Equivalence relations are used in many branches of math. You might have already encountered them in your undergraduate classes or Math 701. We will them in both Maths 703 and 704.

Definition. A ***relation*** on a set A is a subset C of the cartesian product $A \times A$.

If C is a relation on a set A, we use the notation xCy to mean the same thing as $(x, y) \in C$ and we say x is in the relation C to y. Furthermore, for any $x, y \in A$, either $(x, y) \in C$ or $(x, y) \notin C$; thus either xCy (i.e., x is in the relation C to y) or $x \not{C} y$ (i.e., x is not in the relation C to y).

Equivalence Relations and Partitions

An ***equivalence relation*** on a set A is a relation C on A having the following three properties:

(1) (Reflexivity) xCx for every x in A.

(2) (Symmetry) If xCy, then yCx.

(3) (Transitivity) If xCy and yCz, then xCz.

There is no reason one must use a capital letter—or indeed a letter of any sort—to denote a relation, even though it *is* a set. Another symbol will do just as well. One symbol that is frequently used to denote an equivalence relation is the "tilde" symbol $\sim$. Stated in this notation, the properties of an equivalence relation become

(1) $x \sim x$ for every x in A.

(2) If $x \sim y$, then $y \sim x$.

(3) If $x \sim y$ and $y \sim z$, then $x \sim z$.

There are many other symbols that have been devised to stand for particular equivalence relations; we shall meet some of them in the pages of this book.

Given an equivalence relation $\sim$ on a set A and an element x of A, we define a certain subset E of A, called the ***equivalence class*** determined by x, by the equation

$$E = \{y \mid y \sim x\}.$$

Another common notation for the equivalence class determinded by $x \in A$ is $[x]$; thus

$$[x] \;=\; \{y \in A \colon y \sim x\}\,.$$

Rmk 3.0. Let $x \in A$. Then x is in the equivalence class determined by x since $x \sim x$; thus, $x \in [x] \subset A$.

EXAMPLE 3. Define two points in the plane to be equivalent if they lie at the same distance from the origin. Reflexivity, symmetry, and transitivity hold trivially. The collection $\mathcal{E}$ of equivalence classes consists of all circles centered at the origin, along with the set consisting of the origin alone.

EXAMPLE 4. Define two points of the plane to be equivalent if they have the same y-coordinate. The collection of equivalence classes is the collection of all straight lines in the plane parallel to the x-axis.

Lemma 3.1. *Two equivalence classes E and E' are either disjoint or equal.*

Proof. Let E be the equivalence class determined by x, and let E' be the equivalence class determined by x'. Suppose that $E \cap E'$ is not empty; let y be a point of $E \cap E'$. See Figure 3.1. We show that $E = E'$.

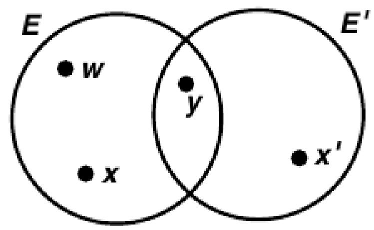

Figure 3.1

By definition, we have $y \sim x$ and $y \sim x'$. Symmetry allows us to conclude that $x \sim y$ and $y \sim x'$; from transitivity it follows that $x \sim x'$. If now w is any point of E, we have $w \sim x$ by definition; it follows from another application of transitivity that $w \sim x'$. We conclude that $E \subset E'$.

The symmetry of the situation allows us to conclude that $E' \subset E$ as well, so that $E = E'$. ■

Lemma 3.1 and Remark 3.0 now give the below key Theorem.

Theorem. *Given an equivalence relation on a set A, let $\mathcal{E}$ be the collection of all distinct equivalence classes determined by this relation. Then A is the disjoint union of the sets from $\mathcal{E}$, i.e.,*

$$A = \biguplus_{E \in \mathcal{E}} E.$$

The below remarks about partition are just for your information.

Definition. A ***partition*** of a set A is a collection of disjoint nonempty subsets of A whose union is all of A.

Studying equivalence relations on a set A and studying partitions of A are really the same thing. Given any partition $\mathcal{D}$ of A, there is exactly one equivalence relation on A from which it is derived.

The proof is not difficult. To show that the partition $\mathcal{D}$ comes from some equivalence relation, let us define a relation C on A by setting xCy if x and y belong to the same element of $\mathcal{D}$. Symmetry of C is obvious; reflexivity follows from the fact that the union of the elements of $\mathcal{D}$ equals all of A; transitivity follows from the fact that distinct elements of $\mathcal{D}$ are disjoint. It is simple to check that the collection of equivalence classes determined by C is precisely the collection $\mathcal{D}$.

To show there is only one such equivalence relation, suppose that C_1 and C_2 are two equivalence relations on A that give rise to the same collection of equivalence classes $\mathcal{D}$. Given $x \in A$, we show that yC_1x if and only if yC_2x, from which we conclude that $C_1 = C_2$. Let E_1 be the equivalence class determined by x relative to the relation C_1; let E_2 be the equivalence class determined by x relative to the relation C_2. Then E_1 is an element of $\mathcal{D}$, so that it must equal the unique element D of $\mathcal{D}$ that contains x. Similarly, E_2 must equal D. Now by definition. E_1 consists of all y such that yC_1x; and E_2 consists of all y such that yC_2x. Since $E_1 = D = E_2$, our result is proved.