2\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is an

[19] thanks to our chose of c_n

[20] Theorem 2.1.5. Recall you ned the target space to be a Banach space.

isometric isomorphism and satisfies the *Plancherel equality*

$$\langle \mathscr{F}_2 f, \mathscr{F}_2 g \rangle_{L^2} \;=\; \langle f, g \rangle_{L^2} \qquad \forall f, g \in L^2(\mathbb{R}^n) \;. \tag{5.15}$$

It is important to observe that the Fourier Plancherel transform cannot be given by the integral (5.7) for all $f \in L_2(\mathbb{R}^n)$ since the integral in (5.7) exists if and only if $f \in L^1(\mathbb{R}^n)$. Furthermore, $\mathscr{F}_2 f$ is an equivalence class of functions whereas (5.7) defines an actual fucntion.

Here is the connection between $\mathscr{F}$ and $\mathscr{F}_2$. Let

$$B_R := \{x \in \mathbb{R}^n : \; |x| \le R\} \;.$$

Theorem 5.2.9.

(a) *For* $f \in L^1(R^n) \cap L^2(R^n)$

$$(\mathscr{F}_2 f)(\cdot) \;=\; \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x)\, e^{-ix(\cdot)}\, dx \qquad \textit{as } L^2 \textit{ functions} \;,$$

i.e.

$$(\mathscr{F}_2 f)(\xi) \;=\; \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x)\, e^{-ix\xi}\, dx \qquad \textit{for } \xi\textit{-a.e.} \;.$$

(b) *For* $f \in L^2(R^n)$

$$\mathscr{F}_2 f(\cdot) \;=\; \lim_{R \to \infty} \frac{1}{(2\pi)^{n/2}} \int_{B_R} f(x)\, e^{-ix(\cdot)}\, dx \;,$$

where the convergence is in the L^2 *norm* $\|\cdot\|_{L^2}$.

The conclusion in (b) is also expressed as[21]

$$(\mathscr{F}_2 f)(\xi) \;=\; \underset{R \to \infty}{\mathrm{l.i.m.}} \frac{1}{(2\pi)^{n/2}} \int_{B_R} f(x)\, e^{-ix\xi}\, dx \;;$$

hereby it is understand that the convergence in *not* pointwise but rather in the square mean (i.e. in L^2). The l.i.m. stands for "limit in the mean".

As in Theorem 2.4.2[22] and so justified by Theorem 5.2.9(a), we will now regard $\mathscr{F}$ and $\mathscr{F}_2$ as the "same" operator. So now we write $\mathscr{F}$ instead of $\mathscr{F}_2$ and the Plancherel inequality becomes

$$\langle \mathscr{F} f, \mathscr{F} g \rangle_{L^2} \;=\; \langle f, g \rangle_{L^2} \qquad \forall f, g \in L^2(\mathbb{R}^n) \;. \tag{5.16}$$

---

[21] This notation is for real - it is not something Prof. Maria the Mad Moose made up.

[22] Riesz-Thorin Interpolation Theorem

Now[23] we determine the Fourier transform on $L_p(\mathbb{R}^n)$ for $1 \leq p \leq 2$.

Theorem 5.2.10 (Hausdorff Young inequality).
Let $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in \mathscr{S}(\mathbb{R}^n)$ then $\mathscr{F} f \in L^q(\mathbb{R}^n)$ and

$$\|\mathscr{F} f\|_{L^q} \leq \frac{1}{(2\pi)^{\frac{n}{p} - \frac{n}{2}}} \|f\|_{L^p} . \tag{5.17}$$

The Fourier transform

$$\mathscr{F} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$$

has an (unique) extension to a continous linear operator

$$\mathscr{F} : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$$

described by

$$(\mathscr{F} f)(\xi) = \underset{R \to \infty}{\text{l.i.m.}} \frac{1}{(2\pi)^{n/2}} \int_{B_R} f(x) e^{-ix\xi} \, dx ;$$

hereby l.i.m. stands for convergence in the $L^q(\mathbb{R}^n)$ norm.

If $f \in L^p(\mathbb{R}^n)$ with $p > 2$ then $\mathscr{F} f$ need not be representable by a function but rather is always representable by a distribution (for this see Section 8.5).

Next we prove the analog to Lemma 5.2.4 for weak derivatives (Def. 5.1.11).

Lemma 5.2.11. *Let $f \in W^m(\mathbb{R}^n)$ and $|\alpha| \leq m$. Then*

$$\mathscr{F}\left(D^{(\alpha)} f\right) = i^{|\alpha|} \xi^\alpha \mathscr{F} f .$$

Now in place are all the tools needed to prove the Sobolev-Rellich theorem. We recall that $C^k(\Omega)$ denotes the vector spcae of all k-times continuously differentiable functions from Ω to $\mathbb{C}$.

Theorem 5.2.12 (Sobolev's lemma).
Let $\Omega \in \mathbb{R}^n$ be open. Let $m, k \in \mathbb{N}_0$ with $m > k + \frac{n}{2}$. If $f \in W^m(\Omega)$ then there exists a k-times continuously differentiable function on Ω that agrees with f almost everywhere. In other words, the equivalence class $f \in W^m(\Omega)$ has a representation in $C^k(\Omega)$.

Roughly said, Sobolev's lemma implies that m-times weakly differentiable functions, with order of differentiablity at least half of the dimension of the domain space, compare to classical differentiable functions. Sobolev's lemma no longer holds for the border exponent $m = k + \frac{n}{2}$; see Problem 5.6.9.

Theorem 5.2.13 (Rellich's embedding theorem).
Let $\Omega \subset \mathbb{R}^n$ be bounded and open. Then the identity operator from $H_0^m(\Omega)$ to $H_0^{m-1}(\omega)$ is compact.

---
[23] wow, finally

Rellich's embedding theorem need not hold for $W^m(\Omega)$; a counterexample can be found in Courant/Hilbert [1968], Volume 2, p. 522. However, if there exists a continuous linear extension from $W^m(\Omega)$ to $W^m(\mathbb{R}^n)$, it easily follows from Theorem 5.2.13 that $W^m(\Omega)$ embeds compactly into $W^{m-1}(\Omega)$. Such an extension exists when Ω has smooth boundary or more generally the "uniform cone embedding" holds (Adam [1975], p. 91).

To close this section, the Sobolev spaces $W^m(\mathbb{R}^n)$ shall be described with the aid of the Fourier transform.

Theorem 5.2.14. *It holds that*

$$W^m(\mathbb{R}^n) = \left\{ f \in L^2(\mathbb{R}^n) : \left(1+|\xi|^2\right)^{m/2} \mathscr{F} f \in L^2(\mathbb{R}^n) \right\} . \tag{5.21}$$

Theorem 5.2.14 creates the possibility to define Sobolev spaces $W^m(\mathbb{R}^n)$ for arbitrary exponents m, with (5.21) serving as the definition.[24] Traditionally one denotes the real exponents by s instead of m.

FT6. Let $f \in L^1(\mathbb{R}^n)$. Fix $y \in \mathbb{R}^n$. Then

(a) $[f(\cdot + y)]\widehat{\ }(\xi) = e^{iy\xi} \widehat{f}(\xi)$

(b) $\left[e^{i(\cdot)y} f(\cdot)\right]\widehat{\ }(\xi) = \widehat{f}(\xi - y)$

for each $\xi \in R^n$. Indeed, just calculate:

$$c_n \int_{\mathbb{R}^n} f(x+y)\, e^{-ix\xi}\, dx = c_n \int_{\mathbb{R}^n} f(w)\, e^{-i(w-y)\xi}\, dw = e^{iy\xi} c_n \int_{\mathbb{R}^n} f(w)\, e^{-iw\xi}\, dw$$

$$c_n \int_{\mathbb{R}^n} e^{ixy} f(x)\, e^{-ix\xi}\, dx = c_n \int_{\mathbb{R}^n} f(x)\, e^{-ix(\xi - y)}\, dx .$$

Proof (Proof of Lemma 5.2.7). Fix $f \in \mathscr{S}(\mathbb{R}^n)$. Fix $\xi_0 \in \mathbb{R}^n$. W.T.S.: ,

$$\left(\mathscr{F}\widehat{f}\right)(\xi_0) = f(-\xi_0) . \tag{1}$$

The left-hand side of (1) is well-defined since the Fourier transform maps the Schwartz class to the Schwartz class (Lemma 5.2.5). Thus left-hand side of (1) can be expressed as a (well-defined) iterated (double) integral:

[24] Also in the case of negative exponents, $W^m(\mathbb{R}^n)$ can be defined via (5.21), however, instead of the space L^2, the space of tempered distributions (Section 8.5) is taken.

$$\begin{aligned}\left(\mathscr{F}\widehat{f}\right)(\xi_0) &= c_n \int_{\mathbb{R}^n} \widehat{f}(x)\, e^{-ix\xi_0}\, dx &(2)\\ &= c_n \int_{\mathbb{R}^n} \left[c_n \int_{\mathbb{R}^n} f(y)\, e^{-iyx}\, dy \right] e^{-ix\xi_0}\, dx \\ &= c_n^2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(y)\, e^{-iyx} e^{-ix\xi_0}\, dy\, dx\,. &(3)\end{aligned}$$

What happens to the integral in (3) if we try interchanging[25] the order of integration:

$$c_n^2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(y)\, e^{-iyx} e^{-ix\xi_0}\, dx\, dy \;=\; c_n^2 \int_{\mathbb{R}^n} f(y) \left[\int_{\mathbb{R}^n} e^{-ix(y+\xi_0)}\, dx \right] dy\,; \qquad (4)$$

the integral in (4) does not exists since $\int_{\mathbb{R}^n}^* |e^{-ix(y+\xi_0)}|\, dx = \infty$ for all $y, \xi_0 \in \mathbb{R}^n$. Indeed, the assumption of the Fubini-Tonelli theorem does not hold here since

$$\int_{\mathbb{R}^n}^* \int_{\mathbb{R}^n}^* \left| f(y)\, e^{-iyx} e^{-ix\xi_0} \right| dy\, dx \;=\; \int_{\mathbb{R}^n}^* \int_{\mathbb{R}^n}^* |f(y)|\, dy\, dx \;=\; \|f\|_{L^1} \int_{\mathbb{R}^n}^* 1\, dx \;=\; \infty\,.$$

To overcome being Fubini-Tonelli-less, we use a *convergence generator*; namely, we will insert into the integrand in (2) the function γ_a for $a > 0$ where

$$\gamma_a(x) := \gamma(ax) \qquad \text{where} \qquad \gamma(x) := e^{-x^2/2}$$

and then calculate. (Where do we use Fubini-Tonelli in the below calculation?)

$$\begin{aligned} c_n \int_{\mathbb{R}^n} \widehat{f}(x)\, e^{-ix\xi_0}\, \gamma_a(x)\, dx &\overset{\text{pre-lemma}}{=} c_n \int_{\mathbb{R}^n} f(x) \left[e^{-i(\cdot)\xi_0}\, \gamma_a(\cdot) \right]^{\widehat{}}(x)\, dx \\ &\overset{\text{FT6b}}{=} c_n \int_{\mathbb{R}^n} f(x)\, \widehat{\gamma}_a(x+\xi_0)\, dx \\ &\overset{\text{Lemma 5.2.6}}{=} c_n \int_{\mathbb{R}^n} f(x)\, \frac{1}{a^n}\, \widehat{\gamma}\left(\frac{x+\xi_0}{a} \right) dx \\ &\overset{au=x+\xi_0}{=} c_n \int_{\mathbb{R}^n} f(au-\xi_0)\, \widehat{\gamma}(u)\, du\,. \end{aligned}$$

Next we want to apply the LDCT to the first and last integral in the above calculation so note that

$$\lim_{a\to 0^+} \widehat{f}(x)\, e^{-ix\xi_0} \gamma_a(x) = \widehat{f}(x)\, e^{-ix\xi_0} \quad \text{and} \quad \left| \widehat{f}(x)\, e^{-ix\xi_0} \gamma_a(x) \right| \le \left| \widehat{f}(x) \right| \quad \text{and} \quad \widehat{f} \in L^1(\mathbb{R}^n)$$

$$\lim_{a\to 0^+} f(au-\xi_0)\, \widehat{\gamma}(u) = f(-\xi_0)\, \widehat{\gamma}(u) \;\; \text{and} \;\; |f(au-\xi_0)\, \widehat{\gamma}(u)| \le \|f\|_\infty\, \widehat{\gamma}(u) \;\; \text{and} \;\; \widehat{\gamma} = \gamma \in L^1(\mathbb{R}^n)\,.$$

Thus we are done since

$$\lim_{a\to 0^+} c_n \int_{\mathbb{R}^n} \widehat{f}(x)\, e^{-ix\xi_0}\, \gamma_a(x)\, dx \;=\; c_n \int_{\mathbb{R}^n} \widehat{f}(x)\, e^{-ix\xi_0}\, dx \;=\; \left(\mathscr{F}\widehat{f}\right)(\xi_0)$$

$$\lim_{a\to 0^+} c_n \int_{\mathbb{R}^n} f(au-\xi_0)\, \widehat{\gamma}(u)\, du \;=\; c_n \int_{\mathbb{R}^n} f(-\xi_0)\, \widehat{\gamma}(u)\, du \;=\; f(-\xi_0)\,. \quad \square$$

---

[25] without justification

5.3 Orthogonality

We now return to analysing general Hilbert spaces. With help of the concept of scalar products, the elementary geometric concept of orthogonality can be abstractly formulated.

Definition 5.3.1. Let X be a pre-Hilbert space. Two vectors $x, y \in X$ are called *orthogonal*, denoted $x \perp y$, provided $\langle x, y\rangle = 0$. Two subsets $A, B \subset X$ are called orthogonal, denoted $A \perp B$, provided $x \perp y$ for each $x \in A$ and $y \in B$. The set

$$A^{\perp} := \{y \in X : x \perp y \quad \forall x \in A\}$$

is called the *orthogonal complement* of A.

The following properties follow directly from the definitions.

- (Pythagoras' Theorem)

$$x \perp y \;\; \Rightarrow \;\; \|x\|^2 + \|y\|^2 = \|x+y\|^2 \; . \tag{5.22}$$

- $A^{\perp}$ is always a closed subspace of X.
- $A \subset \left(A^{\perp}\right)^{\perp}$.
- $A^{\perp} = \left(\overline{\operatorname{lin}} A\right)^{\perp}$.

It will follow from Theorem 5.3.6 that the notation $A^{\perp}$ from Definition 5.3.1 is consist with that from (3.4).

The next theorem is central for Hilbert space theory; completeness is essential.

Theorem 5.3.2 (Projection Theorem).
Let K be a closed convex subset of a <u>Hilbert</u> space H. Let $x_0 \in H$.
Then there exists precisely one $x \in K$ satisfying

$$\|x - x_0\| \; = \; \inf_{y \in K} \; \|y - x_0\| \; .$$

Through Theorem 5.3.2, a (in general nonlinear) map $P\colon H \to K$ can be specified via $P(x_0) = x$. This map can be characterized in the following way.

Lemma 5.3.3. *Let K be a closed convex subset of a Hilbert space H. Let $x_0 \in H$. Then for $x \in K$, T.F.A.E. .*

(i) $\|x_0 - x\| \; = \; \inf_{y \in K} \; \|x_0 - y\|$.

(ii) $\mathit{Re}\, \langle x_0 - x, y - x\rangle \; \le 0 \quad \forall y \in K$.

Geometrically (ii) says that the angle between $x_0 - x$ and $y - x$ is always obtuse (law of cosines![26]). Of upmost importance is the case where K is a closed subspace of X. A projection is, as explained in Section 4.6, a map P such that $P^2 = P$.

[26] $\langle \vec{u}, \vec{v}\rangle = \left\|\vec{u}\right\| \left\|\vec{v}\right\| \cos\theta$

Theorem 5.3.4 (Orthogonal Projection).
Let $U \neq \{0\}$ be a closed subspace of a Hilbert space H.

(1) $\exists$ *a linear projection P_U from H onto U with $\|P_U\| = 1$ and $\ker(P_U) = U^\perp$.*
(2) *$Id - P_U$ is a projection from H onto $U^\perp$ with $\|Id - P_U\| = 1$ (if $U \neq H$).*
(3) *$H = U \oplus_2 U^\perp$.*

P_U is known as the orthogonal projection *of X onto U.*

Corollary 5.3.5. *For a subspace U of a Hilbert space H*

$$\overline{U} \; = \; \left(U^\perp\right)^\perp \; .$$

mgRmk. Let's *pre*-think about the IMPORTANT Fréchet-Riesz Representation Theorem 5.3.6.
Fix $y \in Y$. Define

$$\varphi_y \colon H \to \mathbb{K} \qquad \text{via} \qquad \varphi_y(\cdot) := \langle \cdot, y \rangle \; .$$

In other notation,

$$H \ni x \stackrel{\varphi_y}{\longrightarrow} \langle x \, , \, y \rangle \in \mathbb{K} \; .$$

Note that φ_y is linear since the scalar product is linear in the *first* variable; indeed,

$$\begin{aligned} \varphi_y(x_1 + x_2) &= \langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle = \varphi_y(x_1) + \varphi_y(x_2) \quad && \forall x_1, x_2 \in H \\ \varphi_y(\lambda x) &= \langle \lambda x, y \rangle = \lambda \langle x, y \rangle = \lambda \varphi_y(x) && \forall\, x \in H \;\; \forall \lambda \in \mathbb{K} \, . \end{aligned}$$

Also $\varphi_y \in H'$ and $\|\varphi_y\|_{H'} \; = \; \|y\|_H$ since

$$\begin{aligned} |\varphi_y(x)| &= |\langle x, y \rangle| \stackrel{\text{C-S}}{\leq} \|y\| \; \|x\| \qquad \forall x \in H \\ |\varphi_y(y)| &= |\langle y, y \rangle| \; = \; \|y\| \; \|y\| \; . \end{aligned}$$

So we now can define a (well-defined, norm perserving [27]) map

$$\Phi \colon H \to H' \qquad \text{via} \qquad \Phi(y) := \varphi_y \; .$$

In other notations,

$$H \ni y \stackrel{\Phi}{\longrightarrow} \varphi_y \in H'$$
$$H \ni y \stackrel{\Phi}{\longrightarrow} \langle \cdot \, , \, y \rangle \in H'$$
$$[\Phi(y)](x) \; = \; \langle x, y \rangle \quad \forall\, x, y \in H \; .$$

Φ is *conjugate linear*[28] since the scalar product is conjugate linear in the *second* variable; indeed,

$$\begin{aligned} [\Phi(y_1 + y_2)](\cdot) &= \varphi_{y_1+y_2}(\cdot) = \langle \cdot, y_1 + y_2 \rangle = \langle \cdot, y_1 \rangle + \langle \cdot, y_2 \rangle = \varphi_{y_1}(\cdot) + \varphi_{y_2}(\cdot) \\ &= [\Phi(y_1)](\cdot) + [\Phi(y_2)](\cdot) = [\Phi(y_1) + \Phi(y_2)](\cdot) \\ [\Phi(\lambda y)](\cdot) &= \varphi_{\lambda y}(\cdot) = \langle \cdot \, , \, \lambda y \rangle = \overline{\lambda} \langle \cdot \, , \, y \rangle = \overline{\lambda} [\Phi(y)(\cdot)] = \left[\overline{\lambda} \Phi(y)\right](\cdot) \; . \end{aligned}$$

We have just talked through the proof of the whole of Theorem 5.3.6, *except* for the surjectivity of Φ, which follows from Theorem 5.3.4.

[27] A norm perserving map between normed linear spaces has kernel $\{0\}$ and thus is injective.
[28] i.e. $\Phi(y_1 + y_2) = \Phi(y_1) + \Phi(y_2)$ and $\Phi(\lambda y) = \overline{\lambda} \Phi(y)$ for all $y_1, y_2, y \in H$ and $\lambda \in \mathbb{K}$

Important Thm. 5.3.6 is proved with help of Orthogonal Projection Thm 5.3.4.

Theorem 5.3.6 (Fréchet-Riesz Representation Theorem).
Let H be a Hilbert space. Then the map

$$\Phi : H \;\to\; H' \qquad \textit{given by} \quad y \;\mapsto\; \langle \cdot\,,\, y\rangle$$

is bijective, isometric, and conjugate linear (i.e. Φ *is additive and* $\Phi(\lambda y) \;=\; \overline{\lambda}\,\Phi(y)$)*.*
In other words, for each $x' \in H'$ there exists exactly one $y \in H$ such that

$$x'(x) \;=\; \langle x, y\rangle \qquad \forall x \in H\,; \quad \textit{furthermore} \quad \left\|x'\right\|_{H'} \;=\; \|y\|_H \;.$$

For clarity, one can denote Φ by Φ_H and $\langle\cdot,\cdot\rangle$ by $\langle\cdot,\cdot\rangle_H$. So we have, $\forall x, y \in H$ and $x' \in H'$

$$[\Phi_H(y)](x) \;=\; \langle x, y\rangle_H \qquad \textit{and} \qquad x'(x) \;=\; \langle x, \Phi_H^{-1}(x')\rangle_H \;.$$

mgRmk Let's *post*-think about the IMPORTANT Fréchet-Riesz Representation Theorem 5.3.6.
Let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space. Thus $(H, \|\cdot\|_H)$ is a Banach space with $\|\cdot\|_H = \sqrt{\langle\cdot,\cdot\rangle_H}$.
The dual space $(H', \|\cdot\|_{H'})$ of $(H, \|\cdot\|_H)$, where

$$H' := \{f : H \to \mathbb{K} \mid f \text{ is linear and continuous}\}$$
$$\text{the dual space norm of } f \;:=\; \|f\|_{H'} \;:=\; \sup\{|f(x)| : x \in B(H)\}\;,$$

is also a Banach space. But is $(H', \|\cdot\|_{H'})$ a Hilbert space, i.e. is the dual norm on H' given by some scalar product via $\|\cdot\|_{H'} = \sqrt{\langle\cdot,\cdot\rangle}$? Towards this, with the help of Theorem 5.3.6, let's define a map

$$\langle\cdot,\cdot\rangle_{H'} : H' \times H' \to \mathbb{K} \qquad \text{via} \qquad \langle \Phi(x), \Phi(y)\rangle_{H'} := \langle y, x\rangle_H \;.$$

One quickly verifies $\langle\cdot,\cdot\rangle_{H'}$ is indeed a scalar product; for fun let's verify Def. 5.1.1(b) together[29]

$$\langle \lambda\Phi(x), \Phi(y)\rangle_{H'} \;=\; \langle \Phi\left(\overline{\lambda}x\right), \Phi(y)\rangle_{H'} \;=\; \langle y, \overline{\lambda}x\rangle_H \;=\; \lambda\langle y, x\rangle_H \;=\; \lambda\langle \Phi(x), \Phi(y)\rangle_{H'} \;.$$

Next let's investigate the norm on H' produced by this scalar product:

$$\sqrt{\langle \Phi(x), \Phi(x)\rangle_{H'}} \;=\; \sqrt{\langle x, x\rangle_H} \;=\; \|x\|_H \;=\; \|\Phi(x)\|_{H'} \;.$$

Wunderbar! $(H', \|\cdot\|_{H'})$ is indeed a Hilbert space with the above defined scalar product $\langle\cdot,\cdot\rangle_{H'}$ giving the dual norm $\|\cdot\|_{H'}$.
We are on a roll! So now, by the Fréchet-Riesz Representation Thm. 5.3.6, each of the maps

$$\begin{array}{lll} \Phi_H : H \;\to\; H' & \text{given by} & [\Phi_H(y)](x) := \langle x\,,\, y\rangle_H \\ \Phi_{H'} : H' \;\to\; H'' & \text{given by} & [\Phi_{H'}(g)](f) := \langle f\,,\, g\rangle_{H'} \end{array}$$

is bijective, norm-preserving, and conjugate linear. Thus their composite $\Phi_{H'} \circ \Phi_H$,

$$\Phi_{H'} \circ \Phi_H \;\; : \;\; H \xrightarrow{\Phi_H} H' \xrightarrow{\Phi_{H'}} H''\;,$$

is a bijective, norm-preserving, linear operator from H onto H''. Let's calculate for $x, y \in H$:

$$\begin{aligned} [(\Phi_{H'} \circ \Phi_H)(x)](\Phi_H(y)) \;&=\; [\Phi_{H'}(\Phi_H(x))](\Phi_H(y)) \\ &=\; \langle \Phi_H(y), \Phi_H(x)\rangle_{H'} \;=\; \langle x, y\rangle_H \;=\; [\Phi_H(y)](x)\;. \end{aligned}$$

Thus $\Phi_{H'} \circ \Phi_H : H \to H''$ is the canonical isometric embedding (cf. Section 3.3) of H into H''.

---

[29] This calculation shows why one defines $\langle \Phi(x), \Phi(y)\rangle_{H'}$ as $\langle y, x\rangle_H$ instead of $\langle x, y\rangle_H$.

We have basically talked through the proof of the next corollary.

Weakly convergent sequences and reflexive Banach spaces were introduced in Definitions 3.3.6 and 3.3.3.

Corollary 5.3.7. *Let H be a Hilbert space.*

(a) A sequence (x_n) in H converges weakly to $x \in H$ if and only if

$$\langle x_n - x, y \rangle \ \to \ 0 \quad \forall y \in H \ .$$

(b) H is reflexive.

(c) Each bounded sequence in H contains a weakly convergent subsequence.

Sharper than (c) is the following theorem.

Theorem 5.3.8 (Banach-Saks).
Let (x_n) be a bounded sequence in a Hilbert space. Then there is a weakly convergent subsequence (y_n) of (x_n) whose sequence of arithmetic means

$$\left(\frac{1}{n} \sum_{k=1}^{n} y_k \right)_{n=1}^{\infty}$$

converges with repect to the norm.

mgRmk Let's clear up some notation that might have seemed abusive at first.
Def. 5.3.1 defines the *orthogonal complement* $A^\perp$ of a subset A of a pre-Hilbert space H by

$$A^\perp := \{y \in H : x \perp y = 0 \ \forall x \in A\} \overset{\text{note}}{\subset} H \ .$$

In (3.4) of §3.1, the *annihilator* $U^{\perp \ (\text{in } X')}$ of a subset U of a normed linear space X is defined by

$$U^\perp := \left\{x' \in X' : x'(x) = 0 \ \forall x \in U\right\} \overset{\text{note}}{\subset} X' \ .$$

If H is a pre-Hilbert space, then H is also a normed linear space. These two usages of $^\perp$ are consistent thanks to Theorem 5.3.6; indeed, consider the bijective map $\Phi_H : H \to H'$ given in Thm. 5.3.6 and compute.

$$\begin{aligned} A^\perp := \ & \{y \in H : x \perp y = 0 \ \forall x \in A\} \\ = \ & \{y \in H : \langle x, y \rangle_H = 0 \ \forall x \in A\} \\ = \ & \left\{\Phi_H^{-1} x' \in H : \langle x, \Phi_H^{-1} x' \rangle_H = 0 \ \forall x \in A\right\} \\ = \ & \left\{\Phi_H^{-1} x' \in H : x'(x) = 0 \ \forall x \in A\right\} \\ = \ & \Phi_H^{-1}\left(\left\{x' \in H' : x'(x) = 0 \ \forall x \in A\right\}\right) \ . \end{aligned}$$

5.3 Section 5.3.0: More on Orthogonal Projections

GOAL: Take a closer look at the orthogonal projection given in Thm. 5.3.4.

Theorem 5.3.4 : Revisited. Let $\{0\} \neq U$ be a closed subspace of a Hilbert space H. Define[30] the *orthogonal projection of H onto U*

$$P_U : H \to H$$

via: for $x \in H$

$$\begin{array}{c} P_U(x) \text{ is the unique element in } H \text{ satisfying} \\ P_U(x) \in U \qquad \text{and} \qquad \|x - P_U(x)\| = d(x, U) \ . \end{array} \tag{dP_U}$$

Theorem 5.3.2 gave that P_U is a well-defined map. The proof of Theorem 5.3.4, with the help of Lemma 5.3.3 and some clever (but standard) calculations, gave the following equivalent formulation of the map P_U. For $x \in H$

$$\begin{array}{c} P_U(x) \text{ is the unique element in } H \text{ satisfying} \\ P_U(x) \in U \qquad \text{and} \qquad x - P_U(x) \in U^{\perp} \ . \end{array} \tag{ocP_U}$$

The proof of Theorem 5.3.4 then proceeded to show the following.[31]

(P1)	$(P_U)^2 = P_U$	and so	$(\mathrm{Id}_H - P_U)^2 = \mathrm{Id}_H - P_U$	by (dP_U)
(P2)	P_U is linear	and so	$\mathrm{Id}_H - P_U$ is linear	by (ocP_U)
(P3)	$\operatorname{ran} P_U = U$	and so	$\ker(\mathrm{Id}_H - P_U) = U$	by (dP_U)
(P4)	$\ker P_U = U^{\perp}$	and so	$\operatorname{ran}(\mathrm{Id}_H - P_U) = U^{\perp}$	by (ocP_U)

For $x \in H$, by (P3), (P4) and Pythagoras's Thm (5.22),

$$\begin{array}{c} x = (P_U x) + (\mathrm{Id}_H - P_U)x \\ P_U x \in U \quad \text{and} \quad (\mathrm{Id}_H - P_U)x \in U^{\perp} \\ \|x\|^2 = \|P_U x\|^2 + \|(\mathrm{Id}_H - P_U)x\|^2 \end{array} \tag{P5}$$

Next, (P5) gave more.

(P6)	$\|P_U\| = 1$	(used $U \neq \{0\}$ and Lemma 4.6.1a)	by (P5)
(P7)	$\|\mathrm{Id}_H - P_U\| = 1$	(provided $U \neq H$, used Lemma 4.6.1a)	by (P5)
(P8)	$H = U \oplus_2 U^{\perp}$	i.e. H and $U \oplus_2 U^{\perp}$ are isometric, via	

$$\begin{array}{c} D_U : H \to U \oplus_2 U^{\perp} \quad \text{is an isometry where} \\ D_U(x) := (\, P_U(x) \, , \, (\mathrm{Id}_H - P_U)x \,) \ . \end{array} \tag{P8}$$

□ □

[30] See Definition 5.3.11.

[31] See also the Recall on page 71 of the class notes.

NIFFTY STUFF, let's think about what we have just shown.

Clearly H is the algebraic direct sum of the closed subspaces U and $U^{\perp}$; thus, for each $1 \le p \le \infty$, by Thm. 4.6.3 (OMT), H and $U \oplus_p U^{\perp}$ are isomorphic (as Banach spaces) and so there exists constants $m_p, M_p > 0$ such that

$$m_p \|x\|_H \;\le\; \left[\, \|P_U x\|_H^p + \|(\mathrm{Id}_H - P_U)\, x\|_H^p \,\right]^{1/p} \;\le\; M_p \|x\|_H$$

for each $x \in X$.[32] Unfortunately the OMT provides no other information about m_p and M_p, even for the (obvious) choice here in Hilbert-space-land of $p = 2$. However, (P5) says $m_2 = 1 = M_2$; thus, H and $U \oplus_2 U^{\perp}$ are isometrically isomorphic.

In other words, the *orthogonal projection of X onto U*,

$$P_U \,: H \twoheadrightarrow U \subset H \;,$$

produces an *orthogonal*[33] *decomposition* of H

$$H \;=\; U \oplus_2 U^{\perp}$$
$$\|x\|_H \;=\; \left[\, \|P_U x\|_H^2 + \|(\mathrm{Id}_H - P_U)\, x\|_H^2 \,\right]^{1/2} \qquad \forall\, x \in H \;.$$

This orthogonal decomposition is so useful that it is nice to be able to recognize when a projection is an orthogonal projection. Towards this, a lemma is helpful. It's really more an observation but it is used often so let's explicitly write it down.

Lemma 5.3.9. *If $z \in \mathbb{C}$ and there exists $M \in \mathbb{R}$ such that*

$$Re\;\lambda z \;\le\; M \qquad \forall \lambda \in \mathbb{C}$$

then $z = 0$.

Proof. If z were not zero, then $\lambda = \frac{M+17}{z}$ would provide a contradiction. □

Theorem 5.3.10. *Let H be a Hilbert space.*
Let $P \in L(H,H)$ satisfy $P^2 = P$ and $\operatorname{ran} P := U \neq \{0\}$.[34] *Then T.F.A.E. .*

(1) $P = P_U$ *for* P_U *as defined in* (dP_U).
(2) $P = P_U$ *for* P_U *as defined in* (ocP_U).
(3) $(\operatorname{ran} P)^{\perp} \;=\; \ker P$.
(4) $\operatorname{ran} P \;\perp\; \ker P$.
(5) $\|P\| = 1$.

[32] with obvious modifications when $p = \infty$
[33] in the sense that $U \perp U^{\perp}$
[34] Note U is closed since P is a projection and so $\operatorname{ran} P \;=\; \ker(\mathrm{Id}_H - P)$. See the Recall on page 71.

Proof. The proof of Theorem 5.3.4 shows that (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3).

Clearly (3) $\Rightarrow$ (4) since (4) is equivalent to $\ker P \subset (\operatorname{ran} P)^{\perp}$.

Now let (4) hold. Note that $x - Px \in \ker P$ since $P^2 = P$. So $\forall\, x \in H$

$$\|Px\|^2 \leq \|Px\|^2 + \|x - Px\|^2 \overset{\text{(V.22)}}{=} \|Px + (x - Px)\|^2 \;=\; \|x\|^2 \;.$$

So $\|P\| \leq 1$. Since P is a nonzero projection, $\|P\| \geq 1$ by Lemma 4.6.1a. Thus (5) holds.

Now assume (5). Fix $x \in H$. To show that (2) holds, it is enough to show that $x - Px \in U^{\perp}$. So fix $u \in U$. Then $\forall \lambda \in \mathbb{K}$

$$\begin{aligned}
\|\lambda u\|^2 \;&=\; \|P\left((x - Px) + \lambda u\right)\|^2 && \text{(since } P^2 = P \text{ and } u \in \operatorname{ran} P\text{)}\\
&\leq\; \|(x - Px) + \lambda u\|^2 && \text{(since } \|P\| = 1\text{)}\\
&=\; \langle (x - Px) + \lambda u \;,\; (x - Px) + \lambda u \rangle \\
&= \|x - Px\|^2 + \|\lambda u\|^2 + 2\operatorname{Re} \overline{\lambda} \langle x - Px, u \rangle \;.
\end{aligned}$$

Thus $\forall \gamma \in \mathbb{K}$

$$\operatorname{Re} \gamma \langle x - Px, u \rangle \;\leq\; \|x - Px\|^2 \;.$$

So $\langle x - Px, u \rangle = 0$ by Lemma 5.3.9. Thus (2) holds. $\square$

Compare Theorem 5.3.10 to the to-come Theorem 5.5.9. We just used Lemma 5.3.9 to show that (5) implies (2) in Theorem 5.3.10 in the same way that Lemma 5.3.9 was implicitly used to show that (ii) implies (i) in Theorem 5.5.9.

Definition 5.3.11. Let H be a Hilbert space.
Let $P \in L(H, H)$ satisfy $P^2 = P$ and $\operatorname{ran} P := U \neq \{0\}$. If P satisifies one (and thus all) of the conditions (1)–(5) in Theorem 5.3.10, then P is called *the orthogonal projection of H onto U*.

5.4 Section 5.4.0: Series in Banach Spaces

AN EXCURSION

GOAL: to extend, from the scalars to normed spaces, the theory of (infinite) series.

Towards our goal, let's recall what we already know.

Basics 5.4.0.1. Consider a (formal) series $\sum_{n=1}^{\infty} x_n$ in a normed space X.

(1) The corresponding sequence $(s_n)_{n=1}^{\infty}$ of *partials sums* is given by $s_n := \sum_{j=1}^{n} x_j$.

(2) $\sum_{n=1}^{\infty} x_n$ *converges* IFF $\exists x \in X$ such that $\lim_{N\to\infty} \left\|x - \sum_{n=1}^{N} x_n\right\|_X = 0$.
In this case, we say $\sum_{n=1}^{\infty} x_n$ converges to x and write $\sum_{n=1}^{\infty} x_n = x$.
In other words, the series $\sum_{n=1}^{\infty} x_n$ converges (to x) IFF the sequence $(s_n)_{n=1}^{\infty}$ converges (to x).

(3) $(s_n)_{n=1}^{\infty}$ is Cauchy $\iff$ $\forall \varepsilon > 0\ \exists N \in N$ s.t. if $N < n < m$ then $\left\|\sum_{j=n}^{m} x_j\right\|_X < \varepsilon$.

(4) If $\sum_{n=1}^{\infty} x_n$ converges, then $(s_n)_{n=1}^{\infty}$ is Cauchy.

(5) If $(s_n)_{n=1}^{\infty}$ is Cauchy and X is complete, then $\sum_{n=1}^{\infty} x_n$ converges.

(6) $\sum_{n=1}^{\infty} x_n$ *converges absolutely* IFF $\sum_{n=1}^{\infty} \|x_n\|_X$ converges.

Lemma 1.1.8.$^+$ For a normed space X, T.F.A.E. .

(i) X is complete.

(ii) If $\sum_{n=1}^{\infty} x_n$ in X has a Cauchy partial sum sequence, then $\sum_{n=1}^{\infty} x_n$ converges (to a point in X).

(iii) If $\sum_{n=1}^{\infty} x_n$ in X is absolutely convergent, then $\sum_{n=1}^{\infty} x_n$ converges (to a point in X).

In a theory of series, one would want conditions (ii) and (iii) in Lemma 1.1.8$^+$. So we restrict our attention to normed spaced that are complete (a.k.a. Banach spaces). This is not much of a restiction since each normed space X can be isometrically embedding into a Banach space (namely, it's bidual X''). We now have the following (easy but very useful in proving facts) criterion.

Theorem 5.4.0.2. (Cauchy's criterion for convergence of series.)
Consider a (formal) series $\sum_{n=1}^{\infty} x_n$ in a Banach space. Then $\sum_{n=1}^{\infty} x_n$ converges IFF

$$\forall \varepsilon > 0 \quad \exists N \in N \quad \text{s.t.} \quad \text{if} \quad N < n < m \quad \text{then} \quad \left\|\sum_{j=n}^{m} x_j\right\|_X < \varepsilon . \tag{CC}$$

Next let's think about *unconditional convergence*, which could be plausibly defined by any of the below conditions.

Definition 5.4.0.3. A (formal) series $\sum_{n=1}^{\infty} x_n$ in a normed linear space X is:

(1) *unordered convergent* $\Leftrightarrow \sum_{n=1}^{\infty} x_{\pi(n)}$ converges for each permutation[35] $\pi\colon \mathbb{N} \to \mathbb{N}$

(2) *subseries convergent* $\Leftrightarrow \sum_{k=1}^{\infty} x_{n_k}$ converges for each subsequence $\left(x_{n_k}\right)_k$ of $(x_n)_n$

(3) *randomized convergent* $\Leftrightarrow \sum_{n=1}^{\infty} \varepsilon_n x_n$ converges for each choice $(\varepsilon_n)_n$ of signs $\varepsilon_n = \pm 1$

(4) *bounded multiplier convergent* $\Leftrightarrow \sum_{n=1}^{\infty} a_n x_n$ converges for each choice $(a_n)_n \subset B(\mathbb{K})$.[36]

Which of the above four conditions would you like to choose as the definition of unconditional convergent? Wonderful - you just made the proper choice! (See the next theorem).

[35] i.e. bijective map

[36] $B(\mathbb{K}) = \{a \in \mathbb{K}\colon\ |a| \le 1\}$.

Theorem 5.4.0.4 For a (formal) series $\sum_{n=1}^{\infty} x_n$ in a Banach space X, T.F.A.E. .

(1) $\sum_{n=1}^{\infty} x_{\pi(n)}$ converges for each permutation π of $\mathbb{N}$.

(2) $\sum_{k=1}^{\infty} x_{n_k}$ converges for each subsequence $\left(x_{n_k}\right)_k$ of $(x_n)_n$.

(3) $\sum_{n=1}^{\infty} \varepsilon_n x_n$ converges for each choice $(\varepsilon_n)_n$ of signs $\varepsilon_n = \pm 1$.

(4) $\sum_{n=1}^{\infty} a_n x_n$ converges for each sequence $(a_n)_n$ with $a_n \in \mathbb{K}$ and $|a_n| \leq 1$.

(5) $\sum_{n=1}^{\infty} x_n$ satisfies the following Cauchy criterion for unconditional convergence of series:

$$\forall \varepsilon > 0 \; \exists N \in N \text{ s.t. } \quad \text{if } A \subset \mathbb{N} \cap (N, \infty) \text{ is finite} \quad \text{then } \left\| \sum_{j \in A} x_j \right\|_x < \varepsilon \; . \qquad \text{(CCun)}$$

Furthermore, if $\sum_{n=1}^{\infty} x_n$ satifies condition (2) and $\sum_{n=1}^{\infty} x_n$ converges to $x \in X$, then each rearrangement $\sum_{n=1}^{\infty} x_{\pi(n)}$ of $\sum_{n=1}^{\infty} x_n$ also converges to x.

A proof of Theorem 5.4.0.4 is outlined in Problems 5.6.40–5.6.43.

Definition 5.4.0.5. A (formal) series $\sum_{n=1}^{\infty} x_n$ in a Banach space X is *unconditionally convergent* provided it satisfies one, and hence all, of the conditions in Theorem 5.4.0.4. In this case, if $\sum_{n=1}^{\infty} x_n = x$, we say that $\sum_{n=1}^{\infty} x_n$ *converges unconditionally to* x and write $\sum_{n \in \mathbb{N}} x_n = x$.

In any Banach space X,

$$\text{absolute convergence} \quad \overset{(1)}{\Longrightarrow} \quad \text{unconditional convergence} \quad \overset{(2)}{\Longrightarrow} \quad \text{convergence} \; .$$

An absolutely convergent series clearly satisisfies condition (5) of Theorem 5.4.0.4 so $\overset{(1)}{\Longrightarrow}$ holds. Clearly $\overset{(2)}{\Longrightarrow}$ holds. If $0 \neq x \in X$, then $\sum_{n=1}^{\infty} \frac{(-1)^n x}{n}$ shows that $\overset{(2)}{\Longrightarrow}$ is not reversible. Implication $\overset{(1)}{\Longrightarrow}$ is reversible if $X = \mathbb{K}$ (see Problem 5.6.43 (0)) and thus also if X is finite dimensional (by Theorem 1.2.5). A deep result of Dvoretzky and Rogers in 1940 shows that each infinite dimensional Banach space contains an unconditionally convergent series that fails to be absolutely convergent.

So we now have a through understanding of unconditional convergence of a series $\sum_{n=1}^{\infty} x_n$. We can write $\sum_{n \in \mathbb{N}} x_n$ instead of $\sum_{n=1}^{\infty} x_n$ for unconditionally convergent series since the order of summation is irrevalent. In Section 5.4, we will need to work with series of the form $\sum_{i \in I} x_i$ for an arbitrary index set I (which might be uncountable and so not enumerable). Thus the only convergence that would makes sense is unconditional convergence.

Enough of mg. Onto Section 5.4!

BACK TO DIRK'S BOOK

5.4 Orthonormal Bases

In this section, H is always a Hilbert space. (Some of the following statements, generally those of lesser importance, also hold for pre-Hilbert spaces.)

Definition 5.4.1. A subset $S \subset H$ is an *orthonormal system* provided

$$\|e\| = 1\ {}^{37} \qquad \text{and} \qquad \langle e, f\rangle = 0\ {}^{38} \qquad \forall\, e, f \in S\,,\ e \neq f\ .$$

An orthonormal system S is an *orthonormal basis* provided

$$S \subset T \ \text{ and } \ T \text{ orthonormal system} \quad \Longrightarrow \quad T = S\ .$$

An orthonormal basis is also known as a *complete orthonormal system*.

Why an orthonormal basis is called a basis will be clarified in Theorem 5.4.9. (It is already stressed now that an orthonormal basis is not a vector space basis.)

Let shorten **orthonormal** to **O.N.**

Example (a) In $H = \ell_2$ the (standard) unit vectors $S = \{e_n : n \in \mathbb{N}\}$ form an O.N. system.
Example (b) Let $H = L^2\,[0, 2\pi]$ and

$$S = \left\{\tfrac{1}{\sqrt{2\pi}}\mathbf{1}\right\} \cup \left\{\tfrac{1}{\sqrt{\pi}}\cos n(\cdot) : n \in \mathbb{N}\right\} \cup \left\{\tfrac{1}{\sqrt{\pi}}\sin n(\cdot) : n \in \mathbb{N}\right\}\ .$$

Then S is an orthonormal system, as easily shown through integration by parts.
Example (c) In $H = L^2_{\mathbb{C}}\,[0, 2\pi]$

$$S \ = \ \left\{\tfrac{1}{\sqrt{2\pi}} e^{in(\cdot)} : n \in \mathbb{Z}\right\}$$

is an orthonormal system.
Example (d) In $H = AP^2\,(\mathbb{R})$ (Example 5.1(f)), an orthonormal system is $S = \{f_\lambda : \lambda \in \mathbb{R}\}$.

All of these above O.N. systems are also O.N. bases, as will soon be shown.

Theorem 5.4.2 (Gram-Schmidt Process). *Here,* $\operatorname{card} E$ *is the cardinality of a set* E.

(a) *Let $L \subset H$ be linearly independent and atmost countable.*
Then there is an O.N. system S s.t. $\operatorname{card} S = \operatorname{card} L$ *and* $\operatorname{lin} S = \operatorname{lin} L$.

(b) *Let $D \subset H$ be atmost countable.*
Then there is an O.N. system S s.t. $\operatorname{card} S \leq \operatorname{card} D$ *and* $\operatorname{lin} S = \operatorname{lin} D$.

Proof (Outline). Let $\mathbb{N}_*$ be $\mathbb{N}$ or $\mathbb{N}_N := \{1, 2, \ldots N\}\ \exists\, N \in \mathbb{N}$, as appropriate.

To show (a), given such an $L = \{x_n : n \in \mathbb{N}_*\}$ define $S = \{e_n : n \in \mathbb{N}_*\}$ inductively by $e_1 = \frac{x_1}{\|x_1\|}$ and $e_{k+1} := \frac{f_{k+1}}{\|f_{k+1}\|}$ where $f_{k+1} := x_{k+1} - \sum_{i=1}^{k} \langle x_{k+1}, e_i\rangle e_i$.

For (b), note that there exists a linearly independent subset D_o of D such that $\operatorname{lin} D_o = \operatorname{lin} D$. Indeed, enumerate $\{d_n\}_n$ the nonzero elements of D and define a subsequence $\{d_{n_k}\}_{k \in \mathbb{N}_*}$ inductively as follows. Let $d_{n_1} = d_1$. Given $\{d_{n_1}, \ldots, d_{n_k}\}$, let $J_k = \{j > n_k : \{d_{n_1}, \ldots, d_{n_k}, d_j\}$ is linearly independent$\}$. If $J_k = \emptyset$, stop. If $J_k \neq \emptyset$, let $n_{k+1} = \inf J_k$. Now let $D_o = \{d_{n_k} : k \in \mathbb{N}_*\}$. □

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[37] hence the word *normal*, for normalized

[38] hence the word *ortho*, for orthogonal

Example (e) In $H = L^2[-1,1]$, applying the Gram-Schmidt process to the set $\{x_n \colon n \geq \mathbb{N}_0\}$, with $x_n(t) = t^n$, produces[39] the O.N. system $S = \{e_n \colon n \in \mathbb{N}_0\}$ where

$$e_n(t) = \sqrt{n + \tfrac{1}{2}}\, P_n(t) \qquad \text{with} \qquad P_n(t) = \frac{1}{2^n n!} \left(\frac{d}{dt}\right)^n \left(t^2 - 1\right)^n .$$

$\{P_n\}_n$ are the *Legendre polynomials*. This O.N. system is also an O.N. basis.

Theorem 5.4.3 (Bessel's inequality).
Let $\{e_n \colon n \in \mathbb{N}\}$ be an orthonormal system in H and $x \in H$. Then

$$\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 \leq \|x\|^2 .$$

We will need the following direct consequence.

Lemma 5.4.4. *Let $\{e_n \colon n \in \mathbb{N}\}$ be an orthonormal system in H and $x, y \in H$. Then*

$$\sum_{n=1}^{\infty} |\langle x, e_n \rangle \langle e_n, y \rangle| < \infty .$$

Lemma 5.4.5. *Let $S \subset H$ be an orthonormal system and $x \in H$. Then the set*

$$S_x := \{e \in S \colon \langle x, e \rangle \neq 0\}$$

is at most countable.

Proof. By Bessel's inequality, each set $S_{x,n} := \left\{e \in S \colon |\langle x, e \rangle| \geq \frac{1}{n}\right\}$ is finite; hence, $S_x = \cup_{n \in \mathbb{N}} S_{x,n}$ is finite or countably infinite. □

In order to handle also nonseparable Hilbert spaces, it is necessary to introduce the concept of unconditional convergence for a *family*[40] of vectors.

Definition 5.4.6. Let X be a normed space and I be an index set. Let $x_i \in X$ for each $i \in I$. Then the (formal) series $\sum_{i \in I} x_i$ *converges unconditionally* to $x \in X$ provided

(a) $I_0 = \{i \in I \colon x_i \neq 0\}$ is at most countable (just reduced to countable case)

(b) $\sum_{n=1}^{\infty} x_{i_n} = x$ for each enumeration $\{i_1, i_2, \ldots\}$ of I_0.

In this case, the value of $\sum_{n=1}^{\infty} x_{i_n}$ is independent of the enumeration of I_0 and we write (as in Example 5.1 (e).)

$$\sum_{i \in I} x_i = x .$$

Even for $I = \mathbb{N}$, the symbols $\sum_{n \in \mathbb{N}}$ and $\sum_{n=1}^{\infty}$ differ in the section.[41]

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[39] Cf. Problem 5.6.2. $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

[40] In Definition 5.4.0.5, we defined unconditional convergence for a countable family. We now handle an uncountable family by reducing it to the countable case.

[41] Henceforth, in this section, the notation $\sum_{n \in \mathbb{N}}$ denotes unconditional convergence as defined in Def. 5.4.6; whereas, as usual, $\sum_{n=1}^{\infty}$ denotes convergence as was defined in Basics 5.4.0.1(2).

For $X = \mathbb{K}$ (resp. $\mathbb{K}^n$), the equivalence of absolute and unconditional convergence is well-known. However, in infinite dimensional case, these notions differ since the Dvoretzky-Rogers Theorem says:

- In each infinite dimensional Banach space there exists an unconditionally convergent series that is not absolutely convergent.

Lemma 5.4.5 and Theorem 5.4.3 give the following.

Corollary 5.4.7 (Generalized Bessel Inequality for O.N. systems).
Let $S \subset H$ be a O.N. system and $x \in H$. Then

$$\sum_{e \in S} |\langle x, e\rangle|^2 \;\leq\; \|x\|^2 \;.$$

Theorem 5.4.8 (general case). *Let $S \subset H$ be an O.N. system.*

(a) For each $x \in H$, the series $\sum_{e \in S} \langle x, e\rangle e$ converges unconditionally.

(b) The map $P \colon H \to H$ given by

$$P(x) \;:=\; \sum_{e \in S} \langle x, e\rangle e$$

is THE *orthogonal projection*[42] *of H onto $\overline{\mathrm{lin}} S$.*

Let's see what Thm. 5.4.8 looks like in the case that S is countable.

Theorem 5.4.8 (seperable case). *Let $\{e_n \colon n \in \mathbb{N}\}$ be an enumeration of an O.N. system $S \subset H$.*

(a) For each $x \in H$, the series $\sum_{n=1}^{\infty} \langle x, e_n\rangle e_n$ converges unconditionally.

(b) The map $P \colon H \to H$ given by

$$P(x) \;:=\; \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n$$

is THE *orthogonal projection of H onto $\overline{\mathrm{lin}} S$.*

Let's first show a useful fact.

Domination Lemma 5.4 Let $\sum_{i \in I} x_i$ be a formal series in a Banach space X. Let $\sum_{i \in I} y_i$ be an unconditionally convergent series in a Banach space Y satisfying

$$\Big\|\sum\nolimits_{i \in F} x_i\Big\|_X \;\leq\; \Big\|\sum\nolimits_{i \in F} y_i\Big\|_Y \qquad \forall \text{ finite subsets } F \subset I \;. \tag{DL5.4}$$

Then $\sum_{i \in I} x_i$ converges unconditionally.

[42] See Definition 5.3.11.

Proof. By the domination assumption, $\{i\in I\colon x_i\neq 0\}\subset\{i\in I\colon y_i\neq 0\}:=I_y$. Since $\sum_{i\in I}y_i$ converges unconditionally, I_y is atmost countable. Let $\{i_1,i_2,\ldots\}$ be an enumeration of I_y.[43] Since $\sum_{i\in I}y_i$ converges unconditionally, $\sum_{n=1}^{\infty}y_{i_n}$ satisfies condition (5) of Theorem 5.4.0.4. Thus $\sum_{n=1}^{\infty}x_{i_n}$ also satisfies condition (5) of Theorem 5.4.0.4 thanks to (DL5.4). So $\sum_{i\in I}x_i$ converges unconditionally. □

Proof (Theorem 5.4.8, general case). For an $x\in H$, let $S_x=\{e\in S\colon \langle x,e\rangle\neq 0\}$. Note that, thanks to Lemma 5.4.5, the set S_x is atmost countable.

(a) Fix $x\in H$. We want to show that $\sum_{e\in S}\langle x,e\rangle e$ converges unconditionally.
Way #1, using Domination Lemma 5.4: By Pythagoras' Theorem (5.22)

$$\left\|\sum_{e\in F}\langle x,e\rangle e\right\|^2 \overset{(5.22)}{=} \sum_{e\in F}\|\langle x,e\rangle e\|^2 = \sum_{e\in F}|\langle x,e\rangle|^2 \qquad \forall \text{ finite subsets } F\subset S\,.$$

By Bessel's inequality (Theorem 5.4.7)

$$\sum_{e\in S}|\langle x,e\rangle|^2 \le \|x\|^2\,.$$

Now apply the Domination Lemma 5.4 with $Y=\ell_2(S)$ and $y_e=\langle x,e\rangle$.
Way #2, as in Dirk's book: Let $\{e_1,e_2,\ldots\}$ be an enumeration of S_x.[44] We show first that $\sum_{n=1}^{\infty}\langle x,e_n\rangle e_n$ is Cauchy. By Pythagoras' Theorem (5.22) and Theorem 5.4.3

$$\left\|\sum_{n=N}^{M}\langle x,e_n\rangle e_n\right\|^2 \overset{(5.22)}{=} \sum_{n=N}^{M}\|\langle x,e_n\rangle e_n\|^2 = \sum_{n=N}^{M}|\langle x,e_n\rangle|^2 \overset{\text{Thm. 5.4.3}}{\longrightarrow} 0$$

as $N,M\to\infty$. So there exists $y{:=}\sum_{n=1}^{\infty}\langle x,e_n\rangle e_n$ in H. Similarly, for each permutation $\pi\colon\mathbb{N}\to\mathbb{N}$ of $\{e_1,e_2,\ldots\}$, there exists $y_\pi\in H$ such that $y_\pi=\sum_{n=1}^{\infty}\langle x,e_{\pi(n)}\rangle e_{\pi(n)}$. Towards[45] showing $y=y_\pi$, let $z\in H$ be arbitrary. Then[46]

$$\langle y,z\rangle = \sum_{n=1}^{\infty}\langle x,e_n\rangle\langle e_n,z\rangle = \sum_{n=1}^{\infty}\langle x,e_{\pi(n)}\rangle\langle e_{\pi(n)},z\rangle = \langle y_\pi,z\rangle$$

so $y-y_\pi\in H^\perp=\{0\}$. Thus $y=y_\pi$.

(b) Clearly P is a well-defined linear operator. By Theorem 5.3.10, it is enough to show that $P\in L(H,H)$ and $P^2=P$ and $\operatorname{ran}P=\overline{\operatorname{lin}S}$.

Fix $x\in H$. For each $\varepsilon>0$ there exists a finite subset F_ε of S such that

$$\left\|P(x) - \sum_{e\in F_\varepsilon}\langle x,e\rangle e\right\| = \left\|\sum_{e\in S}\langle x,e\rangle e - \sum_{e\in F_\varepsilon}\langle x,e\rangle e\right\| < \varepsilon\,.$$

[43] WLOG I_y is not finite since the claim is trivial for finite I_y.

[44] WLOG, S_x is not finite for otherwise (a) would be trivial.

[45] By our Thm. 5.4.0.4, we are done here. But by the book's Def. 5.4.6, we must show that each rearrangement of the series in question converges to the SAME element in H. So let's do this here.

[46] The middle equality uses the absolute (thus unconditional) convergence of the series (Lemma 5.4.4) while for the first and last equalities use the continuity of the scalar product (Lemma 5.1.6).

But for any finite subset F of S, thanks again to Pythagoras and Bessel,

$$\left\|\sum_{e\in F}\langle x,e\rangle e\right\|^2 \overset{(V.22)}{=} \sum_{e\in F}|\langle x,e\rangle|^2 \overset{\text{Thm. 5.4.3}}{\leq} \|x\|^2 .$$

Thus $\|P\| \leq 1$. If $e \in S$, then $P(e) = e$ since S is an O.N. system. So for $x \in H$,

$$P^2(x) = P\left(\sum_{e\in S}\langle x,e\rangle e\right) = \sum_{e\in S}\langle x,e\rangle P(e) = \sum_{e\in S}\langle x,e\rangle e = P(x) .$$

Thus $P^2 = P$. Clearly $S \subset \operatorname{ran} P \subset \overline{\operatorname{lin}} S$. Thus, since a projection has closed range, $\operatorname{ran} P = \overline{\operatorname{lin}} S$. □

We now turn our attention to O.N. bases.

Theorem 5.4.9. *Let $S \subset H$ be an O.N. system.*

(a) There exists an O.N. basis S' with $S \subset S'$.

(b) T.F.A.E. .

(i) S is an O.N. basis.

(ii) If $x \in H$ and $x \perp S$, then $x = 0$.

(iii) $H = \overline{\operatorname{lin}} S$.

(iv) $x = \sum_{e\in S}\langle x,e\rangle e \quad \forall x \in H$.

(v) $\langle x,y\rangle = \sum_{e\in S}\langle x,e\rangle\,\langle e,y\rangle \quad \forall x,y \in H$.

(vi) (Parseval's equality)

$$\|x\|^2 = \sum_{e\in S}|\langle x,e\rangle|^2 \qquad \forall x \in H .$$

Rmk: Comments on the series in (iv) and (v).
By Lemma 5.4.5, each series has at most a countable number of nonzero terms.
The notation says that each series converges unconditionally.
By Lemma 5.4.4, the series in (v) also converges absolutely.

Proof. (a) is an immediate consequence of Zorn's Lemma.[47] (If H is separable, then a constructive O.N. basis can be obtained from the Gram-Schmidt process.)

Now onto (b).

(i) ⇒ (ii): Let $x \perp S$. If x were not 0, then $S \cup \{x/\|x\|\}$ would be an O.N. system.

(ii) ⇒ (iii): Corollary 5.3.5. $\overline{\operatorname{lin}S} \overset{5.3.5}{=} \left((\operatorname{lin}S)^{\perp}\right)^{\perp}$ and (ii) says $(\operatorname{lin}S)^{\perp} = \{0\}$.

(iii) ⇒ (iv): Theorem 5.4.8. Recall that the range of a projection P is $\{x\colon P(x) = x\}$.

(iv) ⇒ (v): Fix $x,y \in H$. By (iv) and Lemma 5.4.5, there exists a countable set $S_0 \subset \{e \in S\colon \langle x,e\rangle \neq 0\} \cup \{e \in S\colon \langle y,e\rangle \neq 0\}$ such that, for an (any, but fixed) enumeration $(i_n)_{n=1}^{\infty}$ of S_0,

[47] http://en.wikipedia.org/wiki/Zorns_lemma

$$x \;=\; \sum_{e\in S}\langle x,e\rangle e \;=\; \lim_{n\to\infty} s_n \quad \text{where } \; s_n := \sum_{k=1}^{n}\langle x,e_{i_k}\rangle e_{i_k}$$

$$y \;=\; \sum_{e\in S}\langle y,e\rangle e \;=\; \lim_{n\to\infty} t_n \quad \text{where } \; t_n := \sum_{k=1}^{n}\langle y,e_{i_k}\rangle e_{i_k}$$

$$\sum_{e\in S}\langle x,e\rangle \;\langle e,y\rangle \;=\; \lim_{n\to\infty}\sum_{k=1}^{n}\langle x,e_{i_k}\rangle \;\langle e_{i_k},y\rangle \;\overset{\text{note}}{=}\; \lim_{n\to\infty}\langle s_n,t_n\rangle \;.$$

By Lemma 5.1.6, $\lim_{n\to\infty}\langle s_n,t_n\rangle = \langle x,y\rangle$.
(v) $\Rightarrow$ (vi): Set $x = y$.
(vi) $\Rightarrow$ (i): Otherwise, there would exist $x\in H$ with $\|x\| = 1$ s.t. $S\cup\{x\}$ is an O.N. system; thence, producing the contradiction $0 \overset{\{x\}\perp S}{=} \sum_{e\in S}|\langle x,e\rangle|^2 = \|x\|^2 = 1$. □

Condition (iv) suggests the terminology "basis". It certainly cannot be a vector space (i.e. algebraic) basis (unless dim $H<\infty$), since then all the series could have only finitely many nonzero summands. We now return to the above examples.

Example (a) By definition, $\ell^2 = \overline{\text{lin}}S$, so S is an O.N. basis. In general, $\{e_i\colon\, i\in I\}$ is an O.N. basis for $\ell^2(I)$, where $e_i(j) = \delta_{ij}$.[48]

Example (b) We show here also that $\overline{\text{lin}}S = L^2[0,2\pi]$. W.L.O.G. (why?), $\mathbb{K}=\mathbb{R}$. $\text{lin}\,S$ is the set of trigometric polynomials, which, by Corollary 4.2.12, lies densely in $V = \{f\in C[0,2\pi]\colon\, f(0) = f(2\pi)\}$ w.r.t. $\|\cdot\|_\infty$, so a fortiori w.r.t. $\|\cdot\|_{L^2}$, and V lies densely in $L^2[0,2\pi]$ since $C[0,2\pi]$ does (Theorem 1.2.12). Therefore $\text{lin}\,S$ lies dense in $L^2[0,2\pi]$. (That V is L^2-dense in $L^2[0,2\pi]$ follows also from Lemma 5.1.10.)

Example (c) Since $\cos(nt) = \frac{1}{2}\left(e^{int}+e^{-int}\right)$ and $\sin(nt) = \frac{1}{2}\left(e^{int}-e^{-int}\right)$ and $e^{int} = \cos(nt)+i\sin(nt)$,

$$\overline{\text{lin}}S \;=\; \{f\colon\, f \text{ is a } \mathbb{C}\text{-valued trigometric polynomial}\}\;.$$

From (b), the $\text{lin}\,S$ is dense and so S is a O.N. basis.

Example (d) By definition, $\overline{\text{lin}}S = AP^2(\mathbb{R})$.

Example (e) $L^2[-1,1] = \overline{\text{lin}}\{x_n\colon\, n\in\mathbb{N}_0\} = \overline{\text{lin}}S$ by Thms. 1.2.10, 1.2.12, and 5.4.2.

We return again to (b). Let S be as there and $x\in L^2[0,2\pi]$, so the $\langle x,e\rangle$ are precisely the Fourier coefficients of x appearing in (4.4) and condition (iv) in Theorem 5.4.9(b) says that, for $x\in L^2[0,2\pi]$, the Fourier series converges to x in the quadratic mean. In Fourier analysis, it is often more convenient to work with the expansion from the O.N. basis in Example (c). So, henceforth, it is understood that the Fourier series of an integrable function f is the series

$$\sum_{n=-\infty}^{\infty} c_n e^{int} \qquad \text{with} \qquad c_n \;=\; \tfrac{1}{2\pi}\int_0^{2\pi} f(t)\,e^{-int}\,dt\;.$$

Some authors use the term Fourier series for arbitrary O.N. bases in general Hilbert spaces.

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48 δ_{ij} is the Kronecker symbol which is one when $i=j$ and zero otherwise.

Corollary 5.4.10. *For an infinite dimensional Hilbert space H, T.F.A.E. .*

(a) H is separable.

(b) Each O.N. basis of H is countable.

(c) H has a countable O.N. basis.

Proof. (i)$\Rightarrow$(ii): Let S be an O.N. basis of H. Then $\|e-f\|^2 = \langle e-f, e-f\rangle = 2$ for all $e, f \in S$ with $e \neq f$. Thus S cannot be uncountable. (Cf. Problem 5.6.4(c) or the proof of the inseparability of ℓ^∞ in Section 1.2 Example (c).)

(ii)$\Rightarrow$(iii): This is clear (considering Theorem 5.4.9(a)).

(iii)$\Rightarrow$(i) follows from Theorem 5.4.9(b)(iii) and Lemma 1.2.9. □

More generally, the following is valid, where $|S|$ denotes the cardinality of S.

Lemma 5.4.11. *If S and T are O.N. bases of H, then $|S| = |T|$.*

Proof. For finite S, this is clear from linear algebra. So let $|S| \geq |\mathbb{N}|$. For $x \in S$, set

$$T_x := \{y \in T : \langle x, y\rangle \neq 0\} \ .$$

Then $|T_x| \leq |\mathbb{N}|$ by Lemma 5.4.5. By Theorem 5.4.9(b)(ii)

$$T \setminus \{0\} \ \subset \ \cup_{x\in S} T_x$$

and so $|T| \leq (|S|)\ (|\mathbb{N}|) = |S|$. By symmetry $|S| \leq |T|$. Thus $|S| = |T|$ by the Schröder-Bernstein Theorem[49] from set theory. □

The cardinal number $|S|$ of an O.N. basis is the *Hilbert space dimension* of H; it is well-defined by Lemma 5.4.11.

Theorem 5.4.12. *Let S be an O.N. basis of H. Then $H \cong \ell^2(S)$.*
More specifically, $H \cong \ell^2(S)$ via the isometric isomorphism

$$T\ : H \to \ell^2(S)$$

defined by

$$Tx := (\ \langle x, e\rangle\ )_{e\in S} \qquad \forall x \in H\ .$$

Furthermore, T preserves the scalar product, i.e.

$$\langle Tx, Ty\rangle_{\ell^2(S)} \ = \ \langle x, y\rangle_H$$

for all $x, y \in H$.

[49] http://en.wikipedia.org/wiki/Cantor-Bernstein-Schroder_theorem

Some other notations:

$$(Tx)(e) = \langle x, e\rangle \qquad \forall x \in H\ ,\ \forall e \in S$$

or we can view $\{e\colon e \in S\}$ as the standard unit vectors of $\ell^2(S)$ and so

$$Tx = \sum_{e\in S}\langle x, e\rangle e \qquad \forall x \in H\ .$$

Note that if S is countable and $\{e_n\colon n \in \mathbb{N}\}$ is an enumeration of S, then

$$Tx = \sum_{n=1}^{\infty}\langle x, e_n\rangle e_n \qquad \forall x \in H\ .$$

Proof. T is well-defined by Bessel's Inequality (Cor. 5.4.7). Clearly T is linear. By Parseval's equality (Thm. 5.4.9b(vi)), T is norm preserving[50] (and thus injective).

Towards showing T is surjective, let $(f_e)_{e\in S} \in \ell^2(S)$. Then[51] $x := \sum_{e\in S} f_e e$ converges unconditionally in H; for indeed, by Pythagoras' Theorem (5.22)

$$\left\|\sum\nolimits_{e\in F} f_e e\right\|_H^2 \overset{(5.22)}{=} \sum\nolimits_{e\in F}\|f_e e\|_H^2 = \sum_{e\in F}|f_e|^2 \quad \forall \text{ finite } F \subset S\ .$$

and now apply the Domination Lemma 5.4 with the dominating unconditionally convergent series being $\sum_{e\in S} f_e e \in \ell^2(S)$. Also, $Tx = (f_e)_{e\in S}$ since if $e_0 \in S$ then $\langle x, e_0\rangle = \sum_{e\in S}\langle f_e e, e_0\rangle = f_{e_0}$. Thus T is surjective.

That T preserves the scalar product follows from Thm. 5.4.9b(v). □

Corollary 5.4.13.
If H is a separable infinite-dimensional Hilbert space, then $H \cong \ell^2$. [52]

(The proof of) Corollary 5.4.13 gives that if $\{e_n\colon n \in \mathbb{N}\}$ is any O.N. basis of a separable infinite-dimensional Hilbert space H, then for each $x \in H$,

$$x = \sum_{n\in\mathbb{N}}\langle x, e_n\rangle e_n$$

$$\|x\|_H = \sqrt{\sum_{n=1}^{\infty}|\langle x, e_n\rangle|^2}$$

$$\text{if} \quad x = \sum_{n\in\mathbb{N}}\alpha_n e_n \quad \text{for some} \quad (\alpha_n) \subset \mathbb{K} \quad \text{then} \quad \alpha_n = \langle x, e_n\rangle\ \ \forall n \in \mathbb{N}\ .$$

Corollary 5.4.14 (Fischer-Riesz Theorem).

$$L^2[0,1] \cong \ell^2\ .$$

[50] i.e. $\|Tx\| = \|x\|$ for each $x \in H$

[51] Compare this arguement-to-come with the proof of Theorem 5.4.8(a)

[52] Recall, in Dirk's book, $\ell^2 = \ell^2(\mathbb{N})$ and $\cong$ denotes isometrically isomorphic.

Schauder bases (x_n) for Banach spaces were defined and explored in Problems 4.8.18 and 4.8.19. In particular, Problem 4.8.19-3 defines the basis constant $\mathrm{bc}(x_n)$ of a Schauder basis (x_n). Clearly, $\mathrm{bc}(x_n) \geq 1$. A Schauder basis (x_n) is called

- *monotone* provided $\mathrm{bc}(x_n) = 1$
- *normalized* provided $\|x_n\| = 1$ for each n.

Our theory of Schauder bases was developed for infinite sequence $(x_n)_{n=1}^{\infty}$ in infinite dimensional Banach spaces. There is a parallel (obvious, much easier) theory for finite sequences $(x_n)_{n=1}^{N}$ for N-dimensional Banach spaces where Schauder bases and algebraic bases coincide.

Theorem 5.4.17. Let H be a separable Hilbert space.

(1) (x_n) is a O.N. basis for H if and only if (x_n) is a normalized monotone Schauder basis for H.

(2) H has a normalized Schauder basis that is not an O.N. basis.

A proof of Theorem 5.4.17 is outlined in Problems 5.6.44 and 5.6.45.

5.5 Section 5.5.0: Adjoints: Banach space vs. Hilbert space

THE GOAL OF THIS EXCURSION

To adjust the Banach space adjoint to the Hilbert space setting
by taking into consideration that a Hilbert space is self-dual.

In this section, X and Y are Banach spaces while H and K are Hilbert spaces.

Basics 5.4.0.1. Towards our goal, let's recall what we already know.

- Notation:[53] For $x \in X$ and $x' \in X'$, as well as for $h, h_i \in H$ and $h' \in H'$

$$\begin{aligned} \langle x \mid x' \rangle_X &= x'(x) \\ \langle h \mid h' \rangle_H &= h'(h) \\ \langle h_1, h_2 \rangle_H &= \text{the scalar product of } h_1 \text{ with } h_2 . \end{aligned}$$

- A Hilbert space H is *self-dual* in the sense that the map Φ_H is bijective, norm preserving, and conjugate linear where

$$\Phi_H : H \to H'$$

$$\langle \,\cdot \mid \Phi_H(h) \,\rangle_H \overset{\text{notation}}{=} [\Phi_H(h)](\cdot) \overset{\text{def}}{=} \langle \,\cdot\,, h \,\rangle_H = \langle \,\Phi_H(h)\,,\, \Phi_H(\cdot)\, \rangle_{H'}$$

for each $h \in H$ (where $\cdot \in H$ also).[54]

- If $T \in L(X,Y)$, then the (Banach space) adjoint of T is $T' \in L(Y',X')$ where

$$\begin{aligned} &T : X \to Y && \left(T'y'\right)(x) = y'(Tx) \\ &T' : Y' \to X' && \langle x \mid T'y' \rangle_X = \langle Tx \mid y' \rangle_Y \end{aligned}$$

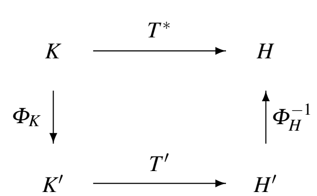
for $y' \in Y'$ and $x \in X$.[55]

- If $T \in L(H,K)$, then the (Banach Space) adjoint of T is $T' \in L(K',H')$ where

$$\begin{aligned} &T : H \to K && \left(T'k'\right)(h) = k'(Th) \\ &T' : K' \to H' && \langle h \mid T'k' \rangle_H = \langle Th \mid k' \rangle_K \end{aligned}$$

for $k' \in K'$ and $h \in H$.

Thoughts 5.4.0.2. So for $T : H \to K$, the Banach space adjoint $T' : K' \to H'$. But in the Hilbert space setting, since a Hilbert space is self-dual, it would be more natural to have a Hilbert space adjoint $T^* : K \to H$. This is easily accomplished by the following commutative diagram.



Let's compute how this Hilbert space adjoint T^* behaves.

---

[53] Slight adjustment to notation introduced in §3.3 since we now use $\langle \cdot, \cdot \rangle$ for the scalar product.

[54] Fréchet-Riesz Representation Thm. 5.3.6. For the last equality, see mgRmk just after Thm. 5.3.6.

[55] Def. 3.4.1.

Computation 5.4.0.3. The behavior of $T^* := \Phi_H^{-1} \circ T' \circ \Phi_K$.

$$\begin{array}{ccc} K & \xrightarrow{\;T^*\;} & H \\ \Phi_K \big\downarrow & & \big\uparrow \Phi_H^{-1} \\ K' & \xrightarrow{\;T'\;} & H' \end{array}$$

For each $h \in H$ and $k \in K$

$$\begin{aligned} \langle\, h\, ,\, T^*k\, \rangle_H &= \langle\, h\, ,\, \Phi_H^{-1} T' \Phi_K k\, \rangle_H && \text{by def. of } T^* \\ &= \langle\, h \;|\; \Phi_H \Phi_H^{-1} T' \Phi_K k\, \rangle_H && \text{by def. of } \Phi_H \\ &= \langle\, Th \;|\; \Phi_K k\, \rangle_K && \text{by def. of } T' \\ &= \langle\, Th\, ,\, k\, \rangle_K && \text{by def. of } \Phi_K\ . \end{aligned}$$

Summary 5.4.0.4. So

$$\begin{array}{rl} T : H \to K & \text{a bounded linear operator} \\ T' : K' \to H' & \text{the Banach space adjoint of } T \\ T^* : K \to H & \text{the Hilbert space adjoint of } T\ . \end{array}$$

with

$$\begin{aligned} \langle h \,|\, T'k' \rangle_H &= \langle Th \,|\, k' \rangle_K \\ \langle\, h\, ,\, T^*k\, \rangle_H &= \langle\, Th\, ,\, k\, \rangle_K \end{aligned}$$

for each $h \in H$ and $k \in K$ and $k' \in K'$.

5.5 Operators on Hilbert Spaces

In this section H, as well as H_i, always denote Hilbert spaces.

Definition 5.5.1. Let $T \in L(H_1,H_2)$ and $\Phi_i \colon H_i \to H_i'$ be the canonical conjugate linear isometric isomorphisms from Theorem 5.3.6. Then the *(Hilbert space) adjoint operator* of T is the operator

$$T^* : H_2 \to H_1 \quad \text{given by} \quad T^* := \Phi_1^{-1} \circ T' \circ \Phi_2 \,,$$

in other words,

$$\langle\, Tx \,,\, y \,\rangle_{H_2} \;=\; \langle\, x \,,\, T^*y \,\rangle_{H_1}$$

for all $x \in H_1$ and $y \in H_2$.

Theorem 5.5.2. *Let $S,T \in L(H_1,H_2)$ and $R \in L(H_2,H_3)$ and $\lambda \in \mathbb{K}$.*

(a) $(S+T)^* \;=\; S^* + T^*$.

(b) $(\lambda S)^* \;=\; \overline{\lambda} S^*$.

(c) $(RS)^* \;=\; S^* R^*$.

(d) $S^* \in L(H_2,H_1)$ *and* $\|S\| \;=\; \|S^*\|$.

(e) $S^{**} = S$.

(f) $\|SS^*\| \;=\; \|S^*S\| \;=\; \|S\|^2$.

(g) $\ker S \;=\; (\operatorname{ran} S^*)^{\perp}$ *and* $\ker S^* \;=\; (\operatorname{ran} S)^{\perp}$ *;*
in particular, S is injective if and only if $\operatorname{ran} S^*$ *is dense.*

Thus the mapping

$$L(H_1,H_2) \;\ni\; S \quad\mapsto\quad S^* \;\in\; L(H_2,H_1)$$

is a conjugate linear surjective isometry. Note the similarities between this map and the mapping $\lambda \mapsto \overline{\lambda}$ on $\mathbb{C}$.

We now define important classes of Hilbert operators.

Definition 5.5.3. Let $T \in L(H_1,H_2)$.

(a) T is *unitary* provided T is invertible with $TT^* = \mathrm{Id}_{H_2}$ and $T^*T = \mathrm{Id}_{H_1}$.

(b) Let $H_1 = H_2$. T is *self-adjoint* (or *hermitian*) provided $T = T^*$.

(c) Let $H_1 = H_2$. T is *normal* provided $TT^* = T^*T$.

There are equivalent characterizations of these classes of operators.

(a) $T \in L(H_1,H_2)$ is unitary IFF T is onto and $\langle Tx,Ty\rangle_{H_2} = \langle x,y\rangle_{H_1}$ $\forall x,y \in H_1$.

(b) $T \in L(H_1,H_1)$ is self-adjoint IFF $\langle Tx,y\rangle \;=\; \langle x,Ty\rangle \quad \forall x,y \in H_1$.

(c) $T \in L(H_1,H_1)$ is normal IFF $\langle Tx,Ty\rangle \;=\; \langle T^*x,T^*y\rangle \quad \forall x,y \in H_1$.

Obviously, if $T \in L(H,H)$ is unitary or self-adjoint, then T is normal.

Example a. Let $H = \mathbb{K}^n$. If T is represented by the matrix $(a_{i,j})_{i,j}$, then T^* is represented by the matrix $(\overline{a_{j,i}})_{i,j}$. Thus Definition 5.5.3 generalizes well-know concepts from linear algebra.

Example b. Let $H = L^2[0,1]$ and $T_k \in L(H)$ be the integral operator[56]

$$(T_k x)(s) \ = \ \int_0^1 k(s,t)\, x(t)\, dt \ .$$

Then $T_k^* = T_{k^*}$ where $k^*(s,t) := \overline{k(t,s)}$. (Cf. Example 3.4(c), note the complex conjugate!) This can be regarded as the continuous analogue to Example (a).

T_k is self-adjoint if and only if $k(s,t) = \overline{k(t,s)}$ almost everywhere; such a k is called a symmetric kernel.

Example c. Let $T\colon \ell^2 \to \ell^2$ be the shift operator $(s_1, s_2, \ldots) \mapsto (s_2, s_3, \ldots)$. Then $T^*((t_1,t_2,\ldots)) = (0,t_1,t_2,\ldots)$ (cf. Example 3.4(a)). T is not normal since $TT^* = \mathrm{Id}$ but $T^*T = P_U$ where $U = \{(s_i) : s_1 = 0\}$.

Example d. T^*T and TT^* are always self-adjoint.

Example e. The Fourier transform $\mathscr{F}\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, which was studied in Section 5.2, is a unitary operator by the Plancherel inequality (5.16) since $\mathscr{F}$ is surjective.

Lemma 5.5.4. *For $T \in L(H_1, H_2)$, T.F.A.E. .*

(i) T *is an isometric embedding (*i.e. $\|Tx\| = \|x\| \ \forall x \in H_1$*).*

(ii) $\langle Tx, Ty\rangle = \langle x, y\rangle \quad \forall x, y \in H_1$.

This lemma geometrically says a length preserving operator is angle preserving.

---

[56] Example 2.1(m)

Next self-adjoint operators will be explored; nontrivial results (with the exception of Theorem 5.5.5) will not come until Chapter 6.

Theorem 5.5.5 (Hellinger-Toeplitz Theorem).
A linear mapping $T: H \to H$ satisfying the symmetric condition

$$\langle Tx, y\rangle \ = \ \langle x, Ty\rangle \qquad \forall x, y \in H$$

is continuous and hence self-adjoint.

Theorem 5.5.6. *Let $\mathbb{K} = \mathbb{C}$ and $T \in L(H)$. Then T.F.A.E. .*

(i) T *is self-adjoint.*

(ii) $\langle Tx, x\rangle \in \mathbb{R} \quad \forall x \in H.$

The proof applied a technique, known as *polarization*, of considering $x + \lambda y$.

Theorem 5.5.7. *For a self-adjoint operator $T \in L(H)$*

$$\|T\| \ = \ \sup_{\|x\| \le 1} |\langle Tx, x\rangle| \ .$$

Corollary 5.5.8. *Let $T \in L(H)$. If*

(1) T *is self-adjoint*

(2) $\langle Tx, x\rangle = 0 \quad \forall x \in H$

then $T = 0$.

A note about the Corollary 5.5.8. For $\mathbb{K} = \mathbb{C}$, (2) implies (1) by Theorem 5.5.6. In the case $\mathbb{K} = \mathbb{R}$, in general one cannot dispense with (1). Indeed, for $H = \mathbb{R}^2$ and

$$T \ = \ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(thus T is rotation through 90^{o}), condition (2) holds but $T \neq T^*$.

In closing is the self-adjoint characterization of projections.[57]

Theorem 5.5.9. *Let $P \in L(H)$ be a projection with $P \neq 0$. Then T.F.A.E. .*

(i) *P is an orthogonal projection (*i.e. $\operatorname{ran} P \perp \ker P$*).* [58]

(ii) $\|P\| = 1.$

(iii) *P is self-adjoint.*

(iv) *P is normal.*

(v) $\langle Px, x\rangle \ge 0 \ \forall x \in H.$

The proof of (iv) $\Rightarrow$ (i) showed the following, which will be used in Chapter 6.

Lemma 5.5.10. *If $T \in L(H)$ is a normal operator, then* $\|Tx\| = \|T^*x\| \quad \forall x \in H$. *In particular,* $\ker T = \ker T^*$.

[57] Compare with Theorem 5.3.10

[58] See Definition 5.3.11.

5.6 Problems

Problem 5.6.1. Recall that the usual scalar product

$$\langle \cdot , \cdot \rangle : C[0,1] \times C[0,1] \to \mathbb{K}$$

is given by

$$\langle x , y \rangle := \int_0^1 x(t)\overline{y(t)}\,dt$$

for each $x, y \in C[0,1] := \{f : [0,1] \to \mathbb{K} \mid f \text{ is continuous}\}$.

Now fix a *weight* $w \in C_{\mathbb{R}}[0,1]$ and consider the mapping

$$\langle \cdot , \cdot \rangle_w : C[0,1] \times C[0,1] \to \mathbb{K}$$

given by

$$\langle x , y \rangle_w := \int_0^1 x(t)\overline{y(t)}w(t)\,dt$$

for each $x, y \in C[0,1]$.

(a) Give necessary and sufficient conditions on w so that $\langle \cdot , \cdot \rangle_w$ is a scalar product.

(b) When is the norm $\|\cdot\|_w$ generated by the scalar product $\langle \cdot , \cdot \rangle_w$ equivalent to the norm $\|\cdot\|$ generated by $\langle \cdot , \cdot \rangle$?

Problem 5.6.4. Fill in the following omitted details to Example 5.1(f).

(a) Show that equation (5.5) defines a scalar product on X.

(b) Show that $\|f_\lambda - f_{\lambda'}\| = \sqrt{2}$ for $\lambda \neq \lambda'$.

(c) Let Z be an uncountable subset of a Banach space X satisfying, for some $\delta > 0$,

$$\|z - \widetilde{z}\| \geq 2\delta \qquad \forall z, \widetilde{z} \in Z \quad \text{s.t.} \quad z \neq \widetilde{z}\,.$$

Show that X is not separable.
Hint. Might be useful to note that: $\forall a \in X$, if $z, \widetilde{z} \in Z \cap N_\delta(a)$ then $z = \tilde{z}$.

(d) Conclude that $AP^2(\mathbb{R})$ is not separable.

Problem 5.6.5. In a Hilbert space H the following equivalence holds.

$$x_n \to x \iff \begin{cases} x_n \xrightarrow{\sigma} x \\ \|x_n\| \to \|x\| \end{cases}.$$

Hint: Cor. 5.3.7(a). Recall that $x_n \to x$ means $\lim_{n\to\infty} \|x_n - x\|_H = 0$.

Problem 5.6.6. Integration by Parts[59]

(a) Let $f\colon \mathbb{R}^n \to \mathbb{C}$ be continuously differentiable with compact support. Then
$$\int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(x)\, dx \;=\; 0$$
for each $j = 1, \dots n$. (Tip: Fubini's Theorem.)

(b) Let $f, g\colon \mathbb{R}^n \to \mathbb{C}$ be continuously differentiable with one of the functions having compact support. Then
$$\int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(x)\, g(x)\, dx \;=\; -\int_{\mathbb{R}^n} f(x)\, \frac{\partial g}{\partial x_j}(x)\, dx$$
for each $j = 1, \dots, n$.

(c) Let $\Omega \subset \mathbb{R}^n$ be open. Let $f, g\colon \Omega \to \mathbb{C}$ be continuously differentiable with one of the functions having compact support (in Ω!). Then
$$\int_{\Omega} \frac{\partial f}{\partial x_j}(x)\, g(x)\, dx \;=\; -\int_{\Omega} f(x)\, \frac{\partial g}{\partial x_j}(x)\, dx$$
for each $j = 1, \dots, n$. (Tip: extend the function fg canonically to $\mathbb{R}^n$.)

Problem 5.6.14. Give an examle of a pre-Hilbert space X and a subspace $U \subset X$ such that both of the following are satisfied.

(a) $\overline{U} \neq U^{\perp\perp}$.

(b) $\overline{U} \oplus U^{\perp} \neq X$.

WLOG, U is a closed subspace of X since $U^{\perp} = \overline{U}^{\perp}$, as noted just after Def. 5.3.1. Remark: this problem shows that completeness is needed in (the fundamental) Thm. 5.3.4.

Problem 5.6.15. Let K be a closed convex **subset** of a Hilbert space H. Let P_K denote the so called *metric projection* map that assigns, in accordance to Theorem 5.3.2, to each $x_0 \in H$ its best approximation $x \in K$. Thus for $x \in H$

$$\begin{gathered} P_K(x) \text{ is the unique element in } H \text{ satisfying} \\ P_K(x) \in K \qquad \text{and} \qquad \|x - P_K(x)\| \;=\; d(x, K)\ . \end{gathered} \tag{$*$}$$

Show
$$\|P_K(x) - P_K(y)\|_H \;\le\; \|x - y\|_H \qquad\qquad \forall x, y \in H\ .$$

(Tip: Lemma 5.3.3)
(Warning: Thms 5.3.4 and 5.3.10 are stated for **subspaces** U of H.)
(Warning: The map P_K need not be linear.)

[59] Cf. page 92 of our class notes.

Problem 5.6.16-1. Use the Projection Theorem (Thm. 5.3.2) to show the Hahn-Banach Separation Theorem (Thm. 3.2.5) in a Hilbert space.
Rmk: Show Thm. 3.2.5 as stated in the book, not as stated more generally in class.

> **Theorem 3.2.5** (Hahn-Banach Separation Thm, Version II) as stated in book.
> Let V be a closed convex subset of a normed space X and $x \notin V$.
> Then there exists $x' \in X'$ and $\varepsilon > 0$ so that
>
> $$\operatorname{Re} x'(x) + \varepsilon \leq \operatorname{Re} x'(v) \quad \forall v \in V \ .$$
>
> mgRemark: Compare with our Thm 3.2.5 from class. The book separated a convex closed set from a point x. In class we separated a convex closed set and a convex compact set. The set $\{x\}$ is a convex compact set.

Problem 5.6.16-2. [60] Use the Orthogonal Projection Theorem (Thm. 5.3.4) to show the Hahn-Banach Extension Theorem (as stated in Thm. 3.1.5) for a norm-closed subspace of a Hilbert space.
Rmk. Do not forget to show the preservation of the norm for the extended functional.

Problem 5.6.17. Consider the orthogonal projections P_U and P_V corresponding to closed subspaces U and V of a Hilbert space H. Show

$$U \subset V \quad \Longleftrightarrow \quad P_U = P_V P_U = P_U P_V \ .$$

Problem 5.6.23. In a normed space X, T.F.A.E. .

(i) $\sum_{i \in I} x_i$ converges unconditionally to x (in the sense of Definition 5.4.6).

(ii) For each $\varepsilon > 0$ there exists a finite subset $F \subset I$ such that for each finite subset G with $F \subset G \subset I$

$$\left\| x - \sum_{i \in G} x_i \right\|_X \leq \varepsilon \ .$$

[60] At Universität Karlsruhe, this problem is known as: Prof. Dr. Weis's favorite comps question.

The purpose of Problems 40–43 is to show Theorem 5.4.0.4 in reasonably sized steps.

Problem 5.6.40. Show that, in Thm. 5.4.0.4, conditions (2) and (3) are equivalent. Hint: Pick your favorite $0 \neq c \in \mathbb{R}$. Then in Theorem 5.4.0.4, T.F.A. obviously E. .

- ∘ $\sum_{k=1}^{\infty} x_{n_k}$ converges for each subsequence $\left(x_{n_k}\right)_k$ of $(x_n)_n$.
- ∘ $\sum_{n=1}^{\infty} \theta_n x_n$ converges for each choice of $(\beta_n)_n$ where each β_n is either 0 or c.

Problem 5.6.41. Let $\sum_{n=1}^{\infty} x_n$ be a series in a Banach space X having condition (5) of Thm. 5.4.0.4.

(a) Give a short argument that $\sum_{n=1}^{\infty} x_n$ satisfies condition (1) of Thm. 5.4.0.4.

(b) Give a short argument that $\sum_{n=1}^{\infty} x_n$ satisfies condition (2) of Thm. 5.4.0.4.

(c) Show that each rearrangement of $\sum_{n=1}^{\infty} x_n$ converges to the same element in X (thereby showing the "Furthermore" of Thm. 5.4.0.4).

Problem 5.6.42. Let $\sum_{n=1}^{\infty} x_n$ be a series in a Banach space X failing condition (5) of Thm. 5.4.0.4.

(a) Show that $\sum_{n=1}^{\infty} x_n$ fails condition (2) of Thm. 5.4.0.4.

(b) Show that $\sum_{n=1}^{\infty} x_n$ fails condition (1) of Thm. 5.4.0.4.

Hint: since $\sum_{n=1}^{\infty} x_n$ fails (5), there exists $\varepsilon > 0$ and a sequence $(A_n)_{n=1}^{\infty}$ of finite subsets of $\mathbb{N}$ with $\max A_n < \min A_{n+1}$ and $\left\|\sum_{i \in A_n} x_i\right\|_X \geq \varepsilon$ for each $n \in \mathbb{N}$.

Problem 5.6.43. Let $K = 2$ if $\mathbb{K} = \mathbb{R}$ and $K = 4$ if $\mathbb{K} = \mathbb{C}$.

(0) **Lemma 0.** If S is a finite set and $\{t_n : n \in S\} \subset \mathbb{K}$, then

$$\left|\sum_{n \in S} t_n\right| \;\leq\; \sum_{n \in S} |t_n| \;\leq\; K \sup_{F \subset S} \left|\sum_{n \in F} t_n\right| \;.$$

Tell me this Lemma is trivial or provide a proof of this trivial Lemma.

(a) Let S be a finite set, $\{x_n : n \in S\}$ be a (finite) subset of a normed space X, and $\{a_n : n \in S\} \subset \mathbb{K}$. Then

$$\left\|\sum_{n \in S} a_n x_n\right\| \;\leq\; K \;\|(a_n)\|_{\ell^\infty(S)} \;\sup_{F \subset S} \left\|\sum_{n \in F} x_n\right\| \;.$$

Hint: Hahn-Banach Thm, Corollary 3.1.7.

(b) Show that, in Theorem 5.4.0.4, condition (5) implies condition (4).

(c) Let $\sum_{n=1}^{\infty} x_n$ be a series in a Banach space X having condition (5) of Thm. 5.4.0.4. Show that the map $T : \ell^\infty \to X$ given by

$$T\left((a_n)_{n=1}^{\infty}\right) := \sum_{n=1}^{\infty} a_n x_n$$

is a bounded linear operator.

The purpose of Problems 44–45 is to show Theorem 5.4.17.

Problem 5.6.44. Let H be a separable Hilbert space.

(a) $\{e_1,\dots,e_N\}\subset H$ is a O.N. system IFF $\{e_1,\dots,e_n\}$ is normalized and

$$\left\|\sum_{k=1}^{n}\alpha_k e_k\right\|_H \;\le\; \left\|\sum_{k=1}^{m}\alpha_k e_k\right\|_H \qquad (*)$$

for all choices (α_k) of scalars and $1\le n<m\le N$.
Hint: Pythagoras' Theorem and Lemma 5.3.9.

(b) Show part (1) of Theorem 5.4.17.
Hint: You may use, without proving, Problem 4.8.19-3.

Problem 5.6.45.

(a) Let $\{e_1,e_2\}$ be an O.N. basis for a two-dimensional Hilbert space H. Fix $0\ne\beta\in\mathbb{K}$ and consider $\{x_1,x_2\}$ where $x_1:=e_1$ and $x_2:=e_1+\beta e_2$. Express the basis constant of $\{x_1,x_2\}$ in terms of β. [61]

(b) Show part (2) of Theorem 5.4.17.

Back to Dirk's book

Problem 5.6.24. Fix $1<p\le\infty$. Give an example of a sequence in Banach space ℓ_p that is unconditionally convergent but not absolutely convergent.

Problem 5.6.27. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of pairwise orthogonal elements from a Hilbert space H. Then the following conditions are equivalent.

(i) $\sum_{i=1}^{\infty}x_n$ is convergent. (convergent in H)

(ii) $\sum_{i=1}^{\infty}\|x_n\|^2$ is convergent. (convergent in $\mathbb{R}$)

(iii) $\sum_{i=1}^{\infty}x_n$ is weakly convergent.
(ie. the sequence $(\sum_{k=1}^{n}x_k)_{n\in\mathbb{N}}$ of partial sums is weakly convergent)

Problem 5.6.28. Let $\{x_1,\dots x_n\}$ be a orthonormal system in a Hilbert space H and $x\in H$. Find scalars $\alpha_1,\dots,\alpha_n$ so that $\|x-\sum_{k=1}^{\infty}\alpha_k x_k\|_H$ is minimal.

Problem 5.6.32. Let H be a Hilbert space. For $T\in L(H)$ T.F.A.E. .

(i) T is normal.

(ii) $\|Tx\| \;=\; \|T^*x\| \qquad \forall x\in H.$

---

[61] Note that (x_1,x_2) and $\left(x_1,\frac{x_2}{\|x_2\|}\right)$ have the same basis constant.

Problem 5.6.33. Fix $k \in L^2\left([0,1]^2\right)$. Let

$$T_k\colon L^2[0.1] \to L^2[0,1]$$

be, as usual, be be integral operator

$$(T_k x)(s) = \int_0^1 k(s,t)\, x(t)\, dt\ .$$

Find[62] conditions on the kernel k so that T_k is normal.

5.7 Remarks and Overviews

[62] necessary nontrival - support with your ideas

Chapter 8
Locally Convex Spaces

8.1 Definition of Locally Convex Spaces; Examples

Thus far we have considered vector spaces in which the convergence of a sequence was defined through a (semi)norm, e.g. uniform convergence in $C[0,1]$ through the supremum norm. The concept of pointwise convergence is subordinate in this system. However, one can take the following standpoint. For $t \in [0,1]$ and a function $x\colon [0,1] \to \mathbb{C}$, set

$$p_t(x) = |x(t)| ,$$

thereby denoting "(x_n) converges pointwise to 0" by nothing more than

$$p_t(x_n) \to 0 \qquad \forall t .$$

The essential observation now is that the p_t's are seminorms[1] and that pointwise convergence is detected through not just one sole seminorm but rather a collection of several seminorms. We will, in what is to follow, have to deal wtih a vector space X and families of seminorms on X (instead of a single norm).

From these ingredients will be built a theory of locally convex spaces, whose elementary building blocks will parallel, in many aspects, those from the theory of normed spaces. The theory of locally convex spaces calls for, indeed, a rudimentary acquaintance of the principles (or at the minimum the vocabulary) of topological spaces, which are presented in Appendix B.

Now comes the definition of locally convex spaces. Let X be a vector space and P be a set of seminorms on X, which, by convension, are denoted by p rather than p_t. For a finite subset F from P and $\varepsilon > 0$, let

$$U_{F,\varepsilon} := \{x \in X\colon\ p(x) \le \varepsilon \ \ \forall p \in F\} ;$$

then consider

$$\mathscr{U} := \{U_{F,\varepsilon} \ : \ \text{finite } F \subset P ,\ \varepsilon > 0\} . \qquad (8.1\text{-}1)$$

[1] $p(\lambda x) = |\lambda|\, p(x)$ and $p(x+y) \le p(x) + p(y)$

The system $\mathscr{U}$ is a substitute for the collection of all balls $\{x\colon \|x\| \le \varepsilon\}$ in a norm space. The system $\mathscr{U}$ has the following properties.

(1) $0 \in U$ for each $U \in \mathscr{U}$.

(2) For $U_1, U_2 \in \mathscr{U}$ $\exists\, U \in \mathscr{U}$ with $U \subset U_1 \cap U_2$ ("$\mathscr{U}$ is downward filtrated"), since $U_{F_1 \cup F_2, \min(\varepsilon_1, \varepsilon_2)} \subset U_{F_1, \varepsilon_1} \cap U_{F_2 . \varepsilon_2}$.

(3) For $U \in \mathscr{U}$ there is $V \in \mathscr{U}$ with[2] $V + V \subset U$; namely, $U_{F,\varepsilon/2} + U_{F,\varepsilon/2} \subset U_{F,\varepsilon}$.

(4) Each $U \in \mathscr{U}$ is absorbing[3] since $x_0 \in \lambda U_{F,\varepsilon}$ if $\lambda > \frac{1}{\varepsilon} \cdot \max_{p \in F} p(x_0)$.

(5) For $U \in \mathscr{U}$ and $\lambda > 0$ there exists $V \in \mathscr{U}$ with $\lambda V \subset U$. Namely, $\lambda U_{F,\varepsilon/\lambda} \subset U_{F,\varepsilon}$, even have "=".

We need the following definition.

Definition 8.1.1. Let A be a subset of a vector space.

(a) A is *circled* provided

$$\{\lambda\colon |\lambda| \le 1\} \cdot A \subset A\ .$$

(b) A is *absolutely convex* provided A is convex and circled.

One can easily show that A is absolutely convex if and only if

$$x, y \in A\ ,\ |\lambda| + |\mu| \le 1 \quad \Rightarrow \quad \lambda x + \mu y \in A\ .$$

In particular, for our system $\mathscr{U}$ we have the following.

(6) Each $U \in \mathscr{U}$ is circled.

(7) Each $U \in \mathscr{U}$ is absolutely convex.

We will, with the help of properties (1)–(6), define a vector space topology with respect to which the algebraic operations of addition and scalar multiplication are continuous. The critical step, namely to obtain a generalized __________ theorem for this topology, requires property (7).

mgRmk As before, let P be a set of seminorms on a vector space X. But now let

$$\begin{aligned} \mathscr{V} &:= \{V_{F,\varepsilon}\ \ :\ \ \text{finite } F \subset P\ ,\ \varepsilon > 0\} \\ V_{F,\varepsilon} &:= \{x \in X\colon\ p(x) < \varepsilon\ \ \forall p \in F\} \end{aligned} \tag{8.1-2}$$

Then the system $\mathscr{V}$ also satisfies properties (1)–(7). Furthermore

$$V_{F,\varepsilon} \subset U_{F,\varepsilon} \subset V_{F,17\varepsilon} \qquad \forall \text{ finite } F \subset P \text{ and } \varepsilon > 0\ .$$

[2] Here we use the suggestive notation
$A + B := \{a + b\colon a \in A\ ,\ b \in B\}$ and $\Lambda A := \{\lambda a\colon \lambda \in \Lambda, a \in A\}$.
Beware: in general $A + A \ne 2A$! (Example?)

[3] Definition 3.2.1: U is *absorbing* IFF $\forall x \in X\ \exists \lambda > 0$ s.t. $x \in \lambda U$.

First let $\mathscr{U}$ be any (nonempty) system of sets with properties (1)–(6).[4] We define a topology on X in the following way.

$$O \subset X \quad \text{is open} \qquad \Longleftrightarrow \qquad \forall x \in O \;\; \exists U \in \mathscr{U} \;\; \text{s.t.} \;\; x + U \subset O\,.^5$$

It still needs to be verified that this does indeed define a topology:

- $\emptyset$ and X are open (clear!)
- Let O_1 and O_2 be open and $x \in O_1 \cap O_2$. Then there exists $U_1, U_2 \in \mathscr{U}$ with $x + U_i \subset O_i$. Using (2) choose $U \in \mathscr{U}$ with $U \subset U_1 \cap U_2$. Then $x + U \subset (x + U_1) \cap (x + U_2) \subset O_1 \cap O_2$. So $O_1 \cap O_2$ is open.
- Let O_i be open for each i in some index set I and $x \in \cup_{i \in I} O_i$, say, $x \in O_{i_0}$. Then there is a $U \in \mathscr{U}$ with $x + U \subset O_{i_0} \subset \cup_{i \in I} O_i$. Thus $\cup_{i \in I} O_i$ is open.

By construction, $\mathscr{U}$ is a nhood base at zero (i.e. each nhood of zero contains a $U \in \mathscr{U}$ and each $U \in \mathscr{U}$ is a nhood of zero). Furthermore, by construction, for each fixed $x \in X$, the map $y \mapsto x + y$ is continuous, it is even a *homeomorphism*[6] as it's inverse is given by $y \mapsto -x + y$. Even more is true: not just continuity in each variable separably but also joint continuity.[7]

Lemma 8.1.2. *Each of the following maps is continuous.*

(a) *Addition:* $X \times X \to X$ *given by* $(x, y) \mapsto x + y$.

(b) *Scalar multiplication:* $\mathbb{K} \times X \to X$ *given by* $(\lambda, x) \mapsto \lambda x$.

Here, the topology on X *is the above described topology generated by* $\mathscr{U}$ *while* $X \times X$ *and* $\mathbb{K} \times X$ *each carry the product topology.*

Definition 8.1.3. Let τ be a topology on a vector space X.

(a) If addition and scalar multiplication are continuous w.r.t. τ, then (X, τ) is a *topological vector space*.

(b) Let P be a set of seminorms on X and τ be the above described topology having of a nhood base at zero

$$\mathscr{U} := \{U_{F,\varepsilon} \;\; : \;\; \text{finite } F \subset P\,,\; \varepsilon > 0\}$$

where

$$U_{F,\varepsilon} := \{x \in X : \; p(x) \le \varepsilon \;\; \forall p \in F\}\;.$$

Then (X, τ) is a *locally convex* topological vector space.[8]

---

[4] These properties are partly redundant, i.e. (1) follows from (4).

[5] Note that the system $\mathscr{U}$ from (8.1-1) and the system $\mathscr{V}$ from (8.1-2) generates precisely the same open sets.

[6] continuous function with a continuous inverse

[7] compare with Problem 4.8.12 and Lemma 5.1.6

[8] Or, for short: locally convex space. Even shorter: LCTVsp.

By Lemma 8.1.2, a locally convex space is necessarily a topological vector space. It should be emphatically observed that not every vector space that is equipped with a topology is automatically a topological vector space. Indeed, in an arbitrary vector space X equipped with the discrete topology (in which each set is open), scalar multiplication is not continuous. (Otherwise $\lim_{n\to\infty}\frac{1}{n}x = 0$ for each $x \neq 0$ but, with the discrete topology, the only convergent sequences are those that are eventually constant.) Although there exists non-LCTVsp, almost all topologies on vector spaces that are important in applications are locally convex.

Example a. Let T be a set and X be a vector space of functions on T (e.g. $T = \mathbb{R}^n$ and $X = C^b(\mathbb{R}^n)$.) Consider the seminorms

$$p_t(x) \;=\; |x(t)| \quad \text{for } t \in T\ .$$

Then the family $P = \{p_t\colon t \in T\}$ generate a locally convex topology called the *topology of pointwise convergence*.

Example b. Let T be a topological space and X be a vector space of continuous functions on T. Consider the seminorms

$$p_K(x) \;=\; \sup_{t\in K}|x(t)| \quad \text{for } K \subset T \text{ compact}\ .$$

Then $P = \{p_K\colon K \subset T$ compact) generates a locally convex topology called the *topology of uniform convergence on compact sets*.

Example c. Let $X = C^\infty(\mathbb{R})$ and

$$p_{K,m}(x) \;=\; \sup_{t\in K}\left|x^{(m)}(t)\right| \quad K \subset \mathbb{R} \text{ compact}\ , m \geq 0\ \ .$$

Then $P = \{p_{K,m}\colon K \subset \mathbb{R} \text{ compact}\ , m \in \mathbb{N}_0\}$ generates a locally convex topology on X. Endowed with this topology, $C^\infty(\mathbb{R})$ is usually written as $\mathscr{E}(\mathbb{R})$. Similarly, one defines a topology for $\mathscr{E}(\mathbb{R}^n)$ and for $\mathscr{E}(\Omega)$ for open $\Omega \subset \mathbb{R}^n$.

Example d. The Schwartz space $\mathscr{S}(\mathbb{R}^n)$, which was introduced in Definition 5.2.3, can be topologized by the seminorms

$$p_{\alpha,m}(\varphi) = \sup_{x\in\mathbb{R}^n}\left(1+|x|^m\right)|(D^\alpha\varphi)(x)|$$

for multi-indices $\alpha \in \mathbb{N}_0^n$ and $m \in \mathbb{N}_0$.

Example e. Let $K \subset \Omega \subset \mathbb{R}^n$ with K compact and Ω open. Let

$$\mathscr{D}_K(\Omega) := \{\varphi\colon \Omega \to \mathbb{C} : \varphi \in C^\infty(\Omega)\ ,\ \operatorname{supp}(\varphi) \subset K\}\ .$$

(The existence of such a function was shown in Lemma 5.1.10). $\mathscr{D}_K(\Omega)$ can be topologized by the seminorms, for multi-indices $\alpha \in \mathbb{N}_0^n$,

$$p_\alpha(\varphi) \;=\; \sup_{x\in\Omega}|(D^\alpha\varphi)(x)|\ .$$

Example f. The just now described topology can also be considered on the space[9] of test functions $\mathscr{D}(\Omega)$; however, for various reasons this is not appropriate. Among other reasons, one wants to consider a topology on $\mathscr{D}(\Omega)$ that reflects the fact that $\mathscr{D}(\Omega) = \cup_K \mathscr{D}_K(\Omega)$, where the union is taken over all compact subsets K of Ω. One such locally convex topology can be defined as follows. Let τ_K be the topology on $\mathscr{D}_K(\Omega)$. Let P be the set of all seminorms p on $\mathscr{D}(\Omega)$ for which all restrictions $p\,|_{\mathscr{D}_K}$ are continuous with respect to τ_K. One considers on $\mathscr{D}(\Omega)$ the locally convex topology generated by P.

Examples (c)–(f) are fundamental for Distribution theory (Section 8.5).

Example g. Let p be a norm on a vector space X. Then the locally convex topology generated by $P = \{p\}$ is the norm topology on X.

Example h. Let X be a normed space. Consider the seminorms

$$p_{x'}(x) = \left|x'(x)\right| \qquad \text{for } x' \in X' .$$

These seminorms generate the *weak topology* $\sigma(X, X')$ on X. The weak topology is almost always differenct from the norm topology. (More on this in Section 8.3).

Example i. On the dual space X' of a normed space X, the seminorms

$$p_x\left(x'\right) = \left|x'(x)\right| \qquad \text{for } x \in X$$

define the *weak* topology* $\sigma(X', X)$, which varies from the norm topology as well as the weak topology $\sigma(X', X'')$ (also for this see Section 8.3).

Example j. On the space $L(X,Y)$, where X and Y are normed spaces, two topologies other than the norm topology are of interest: the *strong operator topology* (sot), which is generated by the seminorms

$$p_x(T) = \|Tx\|_Y \qquad \text{for } x \in X ,$$

as well as the *weak operator topology* (wot), which is generated by the seminorms

$$p_{x,y'}(T) = \left|y'(Tx)\right| \qquad \text{for } x \in X , \ y' \in Y' .$$

(These topologies implicitly appeared in Chapter 7 in the discussion of spectral measures, cf. Theorem 7.1.6(d).)

Example k. In probability theory, on the space $M(\mathbb{R})$ of all finite signed measures, an important locally convex topology is the one generated by the seminorms

$$p_f(\mu) = \left|\int_{\mathbb{R}} f\, d\mu\right| \qquad \text{for } f \in C^b(\mathbb{R}) .$$

In probability theory it is likewise known as the *weak topology*; however, it is different from the functional analytic weak topology from Example (h).

[9] Recall Def. 5.1.9 of C^∞ functions with compact support i.e. test functions:
$\mathscr{D}(\Omega) := \left\{\varphi \in C^\infty(\Omega) : \operatorname{supp}(\varphi) := \overline{\{x : \varphi(x) \neq 0\}}^{\mathbb{R}^n} \subset \Omega \ \text{ is compact}\right\}$.

Example l. $P = \{0\}$ generates, on each vector space X, the *indiscrete* (or chaotic) topology in which the only open sets are $\emptyset$ and X.

The last example shows, in a dramatic way, that a locally convex space need not be Hausdorff (i.e. distinct points belong to disjoint nhoods). This property is easy to characterize for locally convex spaces.

Lemma 8.1.4. *Let τ be the locally convex topology generated by a family P of seminorms on a vector space X. T.F.A.E. .*

(i) (X,τ) *is Hausdorff.*

(ii) *For each $x \neq 0$ there exists a $p \in P$ such that $p(x) \neq 0$.*[10]

(iii) *There exists a nhood base $\mathscr{U}$ of zero such that $\cap_{U\in\mathscr{U}} U = \{0\}$.*

If (ii) holds, the the $\mathscr{U}$ in (iii) can be taken to be the $\mathscr{U}$ in (8.1-1).

The lemma shows that all the above Examples (except for (l)) are Hausdorff spaces; indeed, this follows for (h) (and (j)) from the Hahn-Banach theorem while for (k) from the regularity of μ. For Example (f), we need to entertain still more details.

The next theorem explains why locally convex spaces are called locally convex.

Theorem 8.1.5. *Let (X,τ) be a topological vector space. T.F.A.E. .*

(1) *X is locally convex.*

(2) *τ has a nhood base at zero of the form $\mathscr{U} = \{U_{F,\varepsilon} \colon$ finite $F \subset P$, $\varepsilon > 0\}$ for some family P of seminorms on X.* [11]

(3) *τ has a nhood base $\mathscr{A}$ at zero consisting of sets that are absolutely convex and absorbing.*

Clearly (1) and (2) are equivalent since (2) is just the definition of LCTVsp in Definition 8.1.3.
If (2) holds, then in (3) one can take $\mathscr{A}$ to be $\mathscr{U}$.
If (3) holds, then in (2) the family P of seminorms on X can be taken to be

$$P = \{p_A \colon A \in \mathscr{A}\}$$

where $p_A \colon X \to [0,\infty)$ is the Minkowski functional introduced in Definition 3.2.1 by

$$p_A(x) := \inf\{\lambda > 0 \colon x \in \lambda A\} \ .$$

Theorem 8.1.5 states that locally convex spaces can be described geometrically (through specifications on the nhood base at zero) or analytically (via seminorms). We will follow largely the analytic approach.

mgRMK: Let (X,τ) be a topological vector space, $x_0 \in X$, $0 \neq \lambda_0 \in \mathbb{K}$, and $A \subset X$.
∘ If A is open (resp. closed), then $x_0 + A$ is open (resp. closed). Indeed, $x_0 + A = f^{-1}(A)$ where $X \in x \overset{f}{\mapsto} x - x_0 \in X$.
∘ If A is open (resp. closed), then $\lambda_0 A$ is open (resp. closed). Indeed, $\lambda_0 A = g^{-1}(A)$ where $X \in x \overset{g}{\mapsto} x/\lambda_0 \in X$.
Also, for a seminorm p on a vector space X and $\varepsilon > 0$:
$\{x \in X \colon p(x) \leq \varepsilon\} = \varepsilon\{x \in X \colon p(x) \leq 1\}$ and $\{x \in X \colon p(x) < \varepsilon\} = \varepsilon\{x \in X \colon p(x) < 1\}$.

[10] Such a family P is said to *separate points of X*.

[11] $U_{F,\varepsilon} := \{x \in X \colon p(x) \leq \varepsilon \ \ \forall p \in F\}$.

8.2 Continuous Functionals and the Hahn Banach Theorem

To even a higher degree than for normed spaces, knowledge about the continuous linear functionals on locally convex spaces is important. In fact, it is precisely the existence of *convex* nhoods of zero that will guarantee the existence of continuous functionals in the Hahn Banach theorem.

The following lemma is the linchpin[12] of our investigation. [13]

Lemma 8.2.1. *Let the family P of seminorms generate the LCTVsp (X,τ).*

(a) *For a seminorm $q\colon X \to [0,\infty)$, T.F.A.E. .*

- *(i)* *q is continuous.*
- *(ii)* *q is continuous at zero.*
- *(iii)* *$\{x \in X : q(x) \leq 1\}$ is a nbood of zero.*

(a′) *For a seminorm $q\colon X \to [0,\infty)$, T.F.A.E. .* [14]

- *(1)* *q is continuous.*
- *(2)* *$\{x \in X : q(x) < 1\}$ is open.*
- *(3)* *$0 \in \operatorname{int}\{x \in X : q(x) < 1\}$.*
- *(4)* *$0 \in \operatorname{int}\{x \in X : q(x) \leq 1\}$.* [15]
- *(5)* *q is continuous at zero.*
- *(6)* *$\exists$ a continuous seminorm d on X such that $q \leq d$.* [16]

(b) *Each $p \in P$ is continuous.*

(c) *A seminorm q is continuous IFF[17]*
there exists $M \geq 0$ and a finite subset $F \subset P$ with

$$q(x) \;\leq\; M \max_{p\in F} p(x) \qquad \forall x \in X \,.$$

Corollary 8.2.2. *Let the family P of seminorms generate the LCTVsp (X,τ). Let*

$$P \;\subset\; Q \subset \;\{q\colon\;\; q \text{ is a } \tau\text{-continuous seminorm}\}\;.$$

Then Q likewise generates the topology τ.

Y'al just could not wait so here it is .... reduce convergence in τ_p to convergence in $\mathbb{R}$.

---

[12] something that holds the various elements of a complicated structure together

[13] Keep in mind: the topology on a *topological vector space* looks the same at each point since addition and scalar multiplication are continuous functions.

[14] An expansion of Dirk's (a). Taken from Conway's *Functional Analysis*, 2nd ed., pg 100.

[15] This condition is just saying that $\{x \in X : q(x) \leq 1\}$ is a nhood of zero.

[16] i.e. $q(x) \leq d(x) \quad \forall x \in X$.

[17] You should be expecting some kind of boundedness condition here.

For LCTVsp theory the following result is very important; it shows that the terminology in Examples 8.1(a) and (b) are correctly chosen.

Theorem 8.2.6. *Let (X,τ_P) be a LCTVsp.*
A net $(x_i)_{i\in I}$ in X converges to $x \in X$ if and only if $\lim_i p(x_i - x) = 0$ for each $p \in P$.

We now examine linear mappings. The locally convex topology generated by a family P of seminorms will be denoted by τ_p.

mgRMK: Note that if X and Y are vector spaces and

$$q \circ T : X \xrightarrow{T} Y \xrightarrow{q} [0,\infty)$$

with T linear and q a seminorm, then $q \circ T$ is a seminorm.

Theorem 8.2.3. *Let (X,τ_p) and (Y,τ_Q) be two LCTVsp's and $T\colon X \to Y$ be linear. Then T.F.A.E. .*

(i) T *is continuous.*
(ii) T *is continuous at* 0.
(iii) *A continuous seminorm q on Y gives a continuous seminorm $q \circ T$ on X.*
(iv) *For each $q \in Q$ there exists a finite $F \subset P$ and $M \geq 0$ with*

$$q(Tx) \leq M \max_{p\in F} p(x) \qquad \forall x \in X .$$

Note the analogy between (iv) and the formula $\|Tx\| \leq M\|x\|$.[18] We explicitly note another special case of Theorem 8.2.3.[19]

Corollary 8.2.4. *Let (X,τ_P) be a LCTVsp. A linear map $\ell\colon X \to \mathbb{K}$ is continuous **IFF** there are finitely many $p_1,\dots,p_n \in P$ and $M \geq 0$ with*

$$|\ell(x)| \leq M \max_{i=1,\dots,n} p_i(x) \qquad \forall x \in X .$$

As with normed spaces, one can now introduce the dual space.

Definition 8.2.5.

- The set of continuous linear functionals[20] on a LCTVsp (X,τ) is the *dual space* of X and is denoted by X', or, to indicate the dependence on the topology τ, by $(X_\tau)'$ or $(X,\tau)'$.
- $L(X,Y)$, or $L(X_{\tau_1},Y_{\tau_2})$, denotes the set of continuous linear maps from a LCTVsp (X,τ_1) to a LCTVsp (Y,τ_2).
- So $(X_\tau)' = L\left(X_\tau, \mathbb{K}_{|\cdot|}\right)$.

[18] Furthermore, note that if X and Y are normed spaces and P (resp. Q) consists of just one seminorm $\|\cdot\|_P$ (resp. $\|\cdot\|_Q$), then (iv) is precisely this formula-we-know-so-well.

[19] Compare this corollary to characterizations of continuity of linear functionals on normed spaces.

[20] so functions from X to $\mathbb{K}$

From Theorem 8.2.3 it follows immediately that $L(X,Y)$ and X' are each vector spaces in their own right. We will address the possibility to define appropriate locally convex topologies on X' in Section 8.3.

Another interesting special case of Theorem 8.2.3 is $X = Y$ and $T = \mathrm{Id}$ when there are two locally convex topologies on X. (One can think about, e.g., a normed space X that can be endowed with its norm topology and also its weak topology $\sigma(X,X')$ (c.f. Example 8.1(h)). It is often important to compare topologies.

Let τ_1 and τ_2 be locally convex topologies on X. τ_1 is called *finer* than τ_2 (and τ_2 *coarser* than τ_1) provided each τ_2-open set is also open with respect to τ_1, thus when $\tau_2 \subset \tau_1$. Another equivalent formulation is that

$$\mathrm{Id} \in L(X_{\tau_1}, X_{\tau_2}) \ ;$$

Theorem 8.2.3 gives an easy criterion to verify this. Naturally the two topologies agree when $\mathrm{Id} \in L(X_{\tau_1}, X_{\tau_2})$ and $\mathrm{Id} \in L(X_{\tau_2}, X_{\tau_1})$.

If τ_1 is finer than τ_2, i.e $\tau_2 \subset \tau_1$, then τ_1 has:

- more open sets,
- more closed sets, but
- fewer compact sets,
- more continuous maps into $\mathbb{K}$ (or into any other topological space),
- fewer convergent sequences (since τ_1-convergence implies τ_2-convergence).

(This list naturally applies to not only LCTVsp but also to general topological spaces.)

Example. On $X = C(\mathbb{R}^n)$, consider the topology τ_1 of pointwise convergence and the topology τ_2 of uniform convergence on compact sets (Examples 8.1(a) and (b)). Then τ_2 is finer than τ_1, i.e.

$$\text{topology of ptwse conv.} \quad \subset \quad \text{topology of uniform conv. on compact sets} \ ,$$

as one can immediately verify the criterion in Theorem 8.2.3(iv) for $\mathrm{Id}\colon X_{\tau_2} \to X_{\tau_1}$:

$$p_t(x) \ \le \ 1 \ p_{\{t\}}(x) \ .$$

(Notation is as in Examples 8.1(a) and (b); naturally equality actually holds.)

Another example: the norm topology on a normed space is always finer than the weak[21] topology $\sigma(X,X')$, i.e.

$$\sigma\left(X,X'\right) \ \subset \ \tau_{\|\cdot\|} \ ,$$

since

$$p_{x'}(x) \ = \ \left|x'(x)\right| \ \le \ \left\|x'\right\| \ \|x\| \qquad \forall x \in X \ , \ x' \in X' \ ,$$

and so $\mathrm{Id} \in L\left(X_{\|\cdot\|}, X_{\sigma(X,X')}\right)$.

[21] Example 8.1(h)

Next, convergence in locally convex spaces will be shortly discussed; for the definition of convergence in a topological space see Appendix B.2. There it was noted that, in general, sequences do not suffice in characterizing topological concepts. To illustrate this, we now give an example that traces back to von Neumann.

Example. Consider $X = \ell^2$ with the weak topology $\sigma(X, X')$. [22] Let

$$A := \{e_m + me_n \colon 1 \le m < n\} \ ,$$

where naturally e_n is the n^{th} standard unit vector of ℓ^2. We will show that $0 \in \overline{A}$ but no sequence in A converges to 0. Pictorially, one can imagine A:

$$\begin{array}{lllll}
e_1+e_2 & & & & \\
e_1+e_3 & e_2+2e_3 & & & \\
e_1+e_4 & e_2+2e_4 & e_3+3e_4 & & \\
e_1+e_5 & e_2+2e_5 & e_3+3e_5 & e_4+4e_5 & \\
\vdots & \vdots & \vdots & \vdots & \dots \ .
\end{array}$$

To show $0 \in \overline{A}$, we need to show that $U \cap A \neq \emptyset$ for each nhood U of 0. WLOG, let $U = \{x \in \ell^2 \colon |\langle x, y_i\rangle| \le \varepsilon \text{ for each } i = 1, \dots, r\}$ where $F := \{y_1, \dots, y_r\} \subset \ell_2$. Again, pictorially:

$$\begin{aligned}
y_1 &= (\ y_1(1)\ ,\ y_1(2)\ ,\ y_1(3)\ ,\ \dots\ y_1(m)\ ,\ \dots\ ,\ y_1(n)\ \dots\ ) \\
y_2 &= (\ y_2(1)\ ,\ y_2(2)\ ,\ y_2(3)\ ,\ \dots\ y_2(m)\ ,\ \dots\ ,\ y_2(n)\ \dots\ ) \\
&\vdots \\
y_r &= (\ y_r(1)\ ,\ y_r(2)\ ,\ y_r(3)\ ,\ \dots\ y_r(m)\ ,\ \dots\ ,\ y_r(n)\ \dots\ )\ .
\end{aligned}$$

Note that for each $y_i \in F$

$$|\langle e_m + me_n, y_i\rangle| \ \le\ |y_i(m)| + m|y_i(n)|\ .$$

Choose m so big that $|y_i(m)| \le \frac{\varepsilon}{2}$ for each $y_i \in F$; subsequently, choose $n > m$ satisfying $|y_i(n)| \le \frac{\varepsilon}{2m}$ for each $y_i \in F$. Then $e_m + me_n \in U \cap A$. Thus $0 \in \overline{A}$.

Assume that there exists a weakly null sequence[23] $(e_{m_k} + m_k e_{n_k})_{k\in\mathbb{N}}$ in A. Then by Theorem 8.2.6, for each $y \in \ell_2$,

$$p_y\left(e_{m_k} + m_k e_{n_k}\right) \ =\ \left|\langle e_{m_k} + m_k e_{n_k}\ ,\ y\rangle\right| \ =\ |y(m_k) + m_k y(n_k)| \ \stackrel{k\to\infty}{\longrightarrow}\ 0\ . \tag{8.2}$$

Possibility 1: $(m_k)_{k\in\mathbb{N}}$ is bounded.
Then there exist $l \in \mathbb{N}$ such that $l = m_k$ for infinitely many k. (Thus there is a subsequence of $(e_{m_k} + m_k e_{n_k})_{k\in\mathbb{N}}$ that comes from the " l^{th} column of A".) This produces a contradiction to (8.2) by taking $y = e_l \in \ell^2$ since if $m_k = l$ then $y(m_k) + m_k y(n_k) \ge y(m_k) = 1$.
Possibility 2: $(m_k)_{k\in\mathbb{N}}$ is unbounded.
Wlog (use $m_k < n_k$ and pass to a subsequence) $(n_k)_{k\in\mathbb{N}}$ and $(m_k)_{k\in\mathbb{N}}$ are each strictly montone increasing. This produces a contradiction to (8.2) by taking $y \in \ell^2$ given by $y(j) := \frac{1}{k}\delta_{j,n_k}$ since $y(m_k) + m_k y(n_k) \ge m_k y(n_k) = \frac{m_k}{k} \ge 1$.
Thus: A does not have a weakly null sequence. [24]

---

[22] Recall that the weak topology $\sigma(\ell^2, (\ell^2)')$ on ℓ_2 is generated by the family $P = \{p_y \colon y \in \ell^2\}$ of seminorms on ℓ_2 where $p_y(\cdot) := |\langle \cdot\ ,\ y\rangle|$. Keep in mind: everything in this example is with respect to the weak topology and not with respect to the norm topology.

[23] i.e. a sequence converging to zero in the weak topology

[24] But there must be a weakly null net in A, cf. Problem 8.6.43.

For LCTVsp theory the following result is very important; it shows that the terminology in Examples 8.1(a) and (b) are correctly chosen.

Theorem 8.2.6. *Let (X, τ_P) be a LCTVsp.*
A net $(x_i)_{i\in I}$ in X converges to $x \in X$ if and only if $\lim_i p(x_i - x) = 0$ for each $p \in P$.

Here is an illustration of net technique.

Theorem 8.2.7. *In a LCTVsp,*

(1) the closure of a convex set is convex ,

(2) the closure of an absolute convex set is absolutely convex.

Naturally one can also prove Theorem 8.2.7, without using nets, directly from the Definition 8.1.3 of LCTVsp's (please see Problem 8.6.1). However, this proof using nets goes much smoothly faster.

Now follows yet a few more examples for continuous functionals.

Example (a) Consider the Schwartz class [25] $\mathscr{S}(\mathbb{R}^n)$. Elements of its dual space, which is denoted by $\mathscr{S}'(\mathbb{R}^n)$, are called *tempered distributions* and will be studied in detail in Section 8.5. We show here that $L^P(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$ for $1 \le p \le \infty$, more precisely, for $f \in L^P(\mathbb{R}^n)$, the map

$$T_f : (\mathscr{S}(\mathbb{R}^n) , \tau_P) \to \mathbb{K}$$

given by

$$T_f(\varphi) := \int_{\mathbb{R}^n} f(x)\,\varphi(x)\,dx \qquad \forall \varphi \in \mathscr{S}(\mathbb{R}^n)$$

defines a functional $T_f \in \mathscr{S}'(\mathbb{R}^n)$. By Hölder's Inequality, for $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, keeping with the notation from Example 8.1(d),

$$\begin{aligned} \left|T_f(\varphi)\right| &\le \|f\|_{L_p} \, \|\varphi\|_{L_q} \\ &\le \|f\|_{L_p} \left(\int_{\mathbb{R}^n} \frac{dx}{\left(1+|x|^{n+1}\right)^q}\right)^{1/q} \sup_{x\in\mathbb{R}^n} \left(1+|x|^{n+1}\right) |\varphi(x)| \\ &= M \, p_{0,n+1}(\phi) \; ; \end{aligned}$$

the case $p = 1$ is handled similarly.

[25] Example 8.1(d).

Example (b) Let X be a normed space.

First let's calculate the dual space of the normed space X endowed with its weak topology $\sigma(X,X')$. Claim

$$\left(X \,,\, \sigma\left(X,X'\right) \right)' \;=\; \left(X \,,\, \|\cdot\| \right)' \,,$$

in other words, a linear functional is weakly continuous if and only if it is norm continuous. Proof: in class.

What about the dual space of X', endowed with the weak* topology[26]? Recall[27] that there is a natural isometric embedding $i\colon X \to X'' = \left(X'_{\|\cdot\|}\right)'$ given by point evaluation; specifically, $[i(x)](x') = x'(x)$. Corollary 8.3.4 will show that

$$\left(X' \,,\, \sigma\left(X',X\right) \right)' \;=\; i(X) \,,$$

i.e. the dual space of X', endowed with the weak* topology, consists precisely of the point evaluation functionals.

Example (c) Let X and Y be normed spaces. Endow $L(X,Y)$ with its strong operator topology[28] and denote this topologized operator space by $L_{\text{st}}(X,Y)$. By Corollary 8.2.4, a linear map

$$\Phi\colon L_{\text{st}}(X,Y) \to \mathbb{K}$$

is in $\left(L_{\text{st}}(X,Y)\right)'$ **IFF** there are finitely many $x_1,\ldots,x_n$ from X and $M \geq 0$ such that

$$|\Phi(T)| \;\leq\; M \max_{i=1,\ldots,n} \|Tx_i\|_Y \qquad \forall T \in L_{\text{st}}(X,Y) \,.$$

Claim: $\Phi \in \left(L_{\text{st}}(X,Y)\right)'$ if and only if Φ is of the form

$$\Phi(T) \;=\; \sum_{i=1}^{n} y_i'(Tx_i)$$

for some $n \in \mathbb{N}$ and $x_i \in X$ and $y_i' \in Y'$.
$\Leftarrow$ is clear by Corollary 8.2.4 since, for such a Φ of the above form,

$$|\Phi(T)| \;\leq\; \sum_{i=1}^{n} \left\|y_i'\right\|_{Y'} \|Tx_i\|_Y \;\leq\; \left[\sum_{i=1}^{n} \left\|y_i'\right\|_{Y'}\right] \max_{i=1,\ldots n} \|Tx_i\|_Y \,.$$

$\Rightarrow$ Let $\Phi \in \left(L_{\text{st}}(X,Y)\right)'$. Then there are $x_1,\ldots,x_n$ from X and $M \geq 0$ such that

$$|\Phi(T)| \;\leq\; M \max_{i=1,\ldots,n} \|Tx_i\|_Y \qquad \forall T \in L_{\text{st}}(X,Y) \,. \tag{8.3}$$

To be shown in class: there are $y_1',\ldots y_n'$ from Y' so that $\Phi(T) \;=\; \sum_{i=1}^{n} y_i'(Tx_i)$.

---

[26] Example 8.1(i)

[27] See Section 3.3

[28] Example 8.1(j). The SOT is generated by the seminorms $\{p_x\colon x \in X\}$ where $p_x(T) = \|Tx\|_Y$.

RECALL

Definition 1.1.1. Let X be a $\mathbb{K}$-vector space.
A mapping $p\colon X \to [0,\infty)$ is a *seminorm* provided

(a) $p(\lambda x) = |\lambda|\, p(x) \quad \forall \lambda \in \mathbb{K},\, x \in X$

(b) $p(x+y) \leq p(x) + p(y) \quad \forall x, y \in X$.

Definition 3.1.1. Let X be a $\mathbb{K}$-vector space.
A mapping $p\colon X \to \mathbb{R}$ is called *sublinear* provided

(a) $p(\lambda x) = \lambda p(x)$ for each $\lambda \geq 0$ and $x \in X$,

(b) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$.

Clearly a seminorm is sublinear.

Definition 3.2.1. Let A be a subset of a $\mathbb{K}$-vector space X.
The *Minkowski functional* $p_A\colon X \to [0,\infty]$ is defined by

$$p_A(x) := \inf\left\{\lambda > 0\colon \tfrac{x}{\lambda} \in A\right\} \;\equiv\; \{\lambda > 0\colon x \in \lambda A\} \ .$$

The set A is *absorbing* provided $p_A(x) < \infty$ for each $x \in X$.

Thus A is absorbing IFF $\forall x \in X \; \exists \lambda > 0$ s.t. $x \in \lambda A$.

Each of the above recalled notations is vector space notation. Next came the Lemma 3.2.2: a norm space lemma.

Lemma 3.2.2. *Let U be a convex subset of a normed space X such that $0 \in int\, U$.*

(a) U is absorbing. More precisely: if $N_\varepsilon(0) \subset U$ then $p_U(x) \leq \frac{1}{\varepsilon}\,\|x\| \; \forall x \in X$.

(b) p_U is sublinear.

(c) If U is open, then $U = p_U^{-1}([0,1))$.

We now want a vector space and LCTVsp version of Lemma 3.2.2.

Lemma 3.2.2.VspV. *Let A be an absorbing*[29] *subset of a $\mathbb{K}$-vector space X. Consider its Minkowski functional $p_A\colon X \to [0,\infty)$.*

(1) $p_A(x+y) \leq p_A(x) + p_A(y)$ for all $x, y \in X$ PROVIDED *A is convex.*

(2) $p_A(\lambda x) = \lambda p_A(x)$ for each $\lambda \geq 0$ and $x \in X$.

(3) $p_A(\lambda x) = |\lambda|\, p_A(x) \;\; \forall \lambda \in \mathbb{K},\, x \in X$ PROVIDED *$\lambda A \subset A \; \forall \lambda \in \mathbb{K}$ s.t. $|\lambda| = 1$.*

(4) $\{x \in X\colon p_A(x) < 1\} \subset A$ PROVIDED *$\lambda A \subset A \; \forall 0 < \lambda < 1$.*

(5) $A \subset \{x \in X\colon p_A(x) \leq 1\}$.

So the Minkowski functional of an absolutely convex (i.e. convex and circled) absorbing set is a seminorm.

Lemma 3.2.2.LCTVspV. *Let U be an open convex absorbing subset of a LCTVsp X. Then $\{x \in X\colon p_U(x) < 1) \;=\; U$.*

END RECALL

[29] Note. Absorbing is precisely what we need to assume about A to assure that its Minkowski functional p_A is finite-valued. Also, an absorbing set A must contain zero and $p_A(0) = 0$.

Now, the moment we have been waiting for . . . . .   Hahn-Banach theorems for LCTVsp's.

The proofs to extend the algebraic form of the Hahn-Banach theorem (Theorems 3.1.2 and 3.1.4) to the norm space case are (almost) verbatim to the proofs to extend to the LCTVsp cases when one just replaces *norm* with *continuous seminorm* as well as *ball* with *absolutely convex null nhood*.

Theorem 8.2.8 (Hahn-Banach Theorem: extension version).
Let U be a vector subspace of a LCTVsp X. Each $l \in U'$ has an extension $L \in X'$.

Remark. Let U be a vector subspace of a LCTVsp (X, τ_P) so P is a collection of seminorms generating τ_P. Then the relative topology $\{U \cap B \colon B \in \tau_P\}$ on U is a LCTVsp generated by the seminorms $\{p|_U \colon p \in P\}$.

The upcoming Hahn-Banach separation theorems build the core of LCTVsp theory. First we consider lemmata.[30]

Lemma 8.2.9. *Let W be an absolutely convex, absorbing null-nhood in a LCTVsp X. Then its Minkowski functional $p_W : X \to [0,\infty)$ is a continuous seminorm on X.*

Lemma 8.2.10. *Let V be a convex open subset of a LCTVsp X and $0 \notin V$. There exists $x' \in X'$ with $\quad Re\, x'(x) < 0 \quad \forall x \in V$.*

Theorem 8.2.11 (Hahn-Banach Theorem: Separation Version I). *(beefed-up)*
Let V_1 and V_2 be subsets of a LCTVsp X such that

(1) $V_1 \cap V_2 = \emptyset$

(2) V_1 and V_2 are convex

(3) V_1 is open.

Then there exists $x' \in X'$ and $\gamma \in \mathbb{R}$ with

$$Re\, x'(v_1) < \gamma \le Re\, x'(v_2) \quad \forall v_1 \in V_1 \ , \ v_2 \in V_2 \ .$$

Theorem 8.2.12 (Hahn-Banach Theorem: Separation Version II). *(beefed-up)*
Let V_1 and V_2 be subsets of a LCTVsp X such that

(1) $V_1 \cap V_2 = \emptyset$

(2) V_1 and V_2 are convex

(3) V_1 is closed and V_2 is compact.

Then there exists $x' \in X'$ and $\gamma_i \in \mathbb{R}$ with

$$Re\, x'(v_1) < \gamma_1 < \gamma_2 < Re\, x'(v_2) \quad \forall v_1 \in V_1 \ , \ v_2 \in V_2 \ .$$

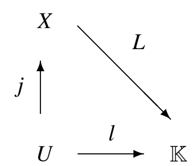
Corollary 8.2.13. *If X is a Hausdorff LCTVsp, then X' separates the points of X (*i.e. *for $x \neq y$ there exists $x' \in X'$ with $x'(x) \neq x'(y)$).*

---

[30] To get from the algebraic HB theorems to the normed spaced HB separation theorems, we used Lemma 3.2.2 and Lemma 3.2.3. Now we want to get from the algebraic HB theorems to the LCTVsp HB separation theorems.
The LCTVsp version of Lemma 3.2.2 (see previous page) is Lemma 8.2.9.
The LCTVsp version of Lemma 3.2.3 (see just below) is Lemma 8.2.10.
Lemma 3.2.3. Let V be a convex open subset of a normed space X and $0 \notin V$.
Then there exists $x' \in X'$ with $\quad \mathrm{Re}\, x'(x) < 0 \quad \forall x \in V$.

8.2 Section 8.2.0: Hahn Banach Reunion

Def. Let U be a vector subspace of a vector space X.
Then $L\colon X \to \mathbb{K}$ is an extension of $l\colon U \to \mathbb{K}$ provided $L\,|_U = l$.



So L is an extension of l IFF the above diagram commutes (where j is the injection map).

Theorem 3.1.2 (Hahn-Banach Theorem: $\mathbb{R}$-linear algebra version).
Let X be a vector space over $\mathbb{R}$.
Let U be a vector subspace of X. Also, let

$$\begin{aligned} \ell\colon U \to \mathbb{R} &\qquad \textit{be linear} \\ p\colon X \to \mathbb{R} &\qquad \textit{be sublinear} \end{aligned}$$

with

$$\ell(x) \le p(x) \quad \forall x \in U \ .$$

Then there exists a linear extension $L\colon X \to \mathbb{R}$ of l with

$$L(x) \le p(x) \quad \forall x \in X \ .$$

Theorem 3.1.4 (Hahn-Banach Theorem: $\mathbb{C}$-linear algebra version).
Let X be a vector space over $\mathbb{C}$.
Let U be a vector subspace X. Also, let

$$\begin{aligned} \ell\colon U \to \mathbb{C} &\qquad \textit{be linear} \\ p\colon X \to \mathbb{R} &\qquad \textit{be sublinear} \end{aligned}$$

with

$$Re\,\ell(x) \le p(x) \quad \forall x \in U \ .$$

Then there exists a linear extension $L\colon X \to \mathbb{C}$, of l with

$$Re\,L(x) \le p(x) \quad \forall x \in X \ .$$

Theorem 3.1.5 (Hahn-Banach Extension Theorem).
Let X be a normed space.
Let U be a vector subspace X.
Each $l \in U'$ has an extension $L \in X'$ with $\|l\| = \|L\|$.

Theorem 8.2.8 (Hahn-Banach Extension Theorem).
Let X be a LCTVsp.
Let U be a vector subspace of X.
Each $l \in U'$ has an extension $L \in X'$.

Theorem 8.2.11 (Hahn-Banach Separation Theorem: Version I). *(beefed-up)*
Let X be a LCTVsp.
Let V_1 and V_2 be subsets of X such that

(1) $V_1 \cap V_2 = \emptyset$

(2) V_1 *and* V_2 *are convex*

(3) V_1 *is open.*

Then there exists $x' \in X'$ and $\gamma \in \mathbb{R}$ with

$$Re\, x'(v_1) < \gamma \leq Re\, x'(v_2) \quad \forall v_1 \in V_1\ ,\ v_2 \in V_2\ .$$

Thus

$$\begin{aligned} \text{the open set } V_1 \ &\subset\ \left\{x \in X : Re\, x'(x) < \gamma\right\} \ \equiv\ \left(Re\, x'\right)^{-1}((-\infty, \gamma)) \ \equiv\ \text{an open set} \\ V_2 \ &\subset\ \left\{x \in X : Re\, x'(x) \geq \gamma\right\} \end{aligned}$$

and we say that the real hyperplane

$$H := \left\{x \in X : Re\, x'(x) = \gamma\right\}$$

separates V_1 and V_2.

Theorem 8.2.12 (Hahn-Banach Separation Theorem: Version II). *(beefed-up)*
Let X be a LCTVsp.
Let V_1 and V_2 be subsets of X such that

(1) $V_1 \cap V_2 = \emptyset$

(2) V_1 *and* V_2 *are convex*

(3) V_1 *is closed and* V_2 *is compact.*

Then there exists $x' \in X'$ and $\gamma_i \in \mathbb{R}$ with

$$Re\, x'(v_1) < \gamma_1 < \gamma_2 < Re\, x'(v_2) \quad \forall v_1 \in V_1\ ,\ v_2 \in V_2\ .$$

Thus, for any $\gamma \in (\gamma_1, \gamma_2)$,

$$\begin{aligned} V_1 \ &\subset\ \left\{x \in X : Re\, x'(x) < \gamma\right\} \\ V_2 \ &\subset\ \left\{x \in X : Re\, x'(x) > \gamma\right\} \end{aligned}$$

and we say that the real hyperplane

$$H := \left\{x \in X : Re\, x'(x) = \gamma\right\}$$

strictly separates V_1 and V_2.

Remark. In the special case that X is a normed space, Theorem 8.2.11 (resp. 8.2.12) is precisely Theorem 3.2.4 (resp. 3.2.5).

8.3 Weak Topology

Examples 8.1(h) and (i) introduced the weak topology on normed spaces and the weak* topology on dual spaces of normed spaces. These topologies will now be studied in detail; thereby we will take a somewhat general viewpoint.

Let X and Y be just vector spaces (without a topology) and

$$X \times Y \ \ni \ (x,y) \ \mapsto \ \langle x,y\rangle \ \in \ \mathbb{K}$$

be a bilinear map. (Not accidentally appears here the same symbol as for a scalar product. But beware: X and Y can now naturally be different and also in the case $\mathbb{K}=\mathbb{C}$ bilinearity[31] and not sesquilinearity[32] is desired.) An element from Y induces a linear map on X by means of the formula

$$\ell_y(x) \ = \ \langle x,y\rangle \ .$$

Similarly, each element of X acts as a linear map on Y.

Definition 8.3.1. (X,Y), or more precisely $(X,Y,\langle\cdot,\cdot\rangle)$, is a *dual pair* provided

$$\begin{array}{llll} \text{(i)} & \forall x\in X\setminus\{0\} & \exists y\in Y & \langle x,y\rangle \ \neq \ 0 \\ \text{(ii)} & \forall y\in Y\setminus\{0\} & \exists x\in X & \langle x,y\rangle \ \neq \ 0\ . \end{array}$$

Often, when given a specific X and Y, it is clear from context which bilinear map is to consider on $X\times Y$ and thus the map is not explicitly specified. Furthermore, given a dual pair (X,Y), we always can (and we will) *identify* Y (resp. X) as a point-separating vector subspace of the algebraic dual of X (resp. Y) since the maps $x\mapsto\langle x,\cdot\rangle$ and $y\mapsto\langle\cdot,y\rangle$ are injective.

Remark. Wow, that last sentence was a mouth full ...think of as

$$Y \ \underset{\text{linear}}{\overset{\text{bijective}}{\longrightarrow}} \ \{\langle\cdot,y\rangle : X\to\mathbb{K} \mid y\in Y\} \ \underset{\text{subspace}}{\overset{\text{vector}}{\subset}} \ \{f: X\to\mathbb{K}\mid f \text{ is linear}\}$$

the map $y\mapsto\langle\cdot,y\rangle$ being linear is precisely the linearity of $\langle\cdot,\cdot\rangle$ in the second variable
the map $y\mapsto\langle\cdot,y\rangle$ being injective is precisely Def. 8.3.1.ii
the space $\{\langle\cdot,y\rangle : X\to\mathbb{K}\mid y\in Y\}$ separating points of X is precisely Def. 8.3.1.i

and similarly

$$X \ \underset{\text{linear}}{\overset{\text{bijective}}{\longrightarrow}} \ \{\langle x,\cdot\rangle : Y\to\mathbb{K} \mid x\in X\} \ \underset{\text{subspace}}{\overset{\text{vector}}{\subset}} \ \{f: Y\to\mathbb{K}\mid f \text{ is linear}\}$$

the map $x\mapsto\langle x,\cdot\rangle$ being linear is precisely the linearity of $\langle\cdot,\cdot\rangle$ in the first variable
the map $x\mapsto\langle x,\cdot\rangle$ being injective is precisely Def. 8.3.1.i
the space $\{\langle x,\cdot\rangle : Y\to\mathbb{K}\mid x\in X\}$ separating points of Y is precisely Def. 8.3.1.ii.

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[31] i.e. linear in each variable separably

[32] i.e. linear in one variable and conjugate linear in the other variable

Example (a) Let X be a Hausdorff LCTVsp and X' be (as usual) its dual space.[33] By our very dear friend[34], (X, X') with the bilinear map $(x, x') \mapsto x'(x)$ is a dual pair.

Example (b) Similarly, in Example (a)'s setting, (X', X) with $(x', x) \mapsto x'(x)$ is a dual pair.

Example (c) Let $X = C^b(\mathbb{R})$ and $Y = M(\mathbb{R})$, the space of all (regular) finite signed (resp. complex)[35] Borel measures, as well as

$$\langle f, \mu \rangle \;=\; \int_{\mathbb{R}} f \, d\mu \; .$$

The regularity of the measures give that (X, Y) is a dual pair.[36]

Example (d) For a function $f\colon \mathbb{R} \to \mathbb{R}$, thus $f \in \mathbb{R}^{\mathbb{R}}$, consider the point evaluation functional $\delta_t\colon f \mapsto f(t)$.[37] Then $\left(\mathbb{R}^{\mathbb{R}}, \operatorname{lin}\{\delta_t : t \in \mathbb{R}\}\right)$ is a dual pair, where naturally

$$\langle f \;,\; \sum_{i=1}^{n} \lambda_i \delta_{t_i} \rangle \;=\; \sum_{i=1}^{n} \lambda_i f(t_i) \; .$$

Example (e) Let X and Y be normed spaces.[38] Write abbreviatorily

$$X \otimes Y' \;=\; \left\{ u\colon X' \to Y' : u\left(x'\right) = \sum_{i=1}^{n} x'(x_i)\, y_i' \;,\; n \in \mathbb{N} \;,\; \left(x_i \;,\; y_i'\right) \in X \times Y' \right\} \; ;$$

by the way, one can understand $X \otimes Y'$ as a tensor product of X and Y'. Then

$$\left(L(X, Y) \;,\; X \otimes Y' \right)$$

is a dual pair under the bilinear map

$$\langle \; T \;,\; u \; \rangle \;=\; \sum_{i=1}^{n} y_i'(T x_i) \; .$$

The Hahn-Banach Theorem shows this.

Example (f) As with Examples (a) and (b), one can always in general interchange the roles of X and Y.

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[33] Here (X, τ), for short X_τ or X, is a T_2LCTVsp. So $\tau = \tau_P$ is generated by a collection of seminorms P on X. Keep in mind that a normed space $(X, \|\cdot\|)$ is a *special case* of (X, τ_P). Def. 8.2.5 defines the dual space of X_τ as $(X_\tau)' = X' = \{f\colon (X, \tau_P) \to \mathbb{K} \mid f \text{ is linear and continuous}\}$.

[34] Hahn-Banach Theorem (Corollary 8.2.13).

[35] It is understood that X and Y are vector spaces over the same scalar field $\mathbb{K}$ and here the continuous functions and measure are $\mathbb{K}$-valued.

[36] See: Example 1.1.10, Def. 1.2.13, Thm. 1.2.14, and Thm. 2.2.5.

[37] We wanted a dual pair $\left(\mathbb{R}^{\mathbb{R}}, Y\right)$. Why didn't we just take $Y = \{\delta_t : t \in \mathbb{R}\}$? Well, because Y needs to be a vector space. So let's take the smallest vector space containing $\{\delta_t : t \in \mathbb{R}\}$.

[38] First note an easy way to get a bounded linear operator $u\colon X' \to Y'$ is to fix $(x_1, y_1') \in X \times Y'$ and define u by $[u(x')](y) := x'(x_1)\, y_1'(y)$. Consider such u's as the basic building blocks of the $X \otimes Y'$, which we new define as the smallest vector space containing all such u's.

Let $(X,Y,\langle\cdot,\cdot\rangle)$ be a dual pair. Towards using Y to produce a topology on X, note that each $y \in Y$ produces a seminorm p_y on X via $p_y(x) := |\langle x,y\rangle|$. Similarly, towards using X to produce a topology on Y, note that each $x \in X$ produces a seminorm p_x on Y via $p_x(y) := |\langle x,y\rangle|$.

Definition 8.3.2. Let (X,Y) be a dual pair. Consider the collection P of seminorms on X where

$$P = \left\{ p_y \colon X \to \mathbb{K} \;\mid\; y \in Y \right\} \qquad \text{with} \qquad p_y(x) := |\langle x,y\rangle|$$

The locally convex topology[39] on X generated by P is called the $\sigma(X,Y)$-topology. Similarly, the $\sigma(Y,X)$-topology is the locally convex topology on Y generated by the seminorms $\{p_x \colon Y \to \mathbb{K} \mid x \in X\}$ where $p_x(y) := |\langle x,y\rangle|$.

The $\sigma(X,Y)$-topology for a dual pair (X,Y) is always Hausdorff by Lemma 8.1.4. Theorem 8.2.6 given that, for a sequence (resp. net) (x_i) in X and $x \in X$

$$x_i \overset{\sigma(X,Y)}{\longrightarrow} x \qquad \Longleftrightarrow \qquad \langle x_i, y\rangle \to \langle x,y\rangle \quad \forall y \in Y\ .$$

Thus the weak convergence from Definition 3.3.6 is just a special case.[40] Also, we can now handle pointwise convergence; indeed, if one considers an $x \in X$ as a functional on Y (via x corresponds to $\langle x,\cdot\rangle \colon Y \to \mathbb{K}$), then x_i converging to x in the $\sigma(X,Y)$-topology corresponds to x_i converging x pointwise on Y.

Let's revisit the above Examples 8.3 (a) – (e). [41]

(a) Here we considered a T_2 LCTVsp (X,τ). The emerging topology $\sigma(X,X')$ is called the *weak topology* of the LCTVsp X. As in the special case that X is a normed space,[42] the weak topology is coarser than the original topology, i.e.

$$\sigma\left(X\ ,\ X'\right) \overset{\text{i.e.}}{=} \sigma\left(X_\tau\ ,\ (X_\tau)'\right) \overset{\text{i.e.}}{=} \sigma\left(\ (X,\tau)\ ,\ (X,\tau)'\right) \subset \tau$$

and, in general, $\sigma(X_\tau,(X_\tau)') \neq \tau$.

(b) $\sigma(X',X)$ is the topology on X' of pointwise convergence on X. As in the normed case, $\sigma(X',X)$ is called the *weak* topology* on X'.

(c) The $\sigma\left(M(\mathbb{R}), C^b(\mathbb{R})\right)$-topology is the weak topology from probabilty theory, which was introduced in Example 8.1.k.

(d) Here we obtain the topology of pointwise convergence from Example 8.1.a.

(e) $\sigma(\ L(X,Y)\ ,\ X \otimes Y'\ )$ is the weak operator topology that was introduced in Example 8.1.j as the LCTVsp on $L(X,Y)$ generated by the seminorms

$$p_{x,y'}(T) = \left|y'(Tx)\right| \qquad \text{for } x \in X\ ,\ y' \in Y'\ .$$

---

[39] See Definition 8.1.3.

[40] In Def.3.3.6, X was a normed space and we had $\sigma(X,X')$.

[41] For a normed space X, the weak $\sigma(X,X')$ and weak* $\sigma(X',X)$ topologies were defined in Examples 8.1.h/i and examined further in 2 examples in Section 8.2. Now, in (a) and (b) below, we extend the notion of weak and weak* topologies from a normed space X to a LCTVsp X.

[42] See the Example shortly after Def. 8.2.5.

Next the dual space of $X_{\sigma(X,Y)}$ will be determined, towards which the following lemma is helpful.

Lemma 8.3.3. *Let X be a vector space and $\ell, \ell_1, \ldots, \ell_n : X \to \mathbb{K}$ be linear. T.F.A.E..*

(i) $\ell \in \operatorname{lin}\{\ell_1, \ldots \ell_n\}$.

(ii) *There exists $M > 0$ so that*

$$|\ell(x)| \leq M \max_{i \leq n} |\ell_i(x)| \qquad \forall x \in X .$$

(iii) $\bigcap_{i=1}^{n} \ker(\ell_i) \subset \ker(\ell)$.

This lemma is the core of the weak topology, as shown by the next theorem.

Corollary 8.3.4. *Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual pair. If $f : X \to \mathbb{K}$ is linear then*

$$f \text{ is } \sigma(X,Y)\text{-continuous} \quad \Longleftrightarrow \quad \exists\, y \in Y \ \text{ s.t. } \ f(\cdot) = \langle \cdot, y \rangle .$$

In other words

$$(\, X \,,\, \sigma(X,Y)\,)' = Y \overset{\text{viewed as}}{=} \{\langle \cdot \,,\, y \,\rangle : X \to \mathbb{K} \mid y \in Y\} .$$

Two special cases are especially worthy of mention (the first was, at least in the normed space case, already shown in Example 8.2.b).
Let $(X, \tau) = X_\tau$ be a T_2LCTVsp (e.g. a normed space).

- A linear functional on X is weakly continuous IFF it is τ-continuous. I.e. $(\, X_\tau \,,\, \sigma(X_\tau, X'_\tau)\,)' = X'_\tau$.
- A linear functional on X' is weak* continuous IFF it is a point evaluation functional $x' \mapsto x'(x)$ for some $x \in X$. I.e. $(\, X'_\tau \,,\, \sigma(X'_\tau, X_\tau)\,)' = X_\tau$.

mgMotivation. Let X be a vector space with two locally convex topologies τ_1 and τ_2 satisfying

$$\tau_2 \subset \tau_1 .$$

Then $\langle$think f^{-1} (of an open subset of $\mathbb{K}$) $\in \tau_2 \subset \tau_1\rangle$

$$(X_{\tau_2})' \subset (X_{\tau_1})'$$

and for $A \subset X$ $\langle$think $A \subset \overline{A}^{\tau_2} \in \{\tau_2\text{-closed sets}\} \subset \{\tau_1\text{-closed sets}\}\rangle$

$$\overline{A}^{\tau_1} \subset \overline{A}^{\tau_2} .$$

In particular, if (X, τ) is a T_2LCTVsp (e.g. a norm topology), then
$\langle$think: weak topology is weaker than the original topology ... and weaker = fewer open sets$\rangle$

$$\text{weak topology on } X = \sigma(X, X') \subset \tau = \text{the original topology on } X$$

$$(X_{\text{weak-top}})' = (X_\tau)' \qquad \text{by Cor. 8.3.4}$$

$$\overline{A}^{\tau} \subset \overline{A}^{\text{weak-top}} .$$

Theorem 8.3.5. *Let X be a vector space with two LC*[43] *topologies τ_1 and τ_2 with*

$$(X_{\tau_1})' = (X_{\tau_2})'$$

Then a convex set C is τ_1-closed IFF *it is τ_2-closed. In other words, if $A \subset X$, then*

$$\overline{\mathrm{co}A}^{\tau_1} = \overline{\mathrm{co}A}^{\tau_2} .$$

In Theorem 8.3.5, the convexity assumption on C in necessary. [44]

Here are some important situations in which Theorem 8.3.5 applies.

- (X,τ) T_2LCTVsp and $\tau_1 = \tau$ and $\tau_2 = \sigma(X,X')$ (Corollary 8.3.4). In the norm case, compare Theorem 8.3.5 with Theorem 3.3.8.

  Theorem 3.3.8. *Let V be a closed convex subset of a normed space X. If a sequence (x_n) from V converges weakly to $x \in X$, then $x \in V$.*

  Corollary 3.3.9. *If $x_n \to x$ weakly $\sigma(X,X')$, then there is a sequence (y_n) of convex combinations of the x_i's such that $\|y_n - x\| \to 0$.*

- $L(X,Y)$ with the weak and strong operator topology. (Example (e), Corollary 8.3.4 and Example 8.2.c.)

We now note an important property of the weak topology.

Theorem 8.3.6. *Let $(X,Y,\langle\cdot,\cdot\rangle)$ be a dual pair.*
Then the weak topology $\sigma(X,Y)$ on X is initial with respect to Y.
I.e. *If T is a topological space, then a function*

$$f\colon T \to X_{\sigma(X,Y)}$$

is continuous IFF *for each $y \in Y$ the composite*

$$y \circ f : T \xrightarrow{f} X \xrightarrow{y} \mathbb{K}$$
$$(y \circ f)(t) = \langle f(t) , y \rangle$$

is continuous.
In particular, $\sigma(X,Y)$ is the coarsest [45] *topology on X s.t. each $y \in Y$ is continuous.*
I.e. *for each $y \in Y$*

$$\langle \cdot , y \rangle \colon (X , \sigma(X,Y)) \to \mathbb{K}$$

is continuous; futhermore, if τ is any topology on X such that for each $y \in Y$

$$\langle \cdot , y \rangle \colon (X,\tau) \to \mathbb{K}$$

is continuous, then $\sigma(X,Y) \subset \tau$.

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[43] Henceforth: LC = locally convex

[44] Come up with an example of a (nonconvex) C in a Banach space X such that $\overline{C}^{\mathrm{weak}} \neq \overline{C}^{\mathrm{norm}}$.

[45] think “smallest, fewest open sets”

mgRmk A null nhood base of the $\sigma(X,Y)$-topology consisting of open, absorbing, absolutely convex sets is

$$\mathscr{V} := \{V_{\varepsilon,y_1,\ldots,y_n} \ : \ \varepsilon > 0 \ , \ n \in \mathbb{N} \ , \ y_i \in Y\}$$

where

$$V_{\varepsilon,y_1,\ldots,y_n} := \{x \in X : \ |\langle x, y_i \rangle| < \varepsilon \ \ \forall i = 1, \ldots n\} \ .$$

A nhood base about $x_0 \in X$ of the $\sigma(X,Y)$-topology consisting of open, convex sets is

$$\mathscr{V}_{x_0} := \{V_{\varepsilon,y_1,\ldots,y_n}(x_0) \ : \ \varepsilon > 0 \ , \ n \in \mathbb{N} \ , \ y_i \in Y\}$$

where

$$V_{\varepsilon,y_1,\ldots,y_n}(x_0) := x_0 + V_{\varepsilon,y_1,\ldots,y_n} = \{x \in X : \ |\langle x - x_0, y_i \rangle| < \varepsilon \ \ \forall i = 1, \ldots n\} \ .$$

For a T_2LCTVsp X, in the weak topology $\sigma(X,X')$ on X these sets that the form

$$V_{\varepsilon,x'_1,\ldots,x'_n}(x_0) = \left\{x \in X : \ \left|x'_i(x - x_0)\right| < \varepsilon\right\} \qquad \text{where } \ \varepsilon > 0 \ , \ n \in \mathbb{N} \ , \ x'_i \in X' \ ;$$

while in the weak* topology $\sigma(X',X)$ on X' these sets that the form

$$V_{\varepsilon,x_1,\ldots,x_n}(x'_0) = \left\{x' \in X' : \ \left|\left(x' - x'_0\right) x_i\right| < \varepsilon\right\} \qquad \text{where } \ \varepsilon > 0 \ , \ n \in \mathbb{N} \ , \ x_i \in X \ .$$

We come to a new notion.

Definition 8.3.7. Let (X,Y) be a dual pair.
For $A \subset X$, the *polar* of A is

$$A^\circ = \{y \in Y : \text{Re}\,\langle x, y \rangle \le 1 \ \ \forall x \in A\} \ .$$

For $B \subset Y$, the *polar* of B is

$$B^\circ = \{x \in X : \text{Re}\,\langle x, y \rangle \le 1 \ \ \forall y \in B\} \ .$$

The polars are always defined with respect to a given fixed dual pair. Inparticular, $A^{\circ\circ} \subset X$.
Warning: some authors define A° as the *absolute polar* $\{y \in Y : \ |\langle x, y \rangle| \le 1 \forall x \in A\}$.
For examples, let X be a normed space and consider the dual pair (X,X'). Then

$$(B_X)^\circ = B_{X'} \qquad \text{and} \qquad (B_{X'})^\circ = B_X \ .$$

For a subspace U of X

$$U^\circ = U^\perp = \left(x' \in X' : x'\,|_U = 0\right\}$$

where the annihilator $U^\perp$ of U was defined in Section 3.4. (Problem 8.6.20).

We come now to a few elementary properties. co B denotes the convex hull of B (that is, the intersection of all convex sets that contain B) while $\overline{\text{co}}B$ is its closure. It follows from Theorem 8.2.7 that $\overline{\text{co}}B$ is the smallest closed convex set containing B.

Lemma 8.3.8. *Let (X,Y) be a dual pair and $A, A_i \subset X$ for i in some index set I.*

(a) A° *is convex and* $\sigma(Y,X)$*-closed.* $A^\circ = (\overline{\text{co}}A)^\circ$.

(b) $0 \in A^\circ$ *and* $A \subset A^{\circ\circ}$.
If $A_1 \subset A_2$ *then* $A_2^\circ \subset A_1^\circ$.

(c) *If A is circled*[46] *then* $A^\circ = \{y \in Y : |\langle x,y\rangle| \le 1 \;\; \forall x \in A\}$.

(d) $(\lambda A)^\circ = \frac{1}{\lambda} A^\circ$ *for* $\lambda > 0$.

(e) $[\cup_{i\in I} A_i]^\circ = \cap_{i\in I} A_i^\circ$.

(f) $[\cap_{i\in I} A_i]^\circ \supset \overline{\text{co}} \cup_{i\in I} A^\circ$ *where the closure is in the* $\sigma(Y,X)$*-topology.*

Theorem 8.3.9 (Bipolar Theorem). *For a dual pair (X,Y) and $A \subset X$*

$$A^{\circ\circ} = \overline{\text{co}}[\, A \cup \{0\}\,]$$

where the closure is taken with respect to the $\sigma(X,Y)$-topology.

Corollary 8.3.10. *Let (X,Y) be a dual pair.*
Let C be a convex subset of X containing 0. Then

$$C \text{ is } \sigma(X,Y)-\text{closed} \quad \Longleftrightarrow \quad \exists\; B \subset Y \text{ s.t. } C = B^\circ .$$

In this case, one can take $B = C^\circ$.

Next we prove a fundamental fact of Functional Analysis:

the polar of a null nhood of a T_2LCTVsp is weak*-compact.

Theorem 8.3.11 (Alaoglu-Bourbaki Theorem).
Let X be a T_2LCTVsp and consider the dual pair (X,X').
Let U be a null nhood of X.
Then U° is $\sigma(X',X)$-compact.

As a corollary we obtain the original Alaoglu Theorem.[47]

Corollary 8.3.12. *For a normed space X, the unit ball*

$$B_{X'} := \left\{ x' \in X' : \left\|x'\right\|_{X'} \le 1 \right\}$$

is $\sigma(X',X)$-compact.

The Alaoglu Theorem does not imply that each bounded sequence of functionals have a $\sigma(X',X)$-convergent subsequence; as a counterexample consider the functionals $x'_n : x \mapsto x(n)$ in ℓ_∞. (Towards this also see Problem 8.6.17(f)). Bear in mind that compactness in topological spaces is characterized by each net having a convergent subnet (see Heather's notes); soever the above sequence has a weak* convergent subnet.

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[46] Definition 8.1.1: $\{\lambda \in \mathbb{K} : |\lambda| \le 1\} \cdot A \subset A$. Equivalently, if $\lambda \in B_{\mathbb{K}}$ then $\lambda A = A$.

[47] Take $U = B_X$. If interested in just this case, go through proof of Thm 8.3.11 with $U = B_X$, $U^\circ = B_{X'}$, and $\lambda_x = \|x\|$.

Corollary 8.3.13. *Each Banach space X is isometrically isomorphic to a closed subspace of a $C(K)$ space of continuous functions on a compact set K.*

Proof. Let X be a Banach space. Let's recall some facts to see if we can piece together the proof.

(1) Recall the niffty Corollary 3.1.7 of the HB theorem.
Corollary 3.1.7. For each normed space X,

$$\|x\|_X = \sup_{x' \in B_{X'}} \left|x'(x)\right| \qquad \forall x \in X . \tag{3.3}$$

Notice the symmetry in formula (3.3) and the definition [48]

$$\left\|x'\right\|_{X'} = \sup_{x \in B_X} \left|x'(x)\right| \qquad \forall x' \in X' . \tag{3.3'}$$

All this lead to Theorem 3.3.1 that the point evaluation map

$$\begin{aligned} &i\colon X \to X'' = \left\{f\colon \left(X', \|\cdot\|_{X'}\right) \to \mathbb{K} \mid f \text{ is continuous and linear}\right\} \\ &[i(x)]\left(x'\right) = x'(x) \end{aligned}$$

is an isometric embedding.

(2) Corollary 8.3.4 (see page 1019) gave that:
A linear functional on X' is weak* continuous IFF it is a point evaluation functional $x' \mapsto x'(x)$ for some $x \in X$. I.e. $(\ X' \ , \ \sigma(X',X)\ )' = X$.

(3) Alaoglu's Thm says that $K := (B_{X'}, \sigma(X',X))$ is compact.

OK - we see it. Let

$$\begin{aligned} &j\colon X \to C(K) := \left\{f\colon \left(B_{X'}, \sigma\left(X', X\right)\right) \to \mathbb{K} \mid \ f \text{ is continuous}\right\} \\ &[j(x)]\left(x'\right) = x'(x) \ . \end{aligned}$$

By Cor 8.3.4, j maps into $C(K)$. Clearly j is linear. By HB, j is norm preserving, indeed

$$\|j(x)\|_{C(K)} = \sup_{k \in K} |[j(x)](x)| = \sup_{x' \in B(X')} \left|x'(x)\right| \overset{\text{HB}}{=} \|x\|_X \ ;$$

and thus j thus has closed range. □

Corollary 8.3.14. *Let X be a normed space an $\varphi\colon B_{X'} \to \mathbb{R}$ be a weak*-continuous function. Then φ obtains its infimum and supremum.*

We come now to a nontrivial criterium for $\sigma(X',X)$-closeness. Let X be a normed space If $C \subset X'$ is $\sigma(X',X)$-closed, then for each $t > 0$ the set $C \cap tB_{X'}$ is also $\sigma(X',X)$-closed. Surprisingly, for a convex set C and a Banach space X, the converse of this statement is true. (Convexity and completeness are essential.)

[48] In contrast: the sup in (3.3) is always obtained but the sup in (3.3') need not be obtained.

Theorem 8.3.15 (Krien-Smulian Theorem).
Let X be a Banach space and $C \subset X'$ be convex. Then

$$C \quad \text{is } \sigma\left(X',X\right) \text{ closed} \qquad \Longleftrightarrow \qquad C \cap tB_{X'} \quad \text{is } \sigma\left(X',X\right) \text{ closed } \forall t > 0\,.$$

The following corollary is known as the Banach-Dieudonné Theorem.

Corollary 8.3.16. *Let X be a Banach space and U be a vector subspace of X'.
Then U is $\sigma(X',X)$ closed if and only if it closed unit ball is.*
I.e. *U is $\sigma(X',X)$ closed iff $U \cap B_{X'}$ is $\sigma(X',X)$ closed.*

After our study of the weak* topology, we close with a short look at the weak topology on a Banach space. A crucial observation is the following. One can identify X as a subspace of X'' (cf. Section 3.3) by the point evaluation map

$$\begin{aligned} i\colon X &\to X'' \\ [i(x)]\left(x'\right) &= x'(x) \end{aligned}$$

Denote

$$i(X) := \widehat{X}\,.$$

We know that, thanks to the HB-thm,

$$i_1\colon\ (X, \|\cdot\|_X) \to \left(\widehat{X}, \|\cdot\|_{X''}\right)$$

is a linear homeomorphism.[49] Also

$$i_2\colon\ \left(X, \sigma\left(X,X'\right)\right) \to \left(\widehat{X}, \sigma\left(X'',X'\right)\right)$$

is a linear homeomorphism (just consider nbood bases).

Theorem 8.3.17 (Goldstine Theorem). *Let X be a normed space.
Then $i(B_X)$ is $\sigma(X'',X')$-dense in $B_{X''}$ and
thus $i(X)$ is $\sigma(X'',X')$-dense in X''. In other words:*

$$\begin{aligned} \overline{i(B_X)}^{\,\sigma\left(X'',X'\right)} &= B_{X''} \\ \overline{i(X)}^{\,\sigma\left(X'',X'\right)} &= X''\,. \end{aligned}$$

Proof. This follows immediately from the Bipoloar Theorem 8.3.9 applied to $i(B_X)$ and the dual pair (X'',X'). Indeed

$$B_{X''} = [i(B_X)]^{\circ\circ} \overset{\text{bipolar thm}}{=} \overline{\operatorname{co}(i(B_X) \cup \{0\})}^{\,\sigma\left(X'',X'\right)} = \overline{i(B_X)}^{\,\sigma\left(X'',X'\right)}\,. \qquad \square$$

As a consequence we have the following reflexivity criterium. (Reflexivity was introduced in Definition 3.3.3.)

[49] i.e. i_1 is bijective, futhermore, i_1 and i_1^{-1} are continuous.

Theorem 8.3.18. *For a Banach space X, T.F.A.E..*

(i) *X is reflexive.*

(ii) *B_X is $\sigma(X,X')$-compact.* I.e. *B_X is weakly compact.*

Compare Theorems 8.3.18 with Theorem 3.3.7.

Theorem 3.3.7. *In a reflexive space X, each bounded sequence has a weakly convergent subsequence.*

Let's compare some more! This is really niffty: compactness of B_X says alot.

Theorem 1.2.7. *For a normed space X, the following are equivalent.*

(i) $\dim X < \infty$.

(ii) $B_X := \{x \in X : \|x\| \le 1\}$ *is norm compact.*

Theorem 8.3.18. *For a Banach space X, T.F.A.E..*

(i) *X is reflexive.*

(ii) *B_X is $\sigma(X,X')$-weak-compact.*

Corollary 8.3.12 (Alaoglu Theorem). *For a normed space X, the unit ball $B_{X'}$ is $\sigma(X',X)$-weak*-compact.*

8.6 Problems

Problem 8.6.1-a. Let X be a topological vector space.

(a) If $O \subset X$ is open, then the convex hull of O is also open.

Problem 8.6.1-b. Let X, more precisely (X, τ), be a topological vector space.

(b) Show without using nets that if $C \subset X$ is convex, then its closure $\overline{C}^{\tau}$ is also convex.

Problem 8.6.2. Let X be a normed space. The unit sphere S_X lies $\sigma(X, X')$-dense in the unit ball B_X provided X is infinite dimensional.

Problem 8.6.3-intro: (no problem to do, just read the intro).
Here is an example of a non-localy convex topological vector space.[50]
Fix $0 < p < 1$. Consider the vector space $L^p[0,1]$, which is naturally defined as the (equivelent class) of measurable functions $f\colon [0,1] \to \mathbb{K}$ so that the

$$\|f\|_p := \left[\int_0^1 |f(t)|^p \, dt\right]^{1/p} < \infty .$$

Define a function $d\colon L^p[0,1] \times L^p[0,1] \to \mathbb{R}$ by

$$d(f,g) := \int_0^1 |f(t) - g(t)|^p \, dt .$$

Problem 8.6.3-0. Show that we have just abused notation in that $\|\cdot\|_p$ is not a norm on $L^p[0,1]$ (naturally for $0 < p < 1$). Hint: triangle inequality.

Problem 8.6.3.a. d defines a metric on $L^p[0,1]$.

Problem 8.6.3.b. Endowed with this metric, $L^p[0,1]$ is a topological vector space.

Problem 8.6.3.c. Let $V \subset L^p[0,1]$ be a open convex nhood of zero.
Then $V = L^P[0,1]$.
(Hint: Choose $\varepsilon > 0$ with $\{f\colon d(f,0) \leq \varepsilon\} \subset V$. Let $f \in L^p[0,1]$. Consider now for a big n a decomposition of $[0,1]$ into n intervals $I_1, \ldots I_n$ and the functions $g_j := n\chi_{I_j} f$.)

Problem 8.6.3.d. Show that $f \mapsto 0$ is the only continuous linear functional of $L^p[0,1]$ for $0 < p < 1$.

Problem 8.6.4-intro: (no problem to do, just read the intro).
Let X be a vector space and P be set of *all* seminorms on X. Then (X, τ_P) is a LCTVsp.

Problem 8.6.4a. (X, τ_P) is a Hausdorff space.

[50] USC Distinguished Professor Emeritus James Roberts made major contributions to the study of these space. See the book "The F-space Sampler" by Kalton, Peck, Roberts.

Problem 8.6.4b. Each linear map into another LCTVsp Y is continuous.

Problem 8.6.4c. Each subspace of X is closed.

Problem 8.6.5a. On $\mathbb{R}^n$, each Hausdorff locally convex topology agrees with the norm topology.

Problem 8.6.5b. A finite dimensional subspace of a T_2 LCTVsp is closed.

Problem 8.6.6-intro: (no problem to do, just read the intro). Let X be a LCTVsp. A subset $B \subset X$ is called *bounded* provided for each null nhood U there exists $\alpha > 0$ so that $B \subset \alpha U$. Consider the following conditions.

(i) B is bounded.

(ii) For each sequence (x_n) in B and each null sequence (α_n) in $\mathbb{K}$, the sequence $(\alpha_n x_n)$ is a null sequence in X.

Problem 8.6.6a. What are the in-this-sense-bounded subsets of a normed space?

Problem 8.6.6b. Show that (i) implies (ii).

Problem 8.6.6c. Show that (ii) implies (i).

Problem 8.6.7a. Consider the Schwartz class $\mathscr{S}(\mathbb{R})$. Show that the differentiation operator $D^k \colon \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ is continuous where $D^k\varphi := \varphi^{(k)}$.

Problem 8.6.7b. Consider the Schwartz class $\mathscr{S}(\mathbb{R})$. Let $g \in \mathscr{S}(\mathbb{R})$. Then the convolution operator $C_g \colon \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ is continuous where

$$(C_g\varphi)(x) = (g * \varphi)(x) = \int_{\mathbb{R}} \varphi(y)\, g(x-y)\, dy .$$

Problem 8.6.8-intro: (no problem to do, just read the intro). Show that the following functionals and operators of the Schwartz class $\mathscr{S}(\mathbb{R})$ are well-defined and continuous.

Problem 8.6.8a. $T(f) = \int_{-\infty}^{\infty} f(t)\, g(t)\, dt$ for a fixed $g \in L^\infty(\mathbb{R})$.

Problem 8.6.8b. $T(f) = f(s)$ for a fixed $s \in \mathbb{R}$.

Problem 8.6.8c. $T(f) = f \cdot g$ for a fixed $g \in \mathscr{S}(\mathbb{R})$.

Problem 8.6.10-intro: (no problem to do, just read the intro). Let X and Y be normed spaces and $T \colon X \to Y$ be linear. Consider the following properties.

(i) T is $\|\cdot\|_X$-to-$\|\cdot\|_Y$ continuous.

(ii) T is $\sigma(X, X',)$-to-$\sigma(Y, Y')$ continuous.

(Tip: Banach-Steinhaus Theorem.)

Problem 8.6.10a. Show (i) implies (ii).

Problem 8.6.10b. Show (ii) implies (i).

Problem 8.6.11-intro: (no problem to do, just read the intro). Let X and Y be normed spaces and $T \in L(Y', X')$. Consider the following properties.

(i) T is $\sigma(Y', Y)$-to-$\sigma(X', X)$ continuous.

(ii) There exists $S \in L(X, Y)$ with $S' = T$.

(Tip: Theorem 8.3.6.)

Problem 8.6.11a. Show (i) implies (ii).

Problem 8.6.11b. Show (ii) implies (i).

Problem 8.6.12-intro: (no problem to do, just read the intro). Let X and Y be normed spaces and $T \in L(X, Y)$. Consider the following properties.

(i) T is $\sigma(X, X')$-to-$\|\cdot\|_Y$ continuous.

(ii) T has a finite dimensional range.

(Hint: Proof of Lemma 8.3.3.)

Problem 8.6.12a. Show (i) implies (ii).

Problem 8.6.12b. Show (ii) implies (i).

Problem 8.6.12c. Why does this not contridict the result from Probelm 3.6.16(b)?

Problem 8.6.13-intro: (no problem to do, just read the intro). Let X and Y be Banach spaces and $T \in L(X, Y)$. Consider the following properties.

(i) T is compact .

(ii) $T'\mid_{B_{Y'}} : B_{Y'} \to X'$ is $\sigma(Y', Y)$-to-$\|\cdot\|_{X'}$ continuous.

Problem 8.6.13a. Show (i) implies (ii).

Problem 8.6.13b. Show (ii) implies (i).

Problem 8.6.14-intro: (no problem to do, just read the intro). Let X and Y be Banach spaces. A bounded linear operator $T : X \to Y$ is called *weakly compact* when the closure of $T(B_X)$ is weakly compact. Let $W(X, Y)$ be the space of all weakly compact operators from X to Y. Consider the following conditions.

(i) T is weakly compact.

(ii) $\operatorname{ran}(T'') \subset i(Y)$.

(iii) T' is $\sigma(Y', Y)$-to-$\sigma(X', X'')$ continuous.

(iv) T' is weakly compact.

Problem 8.6.14a. If X or Y is reflexive, then $W(X, Y) = L(X, Y)$.

Problem 8.6.14b. The inclusion operator $J : C[0,1] \to L^1[0,1]$ is weakly compact but not compact.

Problem 8.6.14c. Show that conditions (i) - (iv) are equivalent.

Problem 8.6.14d. $W(X, Y)$ is a closed subpace of $L(X, Y)$.

Problem 8.6.16. Consider the sequence (x_n) in ℓ^2 given by $x_n = \sqrt{n}e_n$ and the corresponding set $X{:=}\{x_n \colon n \in \mathbb{N}\}$. Show that 0 is in the weak closure of X but no subsequence of (x_n) converges weakly to 0.

Problem 8.6.17. Metrizability of locally convex topologies.

Problem 8.6.17a1. Let (X, τ_p) be a Hausdorff LCTVsp with the topology τ_P being generated by a (countable) family $P = \{p_1, p_2, \ldots\}$ of seminorms. Show that

$$d(x,y) := \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x-y)}{1+p_n(x-y)}$$

is a metric on X.
(Hint: First show that if $0 \le a \le b$ then $\frac{a}{1+a} \le \frac{b}{1+b}$.)

Problem 8.6.17a2. In Problem 8.6.17a1, show that the topology generated by the there-defined metric agrees with the τ_P topology.

Problem 8.6.17b. If a LCTVsp is metrizable then its topology can be generated by a countable family of seminorms.
(Hint: imitate the proof of Thm 8.1.5 for a countable zero nhood base.)

Problem 8.6.17c. A weak topology $\sigma(X,Y)$ is metrizable if and only if Y has a atmost countable vector space basis.

Problem 8.6.17d. If X is an infinite dimensional Banach space, then its weak*-topology on X' is not metrizable.
(Hint: Problem 4.8.2.)

Problem 8.6.17e. If X is a separable normed space, then the weak* topology on $B_{X'}$ is metrizable.
(Hint: $d(x',y') = \sum 2^{-n} |(x'-y')(x_n)|$ for appropriate x_n.)

Problem 8.6.17f. With the help of Problem 8.1.17e and the Alaoglu Theorem, prove again a solution to Problem 3.8.18.

Problem 8.6.8.6.19a. $C[0,1]$ is $\sigma\left(L^\infty[0,1]\ ,\ L^1[0,1]\right)$ dense in $L^\infty[0,1]$.

Problem 8.6.8.6.21. Let X be a reflexive Banach space. Then $B_{L(X)}$ is compact in the weak operator topology.
(Tip: try to imitate the proof on Alaoglu Theorem.)

Appendix A

Measure and Integration Theory

Appendix B
Metric and Topological Spaces

B.1 Metric Spaces

B.2 Topological Spaces

Further References.

- *General Topology* by Willard [1970]
- *Topology* by Dugundji [1966]
- *General Topology* by Kelly [1955]

Definitions. Let T be a set and $M \subset T$.
A *topology* τ on T is a system of subsets of T that have the following properties.

(a) $\emptyset \in \tau$ and $T \in \tau$.

(b) If $O_1 \in \tau$ and $O_2 \in \tau$, then $O_1 \cap O_2 \in \tau$.

(c) If I is an arbitrary index set and $O_i \in \tau$ for each $i \in I$, then $\cup_{i\in I} O_i \in \tau$.

(T, τ), or for short T, is a *topological space*. The collection τ is the *open* sets of T.

(i) M is *closed* provided $T \setminus M$ is open.

(ii) the closure of M is the set $\overline{M} := \cap\{C \;:\; M \subset C\,,\, C \text{ is closed}\}$.

(iii) the interior of M is the set $\operatorname{int} M := \cup\{O \;:\; O \subset M\,,\, O \text{ is open.}\}$

Definitions. Let (T, τ) be a topological space and $t \in T$.

(1) An *Umgebung*[1] *of* t is a set U such that $\exists O \in \tau$ with $t \in O \subset U$.[2]
So: U is a nhood of t IFF $t \in \operatorname{int} U$.

(2) The collection $\mathscr{U}_t$ of all nhoods of t is called the *nbood system at* t.

(3) A subset $\mathscr{B}_t$ of $\mathscr{U}_t$ is called a *nhood base at* t provided
each $U \in \mathscr{U}_t$ contains a $B \in \mathscr{B}_t$ (so $t \in B \subset U$).

(4) A sequence (t_n) from T converges to $t \in T$ (w.r.t. τ) provided

$$\forall\, U \in \mathscr{U}_t \;\; \exists N \in \mathbb{N} \;\; \forall n \geq N \;\; t_n \in U \; .$$

Thus $t_n \xrightarrow{\tau} t \quad \Leftrightarrow \quad$ for some (or equiv. for each) nhood base $\mathscr{B}_t$ at t

$$\forall\, U \in \mathscr{B}_t \;\; \exists N \in \mathbb{N} \;\; \forall n \geq N \;\; t_n \in U \; .$$

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[1] Neighborhood; henceforth, abbreviated *nhood*, as in Willard.

[2] Notice that an Umgebung (nhood) needs not itself be open.

Nets. In set-theoretical topology, it does not always suffice to consider convergence of *sequences*. Often one needs to consider convergence of *filters and nets*. Both of these concepts are equivalent; however, net convergence is easier to explain and seems more adapted to the needs of analysis, thus will be entertained. [3]

A *directed set* a set I endowed with a relation $\leq$ satisfying the following 3 properties.

(a) $\forall i \in I$: $i \leq i$ (reflexive)

(b) $\forall i, j, k \in I$: $i \leq j$ and $j \leq k$ implies $i \leq k$. (transitive)

(c) $\forall i_1, i_2 \in I \;\; \exists j \in I$ s.t. $i_1 \leq j$ and $i_2 \leq j$.

A *net* in (from) a set T is a map from a directed set I to T; one writes $(t_i)_{i \in I}$ or for short (t_i). Obviously $\mathbb{N}$ is directed with its usual ordering; thus, a sequence is a net. Another example of a directed set

$$(\mathscr{B}_t \; , \; \supseteq) \tag{B2}$$

where $\mathscr{B}_t$ is a nhood base at a point t in a topological space and

$$V \leq U \Leftrightarrow V \supseteq U \; ;$$

condition (c) is fulfilled since the intersection of two nhoods is a nhood.

A net (t_i) in a topological space T *converges* to $t \in T$ provided

$$\forall U \in \mathscr{U}_t \;\; \exists j \in I \;\; \forall i \geq j \;\; t_i \in U \; .$$

Clearly, in the above definition of $t_i \xrightarrow{\tau} t$, one can replace the nbood system $\mathscr{U}_t$ at t with any nhood base $\mathscr{B}_t$ at t.

B.3 Nets: by Heather Cheatum

We thank USC graduate student Heather Cheatum for her class presentation on Nets.

Definitions

1. A **topology** on a set X is a collection τ of subsets of X satisfying:
   (a) Arbitrary unions of elements of τ are still in τ
   (b) Finite intersections of elements of τ are still in τ
   (c) X and $\emptyset$ are in τ.
   We call the elements of τ open sets. The complements in X of the members of τ are called closed sets.
2. A topology on a set X is **Hausdorff**, or T_2, if given any two distinct points $x_1, x_2 \in X$ you can find disjoint open sets U and V such that $x_1 \in U$ and $x_2 \in V$.

[3] Also see Wiki: http://en.wikipedia.org/wiki/Net_(mathematics)

3. If X is a topological space and $x \in X$, a **neighborhood** of x is a set U which contains an open set V containing x. A **neighborhood base** at x is a subcollection $\mathscr{B}_x$ taken from the neighborhoods of x where $\mathscr{B}_x$ has the property that for each neighborhood, U, of x there is some $V \in \mathscr{B}_x$ which is contained in U.
4. Given a set X with a topology τ and a subset of X, E, the **closure** of E, denoted $\mathrm{Cl}_X(E)$ or $\overline{E}$, is the intersection of all closed sets containing E. Equivalently, $x \in \overline{E}$ iff every neighborhood of x meets E.
5. Let X and Y be topological spaces and let $f : X \to Y$. Then f is **continuous at** $x_0 \in X$ iff for each neighborhood V of $f(x_0)$ in Y, there is a neighborhood U of x_o in X such that $f(U) \subset V$. We say f is **continuous** on X iff f is continuous at each $x_0 \in X$.
6. A set Λ is a **directed set** iff there is a relation $\leq$ on Λ satisfying:
   (a) $\lambda \leq \lambda$ for each $\lambda \in \Lambda$
   (b) if $\lambda_1 \leq \lambda_2$ and $\lambda_2 \leq \lambda_3$ then $\lambda_1 \leq \lambda_3$
   (c) given $\lambda_1, \lambda_2 \in \Lambda$, there exists $\lambda_3 \in \Lambda$ such that $\lambda_1 \leq \lambda_3$, $\lambda_2 \leq \lambda_3$.
7. A **net** in a set X is a function $P : \Lambda \to X$, where Λ is some directed set. We will denote the point $P(\lambda)$ as x_λ.
8. A net, $(x_\lambda)_{\lambda \in \Lambda}$, in a topological space X **converges** to $x \in X$ provided that for every neighborhood U of x, there is some $\lambda_0 \in \Lambda$ such that $x_\lambda \in U$ for all $\lambda \geq \lambda_0$. We say (x_λ) has a **cluster point** iff for each neighborhood U of x and for each $\lambda_0 \in \Lambda$ there is some $\lambda \geq \lambda_0$ such that $x_\lambda \in U$.

Theorems

1. Let X be a topological space and E be a subset of X. Then TFAE
   (i) $x \in \overline{E}$
   (ii) there exists a net, $(x_\lambda) \subset E$ such that $x_\lambda \to x$
   (iii) there exists a net, $(x_\lambda) \subset E$ which has x as a cluster point.
2. Let X and Y be topological spaces and let $f : X \to Y$. TFAE
   (i) f is continuous
   (ii) for any open set in Y, the inverse image under f is open in X
   (iii) for any closed set in Y, the inverse image under f is closed in X
   (iv) for every $E \subset X$, $f(\mathrm{Cl}_X E) \subset \mathrm{Cl}_Y f(E)$.
3. Let $f : X \to Y$ for topological spaces X and Y. Then f is continuous at $x_0 \in X$ iff for every net (x_λ) converging to x_0 in X, we have $f(x_\lambda) \to f(x_0)$ in Y.
4. Let X be a topological space. TFAE
   (i) X is T_2
   (ii) given a net (x_λ) in X, if (x_λ) has a limit, it is unique.

For an example (for which we ran out of time in class) showing that sequences do not suffice in determining Hausdorff-ness, see
http://www.math.sc.edu/~girardi/FA/09HeatherCheatum/NetsNeededForHausdorff.pdf .