This Homework Set presents the Stone-Weierstrass theorems, which generalize the well-known Weierstrass Approximation theorem, which says that a continuous $f\colon\ [0,1] \to \mathbb{R}$ can be uniformly approximated by polynomials.

A metric space is a Hausdorff topological[1] space. The Stone-Weierstrass theorems consider continuous functions defined on a compact Hausdorff topological space. Results are stated for an arbitrary compact Hausdorff topological space although you should just prove the results for an arbitrary compact metric space. Once you take some topology, you will see that your proofs for a metric space carry right over to a Hausdorff topological space. Let's begin.

View Point. Let $\mathbb{K}$ be the field $\mathbb{R}$ or $\mathbb{C}$. View the collection of all functions from a (nonempty) set S into $\mathbb{K}$ as vector space (over $\mathbb{K}$) where the vector space operations are given by the induced pointwise operations over $\mathbb{K}$.[2] Thanks to the product being defined in $\mathbb{K}$, we can even make sense of the product of two functions by defining fg as $(fg)(x) := [(f(x))] \cdot [(g(x))]$.

Recall 1. Let K be a compact Hausdorff topological space and $\mathbb{K}$ be the field $\mathbb{R}$ or $\mathbb{C}$.
Let $C(K,\mathbb{K})$ be the vector space (over $\mathbb{K}$) of all continuous functions from K into $\mathbb{K}$.
Equip $C(K,\mathbb{K})$ with the uniform[3] norm $\|\cdot\|_\infty$, which generates the uniform metric d_∞, where

$$\|f\|_\infty := \sup_{x\in K} |f(x)| \qquad \text{and} \qquad d_\infty(f,g) := \|f-g\|_\infty \overset{\text{i.e.}}{=} \sup_{x\in K} |f(x) - g(x)|$$

Also, $(C(K,\mathbb{K}), d_\infty)$ is a complete metric space. Since K is compact, $C(K,\mathbb{K}) \subset B(K,\mathbb{K})$ □

The Weierstrass Approximation Theorem can be phrased as:

Theorem 2 (Weierstrass Approximation Theorem)**.** .
The collection of polynominal[4] *is dense in* $(C([0,1],\mathbb{R}), d_\infty)$.

We now isolate some of the key properties of the collection of polynomial that help make approximation possible.

Definition 3. Let G be a collection of functions from S into $\mathbb{K}$, where S is a (nonempty) set and $\mathbb{K}$ is the field $\mathbb{R}$ or $\mathbb{C}$.

(1) G separates points (of S) provided for all $x, y \in S$ with $x \neq y$ there is a function $g \in G$ such that $g(x) \neq g(y)$.
(2) G is an algebra (of functions over $\mathbb{K}$) provided if $f, g \in G$ and $\alpha \in \mathbb{K}$, then $f+g$, fg, and αf belong to G. □

We can now state the Stone-Weierstrass theorems. The Weierstrass Approximation Thm. (Thm. 2) follows directly from Thm. 4. In fact, we will prove Thm. 4 without using Thm. 2.

[1] Let X be a topological space. X is Hausdorff provided for all $x, y \in X$ with $x \neq y$ there exists disjoint open sets U_x and U_y with $x \in U_x$ and $y \in U_y$.
[2] so the function $f+g$ is defined by $(f+g)(\cdot) := (f(\cdot)) + (g(\cdot))$ and αf is defined by $(\alpha f)(\cdot) := \alpha(f(\cdot))$.
[3] The term *uniform* comes from: $\|f_n - f\|_\infty \to 0$ iff f_n converges unformly to f on K.
[4] with real coefficients

Theorem 4 (Stone-Weierstrass theorem, real version so $\mathbb{K} = \mathbb{R}$)**.**
*Let K be a compact Hausdorff topological space. Let G be a **subalgebra** of $C(K, \mathbb{R})$ having the following two properties.*

(1) G separates points.
(2) G contains the constant function 1_K (where $1_K \colon K \to \mathbb{R}$ is defined by $1_K(x) = 1$ for each $x \in K$).

Then G is dense in $C(K, \mathbb{R})$.

The complex version of the Stone-Weierstrass theorem requires an additional hypothesis, namely, closure of G under complex conjugation.

Theorem 5 (Stone-Weierstrass theorem, complex version so $\mathbb{K} = \mathbb{C}$)**.**
*Let K be a compact Hausdorff topological space. Let G be a **subalgebra** of $C(K, \mathbb{C})$ having the following three properties.*

(1) G separates points.
(2) $1_K \in G$.
(3) $\overline{f}$ belongs to G for each $f \in G$ (where $\overline{f}$ is defined by $\overline{f}\,(x) := \overline{f\,(x)}$ for each $x \in K$).

Then G is dense in $C(K, \mathbb{C})$.

Our proof of the Stone-Weierstrass theorems will relies on one simple approximation of the square root function on the unit interval.

Lemma 6. *Let $v : [0, 1] \to \mathbb{R}$ be given by $v(\cdot) = \sqrt{\cdot}$. There exists an increasing[5] sequence $\{u_n\}_n$ of polynomials[6] that converges uniformly (i.e., in $(C([0, 1], \mathbb{R}), d_\infty)$) to v.*

Proof. Define $u_n \colon [0, 1] \to \mathbb{R}$ inductively by $u_1 \equiv 0$ (i.e., $u_1(t) = 0$ for each $t \in [0, 1]$) and

$$u_{n+1}(t) := u_n(t) + \frac{1}{2}\left(t - [u_n(t)]^2\right) \quad \text{for each } t \in [0,1] \text{ and } n \in N\ . \tag{1}$$

Then

$$u_n(t) \le \sqrt{t} \quad \text{for each } t \in [0,1] \text{ and } n \in N \tag{2}$$

follows by induction on n. Indeed, for a fixed $t \in [0, 1]$, clearly $u_1(t) \le \sqrt{t}$ and the inductive step follows from the observations that, for each $n \in \mathbb{N}$ with $n \ge 1$, if $u_n(t) \le \sqrt{t}$ then

$$\sqrt{t} + u_n(t) \le \sqrt{t} + \sqrt{t} \le 2 \qquad \text{so} \qquad \frac{1}{2}\left(\sqrt{t} + u_n(t)\right) \le 1$$

and so

$$\sqrt{t} - u_{n+1}(t) \overset{\text{by (1)}}{=} \sqrt{t} - u_n(t) - \frac{1}{2}\left(t - u_n^2(t)\right) \underset{\text{algebra}}{\overset{\text{by}}{=}} \left(\sqrt{t} - u_n(t)\right)\left(1 - \frac{1}{2}\left(\sqrt{t} + u_n(t)\right)\right) \ \ge 0.$$

Next note that (1) and (2) implies that, for each $t \in [0, 1]$, the sequence $\{u_n(t)\}_n$ is increasing (i.e., $u_n(t) \le u_{n+1}(t)$ for each $n \in N$) and bounded above; thus converges to some point $v(t) \in \mathbb{R}$. Equation (1) implies that $\frac{1}{2}\left(t - [v(t)]^2\right) = 0$, and as $v(t) \ge 0$, we have that $v(t) = \sqrt{t}$.

The remainder of the proof is an exercise. □

[5] i.e., if $t \in [0, 1]$, then $u_n(t) \le u_{n+1}(t)$ for each $n \in N$
[6] with real coefficients

Stone-Weierstrass Exercises

In these exercises, let K be a compact Hausdorff topological space (or just a compact metric space) and let G be a **subalgebra** of $C(K,\mathbb{R})$ having the following two properties.

(1) G separates points.

(2) G contains the constant function 1_K.

Let $\overline{G}$ be the closure of G in the metric space $(C(K,\mathbb{R}), d_\infty)$.

SW 1. Finish the proof of in Lemma 6.

⟨Hint: you do not need to reprove any prior results from class provided that you give the full statement of the result used.⟩

SW 2. Show that if $f \in G$ then $|f| \in \overline{G}$ (by using Lemma 6).

SW 3. Show $\overline{G}$ is an algebra that separated points of K and contains the constant function 1_K (by using the sequential characterization of closure).

SW 4. Show if $f,g \in \overline{G}$, then $\inf(f,g), \sup(f,g) \in \overline{G}$. ⟨Hint. If $a,b \in \mathbb{R}$, then $\sup(a,b) = \frac{a+b}{2} + \frac{|a-b|}{2}$.⟩

SW 5. Let $\alpha, \beta \in \mathbb{R}$ and $x,y \in K$ with $x \neq y$. Show there is a $f \in \overline{G}$ with $f(x) = \alpha$ and $f(y) = \beta$.

SW 6. Let $f \in C(K,\mathbb{R})$, $x \in K$, and $\epsilon > 0$.
Show that there exists a function g in $\overline{G}$ such that for all $y \in K$

$$g(x) = f(x)$$
$$g(y) \leq f(y) + \epsilon$$

Hint. (Please follow the notation I set up in this hint so it's not a nightmare to grade. Thanks!) First show that for each $z \in K$ there is a function $h_z \in \overline{G}$ such that

$$h_z(x) = f(x)$$
$$h_z(z) \leq f(z) + \frac{\varepsilon}{2}.$$

Next, for each $z \in K$, find an open set $V(z)$ containing z that that is particularly nice (you decide what nice is). Note $K \subset \bigcup_{z \in K} V(z)$.

SW 7. Show that G is dense in $(C(K,\mathbb{R}), d_\infty)$. I.e., show that $\overline{G}^{d_\infty} = C(K,\mathbb{R})$.

SW 8. Show the $\mathbb{C}$-version of the Stone-Weierstrass Theorem by reducing the $\mathbb{C}$-version (Thm. 5) to the $\mathbb{R}$-version (Thm. 4). Hint. For a $H \subset C(K,\mathbb{C})$, consider $G := \{\operatorname{Re} h : h \in H\} \cup \{\operatorname{Im} h : h \in H\}$. Recall: $\operatorname{Re} z = \frac{z+\bar{z}}{2}$ and $\operatorname{Im} z = \frac{z-\bar{z}}{2i}$.