Def. A branch of the logarithm on G is a continuous function $\log\colon G \to \mathbb{C}$, where G is an open connected subset of $\mathbb{C}$, such that Def I.2.1 p4

$$e^{\log z} \;=\; z \qquad \text{, for each} \quad z \in G.$$

⟨Note G cannot contain zero since for all $z \in \mathbb{C}$ we have $\left|e^{\log z}\right| = e^{\operatorname{Re}(\log z)} \neq 0$.⟩

Thm. If $\log\colon G \to \mathbb{C}$ is a branch of the logarithm on (an open connected set) G, then $\log(\cdot) \in H(G)$ and Cor I.3.16 p8

$$[\log(\cdot)]'(z) \;=\; \frac{1}{z} \qquad \text{, for each} \quad z \in G.$$

. . . .

The argument of $z \in \mathbb{C} \setminus \{0\}$, denoted $\arg z$, is the set

$$\arg z := \left\{\theta \in \mathbb{R}\colon z = |z|\, e^{i\theta}\right\}.$$

Thus, if $z = re^{i\theta}$ with $r > 0$, then

$$\arg z \;=\; \{\theta + 2k\pi \in \mathbb{R}\colon k \in \mathbb{Z}\}.$$

When we want to think of $\arg z$ as the unique real number from the set $\arg z$ that is also in a clopen interval of length 2π, we (abusively) denote the desired unique real number as, e.g., "$\arg z$ where $17 < \arg z \le 17 + 2\pi$".

The principle value of the argument of $z \in \mathbb{C} \setminus \{0\}$, denoted $\operatorname{Arg} z$, is the (unique) real number such that

$$\operatorname{Arg} z \in \arg z \qquad \text{and} \qquad -\pi < \operatorname{Arg} z \le \pi,$$

which gives rise to the function $\operatorname{Arg}\colon \mathbb{C} \setminus \{0\} \to (-\pi, \pi]$ defined by $\operatorname{Arg}(z) = \operatorname{Arg} z$.

The branch cut for the principle branch of the logarithm function is the ray

$$B = \left\{re^{i\pi}\colon r \ge 0\right\} \overset{\text{i.e.}}{=} \{z \in \mathbb{C}\colon \operatorname{Im} z = 0, \operatorname{Re} z \le 0\}.$$

The principal branch of the logarithm function is the function

$$\operatorname{Log}\colon \mathbb{C} \setminus B \to \mathbb{C} \qquad \text{given by} \qquad \boxed{\operatorname{Log}(z) := \ln|z| + i \operatorname{Arg} z}$$

where $\ln\colon (0, \infty) \to \mathbb{R}$ is the ususal (real) natural logarithm function.

Towards forming another branch of the logarithm function, fix $\alpha \in \mathbb{R}$. Form the branch cut

$$B_\alpha = \left\{re^{i\alpha}\colon r \ge 0\right\}.$$

Then another branch of the logarithm function is the function

$$\log\colon \mathbb{C} \setminus B_\alpha \to \mathbb{C} \quad \text{given by} \quad \log(z) := \ln|z| + i \arg z \quad \text{where} \quad \alpha < \arg z < \alpha + 2\pi \;,$$

which satisfies

$$\log\left(e^z\right) = z \qquad \text{, for each} \quad z \in \mathbb{C} \setminus B_\alpha \quad \text{such that} \quad \alpha < \operatorname{Im} z < \alpha + 2\pi$$

⟨for $x+iy \in \mathbb{C}\setminus\{0\}$ note: $\ln\left|e^{x+iy}\right| = \ln\left(\left|e^x\right|\left|e^{iy}\right|\right) = \ln e^x = x$ and $\arg\left(e^{x+iy}\right) = \arg\left(e^x e^{iy}\right) = \arg\left(e^{iy}\right) = \{y + 2\pi k\colon k \in \mathbb{Z}\}$⟩.