Set-up/Notation

Throughout this handout:

- (X, d) is a metric space
- $S \subset X$, i.e., $S \in \mathcal{P}(X) =$ the power set of X.

We write "A is (X, d)-open" to mean that

$$A \subset X \qquad \text{and} \qquad A \text{ is open in the metric } (X, d).$$

When d and/or X are understood, we often say just "A is X-open" or "A is d-open" or "A is open".

Definitions

1. $\mathcal{U}$ is a <u>covering</u> of S provided

$$\mathcal{U} \subset \mathcal{P}(X) \qquad \text{and} \qquad S \subset \bigcup_{U \in \mathcal{U}} U \,.$$

2. $\widetilde{\mathcal{U}}$ is a <u>subcovering</u> of a covering $\mathcal{U}$ of S provided

$$\widetilde{\mathcal{U}} \subset \mathcal{U} \qquad \text{and} \qquad S \subset \bigcup_{U \in \widetilde{\mathcal{U}}} U \,.$$

3. A covering $\mathcal{U}$ is a <u>finite covering</u> provided $\mathcal{U}$ has a finite number of elements.

4. A covering $\mathcal{U}$ is an <u>X-open covering</u> provided each element of $\mathcal{U}$ is X-open.

5. S is <u>compact</u> in X provided each X-open covering of S has a finite subcovering.

6. S is <u>totally bounded</u> provided, for each $\varepsilon > 0$, there is a finite covering of S by (open) ε-balls.

7. S is <u>sequentially compact</u> provided each sequence from S has a convergent subsequence (which convergences to a point in S).

8. S has the <u>Bolzano-Weierstrass property</u> (also pharsed S is <u>limit point compact</u>) provided each infinite subset of S has a limit point in S.

Remark

Definition 5 of compact varies slightly from the book's def'n. The two definitions are equivalent, as shown by Lemma 2.3.14, which follows from Lemma 2.1.5, which says, for a $U \subset S$,

$$U \text{ is open in } (S, d\,|_S) \qquad \Longleftrightarrow \qquad U = S \cap V \text{ for some } (X, d)\text{-open subset } V \text{ of } X.$$

Lemma. *2.3.14. Let $S \neq \emptyset$. ⟨So $(S, d\,|_S)$ is a metric space.⟩ Then*

$$S \text{ is compact in } (X, d) \qquad \Longleftrightarrow \qquad S \text{ is compact in } (S, d\,|_S) \,.$$

Goal: characterize compactness in a metric space

The Heine-Borel Thm. says that a subset of $\mathbb{R}$ is compact if and only if it is closed and bounded. Now we extend Heine-Borel to general metric spaces.

<u>Theorem 2.3.8</u>. TFAE:

a. X is compact.

b. $C(X) = \mathcal{C}(X)$, i.e., each continuous map $f : X \to \mathbb{R}$ is bounded.

c. X has the Bolzano-Weierstrass property.

d. X is sequentially compact.

e. X is complete and totally bounded.