

- ▷. Throughout this handout, $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are positive measure spaces.¹ Goal:

$$\int_X^* \left[\int_Y^* f(x, y) d\nu(y) \right] d\mu(x) = \int_{X \times Y}^* f d(\mu \times \nu) = \int_Y^* \left[\int_X^* f(x, y) d\mu(x) \right] d\nu(y). \quad (\text{FT})$$

- ▷. We begin by forming the product measurable space $(X \times Y, \mathcal{M} \otimes \mathcal{N})$ where

$$X \times Y = \{(x, y) : x \in X, y \in Y\}$$

$$\mathcal{M} \otimes \mathcal{N} = \sigma(\{M \times N : M \in \mathcal{M}, N \in \mathcal{N}\}),$$

The collection $\mathcal{R}$ of finite disjoint union of rectangles² of $X \times Y$ forms a field that generates $\mathcal{M} \otimes \mathcal{N}$. By the extension theorem, there is an unique measure, called the product measure,

$$\mu \times \nu : \mathcal{M} \otimes \mathcal{N} \rightarrow [0, \infty]$$

such that $(\mu \times \nu)(A \times B) = \mu(A) \nu(B)$ for each rectangle $A \times B \in \mathcal{M} \times \mathcal{N}$.

Also, if ν and μ are σ -finite, then $\mu \times \nu$ is also σ -finite.

Notation in Folland. $L^+(X, \mathcal{M}, \mu)$ is the space of $\mathcal{M}$ -measurable functions $f : X \rightarrow [0, \infty]$.

2.37. Fubini-Tonelli Theorem. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be positive σ -finite measure spaces.

Lemma. Let $f : X \times Y \rightarrow \widehat{\mathbb{R}}$ be $\mathcal{M} \otimes \mathcal{N}$ -measurable. Then

$$\begin{array}{llll} \forall x \in X & \text{the } x\text{-section} & f_x : Y \rightarrow [0, \infty] & \text{given by } f_x(y) = f(x, y) \text{ is } \mathcal{N}\text{-measurable} \\ \forall y \in Y & \text{the } y\text{-section} & f^y : X \rightarrow [0, \infty] & \text{given by } f^y(x) = f(x, y) \text{ is } \mathcal{M}\text{-measurable.} \end{array}$$

Tonelli. Let $f \in L^+(X \times Y)$. Then

$$\begin{array}{llll} g : X \rightarrow [0, \infty] & \text{given by} & g(x) = \int_Y f_x d\nu & \text{is in } L^+(X) \\ h : Y \rightarrow [0, \infty] & \text{given by} & h(y) = \int_X f^y d\mu & \text{is in } L^+(Y) \end{array}$$

and (FT) holds.

Fubini. Let $f \in L^1(\mu \times \nu)$. Then

$$\begin{array}{ll} f_x \in L^1(\nu) & \text{for a.e. } x \in X \\ f^y \in L^1(\mu) & \text{for a.e. } y \in Y \end{array}$$

and

$$\begin{array}{llll} g : X \rightarrow \widehat{\mathbb{R}} & \text{defined a.e. by} & g(x) = \int_Y f_x d\nu & \text{is in } L^1(\mu) \\ h : Y \rightarrow \widehat{\mathbb{R}} & \text{defined a.e. by} & h(y) = \int_X f^y d\mu & \text{is in } L^1(\nu) \end{array}$$

and (FT) holds.

- ▷. **Goal:** to reverse the order of integration in an iterated integral $\int \int f d\mu d\nu$ of a $(\mu \times \nu)$ -measurable f .

Attack: use the Fubini and Tonelli theorems in tandem.

First: Apply Tonelli to $|f|$ to get $f \in L^1(X \times Y)$. Do this by showing $\int_{X \times Y} |f| d(\mu \times \nu) < \infty$ by evaluating the integral as an iterated integral (in whichever order is more do-able).

Second: Then can apply Fubini to conclude that $\int_Y \int_X f d\mu d\nu = \int_X \int_Y f d\nu d\mu$.

¹So $\mathcal{M}$ and $\mathcal{N}$ are σ -fields and the measures $\mu : \mathcal{M} \rightarrow [0, \infty]$ while $\nu : \mathcal{N} \rightarrow [0, \infty]$.

²a rectangle is a set $M \times N$ where $M \in \mathcal{M}$ and $N \in \mathcal{N}$

- ▷. Even if $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are complete positive finite measure spaces, their product measure space $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$ is rarely complete.

2.39. Fubini-Tonelli Theorem for complete measures.

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be complete positive σ -finite measure spaces.

Let $(X \times Y, \mathcal{L}, \lambda)$ be the completion of $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$.

Lemma. Let $f: X \times Y \rightarrow \widehat{\mathbb{R}}$ be $\mathcal{L}$ -measurable. Then

for a.e. $x \in X$ the x -section $f_x: Y \rightarrow [0, \infty]$ given by $f_x(y) = f(x, y)$ is $\mathcal{N}$ -measurable
 for a.e. $y \in Y$ the y -section $f^y: X \rightarrow [0, \infty]$ given by $f^y(x) = f(x, y)$ is $\mathcal{M}$ -measurable.

Tonelli. Let $f \in L^+((X \times Y, \mathcal{L}, \lambda))$. Then

$$g: X \rightarrow [0, \infty] \quad \text{defined a.e. by} \quad g(x) = \int_Y f_x d\nu \quad \text{is in} \quad L^+(X)$$

$$h: Y \rightarrow [0, \infty] \quad \text{defined a.e. by} \quad h(y) = \int_X f^y d\mu \quad \text{is in} \quad L^+(Y)$$

and

$$\int_X \left[\int_Y^* f(x, y) d\nu(y) \right] d\mu(x) = \int_{X \times Y}^* f d\lambda = \int_Y \left[\int_X^* f(x, y) d\mu(x) \right] d\nu(y). \quad (\overline{\text{FT}})$$

Fubini. Let $f \in L^1((X \times Y, \mathcal{L}, \lambda))$. Then

$$f_x \in L^1(\nu) \quad \text{for a.e. } x \in X$$

$$f^y \in L^1(\mu) \quad \text{for a.e. } y \in Y$$

and the a.e. defined functions

$$g: X \rightarrow \widehat{\mathbb{R}} \quad \text{defined a.e. by} \quad g(x) = \int_Y f_x d\nu \quad \text{is in} \quad L^1(\mu)$$

$$h: Y \rightarrow \widehat{\mathbb{R}} \quad \text{defined a.e. by} \quad h(y) = \int_X f^y d\mu \quad \text{is in} \quad L^1(\nu)$$

and $(\overline{\text{FT}})$ holds.

A good place to read up on the Fubini-Tonelli Theorem is *Real Analysis. Modern Techniques and Their Applications* by Gerald R. Folland. In the second edition, §1.2 talks about the σ -field generated by a product, see

<http://people.math.sc.edu/girardi/m7034/1617/hw/FollandRA2edSigmaField.pdf>

(esp. Prop. 1.3) while §2.5 talks about product measures, see

<http://people.math.sc.edu/girardi/m7034/1617/hw/FollandRA2edProdMsr.pdf>

(esp. Prop 2.34, Thm 2.37, and Thm 2.39).