Def. $f\colon G\setminus\{a\}\to\mathbb{C}$ has an isolated singularity at a provided $f\in H(G\setminus\{a\})$ & G open & $a\in G$.

Thoughout this handout, $f\colon G\setminus\{a\}\to\mathbb{C}$ where G is an open subset of $\mathbb{C}$ and $a\in G$ and

$$f\in H(G\setminus\{a\})\ ;$$

i.e., a **is an isolated singularity of** f.

Def. Types of isolated singularities:

(1) a is a removable singularity of f provided
  - (i) a is an isolated singularity of f
  - (ii) there exists $\tilde f\in H(G)$ such that $\tilde f\big|_{G\setminus\{a\}} = f\big|_{G\setminus\{a\}}$.

(2) a is a pole of f provided
  - (i) a is an isolated singularity of f
  - (ii) $\lim_{z\to a}|f(z)|=\infty$.

(3) a is an essential singularity of f provided
  - (i) a is an isolated singularity of f
  - (ii) a is neither a removable singularity nor a pole of f.

Summary. Let a be an isolated singularity of f.
Let $\{c_n\}_{n\in\mathbb{Z}}$ be f's Laurent Series Expansion Coefficients (see (LSE)).

1. **TFAE** to a is a removable singularity of f.
   - (1a) Def: $\exists\ \tilde f\in H(G)$ such that $\tilde f\big|_{G\setminus\{a\}} = f\big|_{G\setminus\{a\}}$.
   - (1b) $\lim_{z\to a} f(z)$ exists.
   - (1c) f is bounded on $B_r'(a)$ for some $r>0$.
   - (1d) $c_n=0$ for each $n\le -1$

2. **TFAE** to a is a pole of f.
   - (2a) Def: $\lim_{z\to a}|f(z)|=\infty$
   - (2b) $\exists!\ m\in\mathbb{N}$ & $g\in H(G)$ with $g(a)\neq 0$ s.t. $\forall z\in G\setminus\{a\}$: $f(z)=\frac{g(z)}{(z-a)^m}$. $\langle$so $g(z)=(z-a)^m f(z)\rangle$.
   - (2c) $\exists!\ m\in\mathbb{N}$ and $\{c_n\}_{n=-m}^{-1}$ from $\mathbb{C}$ with $c_{-m}\neq 0$ s.t. a is a removable singularity of $f(z)-\sum_{n=-m}^{-1} c_n (z-a)^n$.
   - ∘ The above m is the order of the pole and $\sum_{n=-m}^{-1} c_n(z-a)^n$ is the principal part of f.
   - ∘ Also, [f has a pole at a of order m] $\iff$ [$c_k=0$ if $k<-m$ and $c_{-m}\neq 0$] .

3. **TFAE** to a is an essential singularity of f.
   - (3a) Def: a is neither a removable singularity nor a pole of f.
   - (3b) $\forall r>0$ s.t. $B_r(a)\subset G$, $f(B_r'(a))$ is dense in $\mathbb{C}$, i.e., $\overline{f(B_r'(a))}=\mathbb{C}$ (Casorati-Weierstrass Thm.)
   - (3c) $\exists$ sequences $\{z_n^1\}_{n\in\mathbb{N}}$ & $\{z_n^2\}_{n\in\mathbb{N}}$ in $G\setminus\{a\}$, each converging to a, s.t. $\{f(z_n^1)\}_n$ & $\{f(z_n^2)\}_n$ each converge but to different points in $\mathbb{C}$.
   - (3d) $c_n\neq 0$ for infinitly many $n<0$.

Examples. of $f\colon\mathbb{C}\setminus\{0\}\to\mathbb{C}$ with isolated singularity at $z=0$.

(1) $f(z)=\dfrac{z^2}{z}$ (2) $f(z)=\dfrac{1}{z}$ (3) $f(z)=e^{1/z}$

<u>Thm. 2.8</u>. <u>Laurent Series Expansion</u>. Still have $f \in H(G \setminus \{a\})$.

There exists $R > 0$ and $\{c_n\}_{n\in\mathbb{Z}}$ from $\mathbb{C}$ s.t. for each $z \in B'_R(a)$,

$$f(z) \;=\; \underbrace{\sum_{n=-\infty}^{-1} c_n (z-a)^n}_{\text{singular part}} \;+\; \underbrace{\sum_{n=0}^{\infty} c_n (z-a)^n}_{\text{analytic part}} \;\overset{\text{in short}}{=}\; \sum_{n=-\infty}^{\infty} c_n (z-a)^n \tag{LSE}$$

where each series in (LSE) converge absolutely for each (fixed) $z \in B'_R(a)$.

For $k \in \mathbb{N}$, let

$$S_k(z) := \sum_{n=-k}^{-1} c_n (z-a)^n \qquad \text{and} \qquad A_k(z) := \sum_{n=0}^{k} c_n (z-a)^n$$

Let $0 < r_1 < r_2 < R$. Futhermore:

(A1) $A_k \in H(\mathbb{C})$ for each $k \in N$

(A2) $\{A_k\}_{k\in\mathbb{N}}$ converges uniformly to the analytic part on $\{z \in \mathbb{C} \colon |z-a| \le r_2\}$

(A3) the analytic part is in $H(B_R(a))$

(S1) $S_k \in H(\mathbb{C} \setminus \{a\})$ for each $k \in N$

(S2) $\{S_k\}_{k\in\mathbb{N}}$ converges uniformly to the singular part on $\{z \in \mathbb{C} \colon r_1 \le |z-a|\}$

(S3) the singular part is in $H(\mathbb{C} \setminus \{a\})$

Thus each series in (LSE) converges uniformly on each closed annulus $\{z \in \mathbb{C} \colon r_1 \le |z-a| \le r_2\}$.

Also, if $0 < r < R$ and $\gamma_r(t) = a + re^{it}$ with $0 \le t \le 2\pi$, then

$$c_n \;=\; \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta\ . \tag{LSC}$$

Any pointwise convergent expansion of f on $B'_R(a)$ of this form equals the Laurent expansion.

<u>Lemma for (1)+(2)=(3)</u>. Let $\{h_n\}_{n\in\mathbb{N}}$ be a sequence from $H(B_\varepsilon(z_0))$ that convergences uniformly to a function h on each $\overline{B_{\varepsilon_0}(z_0)}$ with $0 < \varepsilon_0 < \varepsilon$. Then $h \in H(B_\varepsilon(z_0))$.

Proof. Let γ be a closed contour in $B_\varepsilon(z_0)$. Then

$$0 \;\underset{\text{Thm 2.12}}{\overset{\text{Cauchy Thm for } \star}{=}}\; \int_\gamma h_n(z)\, dz \xrightarrow{n\to\infty,\text{by uniform conv.}} \int_\gamma h(z)\, dz\ .$$

Note $h \in C(B_\varepsilon(z_0))$ since the uniform limit of continuous functions is continuous. So by Morea's theorem ⟨Thm II.2.25⟩, $h \in H(B_\varepsilon(z_0))$. □

<u>Comment about uniqueness</u>. If, for some $\{c_n\}_{n\in\mathbb{Z}}$, f has an expansion of the form (LSE) that converges pointwise on $B'_R(a)$, then from the proof (A2) and (S2) hold, so we get uniform convergence on γ_r^*. For $n, k \in \mathbb{Z}$, $\int_{\gamma_r} \frac{(z-a)^k}{(z-a)^{n+1}}\, dz$ is $2\pi i$ if $n = k$ and 0 otherwise. So for any $n \in \mathbb{Z}$

$$\int_{\gamma_r} \frac{f(z)}{(z-a)^{n+1}}\, dz \;=\; \int_{\gamma_r} \frac{\sum_{k=-\infty}^{\infty} c_k (z-a)^k}{(z-a)^{n+1}}\, dz \;\overset{\text{by unif. conv.}}{=}\; \sum_{k=-\infty}^{\infty} c_k \left[\int_{\gamma_r} \frac{(z-a)^k}{(z-a)^{n+1}}\, dz\right] \;=\; c_n\,[2\pi i]\ .$$

So (LSC) holds.