Throughout, X and Y are metric spaces and

$$X = U \biguplus V$$

denotes that X is the disjoint union of U and V. Also, $a, b \in \mathbb{R}$ with $a < b$ and

$$S, C, D, P \subset X \ .$$

Def. (U, V) is a separation of X provided

(1) U and V are X-open subsets of X
(2) $U \neq \emptyset$ and $V \neq \emptyset$
(3) $X = U \biguplus V$.

Def. (U, V) is a X-separation of D provided

(1) U and V are X-open subsets of X
(2) $U \cap D \neq \emptyset$ and $V \cap D \neq \emptyset$
(3) $D \subset U \cup V$
(4) $U \cap V \subset D^C$

Note that (4) is equivalent to (4′) $U \cap V \cap D = \emptyset$.
Def. 2.4.3/4.

X is connected	$\Leftrightarrow$	$\nexists$ a separation of X .
X is disconnected	$\Leftrightarrow$	$\exists$ a separation of X .
C a is connected set in (X, d)	$\Leftrightarrow$	$\left[C = \emptyset \ \text{ or } \ \left(C, d\big|_C\right) \text{ is connected} \right]$.
D a is disconnected set in X	$\Leftrightarrow$	D is not a connected set in X .

Thm. 2.4.5. D a is disconnected set in X $\Leftrightarrow$ $\exists$ a X-separation of D.
Thm. 2.4.6. D a is disconnected set in X $\Leftrightarrow$ $\exists$ a X-separation (U, V) of D with $U \cap V = \emptyset$.
Thm. 2.4.2. TFAE.

(1) The only subsets of X with are both open and closed are $\emptyset$ and X.
(2) X is connected.

Thm. 2.4.8. The connected subsets of $\mathbb{R}$ are the intervals. ⟨here, consider $\emptyset$ as the degenerate interval⟩
Example 2.4.7. Easy but useful comments.

(iv) If X has a separation (U, V) and C is a connected set in X, then either $C \subset U$ or $C \subset V$.
(v) X is connected $\Leftrightarrow \forall x, y \in X$ there is a connected subset C in X with $x, y \in C$.

Def. 2.4.13. A path in S is a function $\gamma \colon [a, b] \xrightarrow{\text{cont.}} X$ such that it's track $\gamma^* {:=} \gamma\left([a, b]\right) \subset S$.

- A path is simple provided "it does not cross itself expcept possibly at the endpoints".
- A path γ in $\mathbb{R}^n$ is a polygonal path provided "γ^* is the finite union of line segments".
- A path γ in $\mathbb{R}^n$ is a p-path provided "γ^* is the finite union of line segments $\|$ to coord. axes".
- ⟨For a polygonal path γ, we can write: $\gamma^* = \cup_{j=1}^{k} [x^{(j-1)}, x^{(j)}]$⟩

Def. 2.4.16. P is path-connected $\iff \forall x, y \in P$, there is a path in P from x to y.
Thm. 2.4.11/Exercise 2.4.33:6. Let $f \colon X \xrightarrow{\text{cont.}} Y$.

$$[\, C \text{ connected in } X \,] \implies [\, f(C) \text{ connected in } Y \,]$$
$$[\, P \text{ path connected in } X \,] \implies [\, f(P) \text{ path connected in } Y \,]$$

Thm. 2.4.20. P path-connected set $\implies$ P is connected. ⟨converse is false, example 2.4.21ii⟩
Exer. 2.4.30:5. If P_1 & P_2 are path-connected and not disjoint, then $P_1 \cup P_2$ is path-connected.

Connected and Path-Connected. Let $C \subset C_0 \subset \overline{C}$.

$$[\,C \text{ connected}] \implies [\,C_0 \text{ connected}] \tag{2.4.33:1}$$
$$[\,C \text{ connected}] \implies [\,\overline{C} \text{ connected}]$$
$$[\,P \text{ path-connected}] \not\Rightarrow [\,\overline{P} \text{ path-connected}] \tag{2.4.21iii}$$
$$[\,S \text{ connected}] \not\Rightarrow [\,S \text{ path-connected}] \tag{2.4.21ii}$$
$$[\,S \text{ path-connected}] \implies [\,S \text{ connected}] \tag{Thm. 2.4.20}$$
$$[\,x \in \mathbb{R}^n\,] \implies [\,B_\varepsilon(x) \text{ and } \mathbb{R}^n \text{ are path connected}] \tag{2.4.21i}$$

Thm. 2.4.22. Let G be an open subset of $\mathbb{R}^n$. TFAE.

(1) G is connected.
(2) G is path-connected.
(3) G is p-path-connected ⟨i.e., can even take the path to be a p-path⟩.

Components

Def. 2.4.24/29. Let $S \neq \emptyset$. ⟨maximal is in the sense of set containment⟩

- C is a component of S provided C is a maximal connected subset of S.
  ⟨i.e., $C \subset S$ and C is connected and if $C \subset C_1 \subset S$ and C_1 is connected, then $C = C_1$⟩
- P is a path-component of S provided P is a maximal path-connected subset of S.

Lemma 2.4.2$^+$. Let $s \in S$. ⟨One uses Exercise 2.3.33:5. to show (2).⟩

(1) The union of a non-empty family of connected subsets of S containing s is connected.
(2) The union of a non-empty family of path-connected subsets of S containing s is path-connected.

Thm. 2.4.26$^+$. Let $S \neq \emptyset$. For each $s \in S$, let

$$C_s := \bigcup \{C \subset S \colon s \in C \text{ and } C \text{ is connected}\}$$
$$P_s := \bigcup \{P \subset S \colon s \in P \text{ and } P \text{ is path-connected}\}$$

(1) Each C_s is a component of S. Each P_s is a path-component of S.
(2) Any two C_s's are either equal or disjoint. Any two P_s's are either equal or disjoint.
(3) From (2) it follows that there exist $\Gamma_c, \Gamma_p \subset S$ such that

$$S = \biguplus_{s \in \Gamma_c} C_s \qquad \text{and} \qquad S = \biguplus_{s \in \Gamma_p} P_s\,. \tag{3}$$

Thm. 2.4.27. If $\emptyset \neq G \subset \mathbb{R}^n$ and G is open, then each component of G is open and the Γ_c in (3) is countable.

Remark. By taking $S = X$ in Thm 2.4.26$^+$ part (3), it follows that if each component (resp. path-component) of X is X-open, then each component (resp. path-component) of X is X-closed.

Thm 2.4.30a. TFAE.

- Each path-component of X is open in X.
- $\forall x \in X$ there is a path-connected neighborhood containing x.

Thm 2.4.30b. TFAE.

- X is path-connected of X.
- X is connected and $\forall x \in X$ there is a path-connected neighborhood containing x.
- X is connected and each path-component of X is open in X.

Thm 2.4.31. Let (X, d_X) and (Y, d_Y) be connected (resp. path-connected). Then $\left(X \times Y, \left[(d_x)^2 + (d_Y)^2\right]^{1/2}\right)$ is connected (resp. path-connected).