Complex Script

for the

Complex Variables

portion of

Math 703/704

Notes prepared by Prof. Schep (Fall 2011) for quals preparation.

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CHAPTER 1

Holomorphic (or Analytic) Functions

1. Definitions and elementary properties

In complex analysis we study functions $f : S \to \mathbb{C}$, where $S \subset \mathbb{C}$. When referring to open sets in $\mathbb{C}$ and continuity of functions f we will always consider $\mathbb{C}$ (and its subsets) as a metric space with respect to the metric $d(z_1, z_2) = |z_1 - z_2|$, where $|\cdot|$ denotes the complex modulus, i.e., $|z| = \sqrt{x^2 + y^2}$ whenever $z = x + iy$ with $x, y \in \mathbb{R}$. An open ball with respect this metric will be also referred to as an open disc and denoted by

$$B(a, B(a, r) = \{z \in \mathbb{C} : |z - a| < r\},$$

where a is the center and $r > 0$ is the radius of the open ball. The closed disc with center a and radius r is denoted by $\overline{B(a, r)}$, so

$$\overline{B(a, r)} = \{z \in \mathbb{C} : |z - a| \leq r\}.$$

Recall that $G \subset \mathbb{C}$ is called open if for all $a \in G$ there exists $r > 0$ such that $B(a, r) \subset G$.

If $z = x + iy$, then the conjugate $\overline{z}$ of z is defined by $\overline{z} = x - iy$. Now $z\overline{z} = |z|^2$, so that $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$ for $z \neq 0$. Elementary properties of complex numbers are given by:

(1) The real part $\operatorname{Re} z$ of z satisfies $\operatorname{Re} z = \frac{1}{2}(z + \overline{z})$, while the imaginary part $\operatorname{Im} z$ of z is given by $\operatorname{Im} z = \frac{1}{2i}(z - \overline{z})$.
(2) For all $z_1, z_2 \in \mathbb{C}$ we have $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ and $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$.
(3) For all $z_1, z_2 \in \mathbb{C}$ we have $|z_1 z_2| = |z_1|\,|z_2|$.

2. Elementary transcendental functions

Recall also that if $z = x+iy \neq 0$, then, using polar coordinates, we can write $z = r\cos\theta + ir\sin\theta$. In this case we write $\arg z = \{\theta + 2k\pi : k \in \mathbb{Z}\}$. By $\operatorname{Arg} z$ we will denote the principal value of the argument of $z \neq 0$, i.e. $\theta = \operatorname{Arg} z \in \arg z$ if $-\pi < \theta \leq \pi$. Note that if $z_1 = |z_1|(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = |z_2|(\cos\theta_2 + i\sin\theta_2)$, then we have $z_1 z_2 = |z_1||z_2|(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)) = |z_1 z_2|(\cos(\theta_1 + \theta_2) + i(\sin(\theta_1 + \theta_2))$. Hence we have $\arg(z_1 z_2) = \arg z_1 + \arg z_2$. Define now $e^z = e^x(\cos y + i\sin y)$. Then $|e^z| = e^x$ and $\arg e^z = y + 2k\pi$. In particular $e^{2\pi i} = 1$ and the function e^z is $2\pi i$-periodic, i.e., $e^{z+2\pi i} = e^z e^{2\pi i} = e^z$ for all $z \in \mathbb{C}$. We want now to define $\log w$ such that $w = e^z$ where $z = \log w$, but we can not define it as just the inverse of e^z as e^z is not one-to-one. Consider therefore the equation $w = e^z$ for a given w. We must assume that $w \neq 0$ as $e^z \neq 0$ (and thus $\log 0$ is not defined). Then $|w| = |e^z| = e^x$ and $y = \operatorname{Arg} w + 2k\pi$ $(k \in \mathbb{Z})$. Hence $\{\log|w| + i(\operatorname{Arg} w + 2k\pi) : k \in \mathbb{Z}\}$ is the set of all solutions z of $w = e^z$. We write $\log w$ for any w in the set $\{\log|w| + i(\operatorname{Arg} w + 2k\pi) : k \in \mathbb{Z}\}$.

Definition 2.1. Let $G \subset \mathbb{C}$ be an open connected set and $f : G \to \mathbb{C}$ a continuous function such that $z = e^{f(z)}$ for all $z \in G$. Then f is called a *branch of the logarithm* on G.

It is clear that if f is a branch of the logarithm on G, then $0 \notin G$ and $f(z) = \log|z| + i(\text{Arg } z + 2k\pi)$ for some $k \in \mathbb{Z}$, where k can depend on z. Also, if f is a branch of the logarithm on G, then for fixed k also $g(z) = f(z) + 2k\pi i$ is a branch of the logarithm on G. The converse also holds.

Proposition 2.2. *Let $G \subset \mathbb{C}$ be an open connected set and $f : G \to \mathbb{C}$ a branch of the logarithm on G. Then every other branch of the logarithm on G is of the form $f + 2k\pi i$ for some fixed $k \in \mathbb{Z}$.*

Proof. Suppose g is another branch of the logarithm on G. Then define $h = \frac{1}{2\pi i}(f - g)$. Then h is continuous on G, $h(G) \subset \mathbb{Z}$, and G connected implies that $h(G) = \{k\}$ for some $k \in \mathbb{Z}$. $\square$

To find a branch of $\log z$ for a given open and connected set G requires finding (as $\log|z|$ is continuous on $\mathbb{C}\setminus\{0\}$) a continuous selection of arg z in $\{\text{Arg } z + 2k\pi\}$. As G is connected, the range of this continuous selection has to be an interval of length at most 2π, but such a selection does not always exist! This happens e.g. in case $G = \mathbb{C} \setminus \{0\}$, then G is open and connected, but there does not exist a branch of $\log z$ on G, i.e., Arg z is discontinuous on the negative x-axis. in the next examples we construct some branches of $\log z$.

Example 2.3.

(i) Let $G = \mathbb{C}\setminus\{z \in \mathbb{R} : z \leq 0\}$. Then Arg z is continuous on G, so $f(z) = \log|z| + i\text{Arg } z$ is a branch of $\log z$ on G. This branch is called the *principal branch* of $\log z$ and denoted by Log z.

(ii) Let $G = \mathbb{C} \setminus \{z \in \mathbb{R} : z \geq 0\}$. Let $\theta(z)$ denote the unique value of arg z such that $0 < \theta(z) < 2\pi$. Then $f(z) = \log|z| + i\theta(z)$ is a branch of $\log z$ on G.

3. Differentiable functions

Definition 3.1. Let $G \subset \mathbb{C}$ be an open set and $f : G \to \mathbb{C}$. Then f is *differentiable* at $z \in G$ if

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists. When this limit exists we denote it by $f'(z)$ and call it the (complex) derivative of f at z. If $f'(z)$ exists at every point of G, then we call f *analytic* or *holomorphic* on G.

Notation. $H(G) = \{f : g \to \mathbb{C}; f \text{ holomorphic in } G\}$.

If $S \subset \mathbb{C}$ is any set, then we say that f is holomorphic in S if $f \in H(G)$ for some open set $G \supset S$.

Remarks 3.2.

1. The function f is differentiable at $z \in G$, if for $|h|$ small enough we can write $f(z + h) = f(z) + f'(z)h + \epsilon(h)h$, where $\epsilon(h) \to 0$ as $h \to 0$. From this it follows directly that if f is differentiable at z, then f is continuous at z.

2. Note that f is differentiable at $z_0 \in G$ with derivative equal to $f'(z_0)$ is equivalent to saying that for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\left| \frac{f(z+h) - f(z)}{h} - f'(z_0) \right| < \epsilon$$

for all $h \in \mathbb{C}$ with $0 < |h| < \delta$. In particular we can take $h = x$ with x real and $0 < |x| < \delta$ or $h = iy$ with y real and $0 < |y| < \delta$. This fact will be exploited in the proof of the next theorem.

THEOREM 3.3. *(Cauchy–Riemann equations) Let $G \subset \mathbb{C}$ be an open set and $f : G \to \mathbb{C}$ be differentiable at $z = x + iy \in G$. Let $f(z) = u(x, y) + iv(x, y)$, where u and v are real valued functions on G. Then the first order partials $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ exist at (x, y) and satisfy the Cauchy–Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at the point (x, y).

PROOF. In the definition of the derivative we can restrict ourselves first to real valued $h \to 0$. We get then that

$$f'(z) = \lim_{h\to 0, h\in\mathbb{R}} \left\{ \frac{u(x+h, y) - u(x, y)}{h} + i\frac{v(x+h, y) - v(x, y)}{h} \right\} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

exists at $z = x + iy$ and similarly by restricting to $h = ik$ with k real valued and $k \to 0$, we get

$$f'(z) = \lim_{k\to 0, k\in\mathbb{R}} \left\{ \frac{u(x, y+k) - u(x, y)}{ik} + i\frac{v(x, y+k) - v(x, y)}{ik} \right\} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Equating the two expressions for $f'(z)$ we get that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at the point (x, y). □

EXAMPLE 3.4.

(i) Let $f(z) = \overline{z}z = x^2 + y^2$. Then $\frac{\partial u}{\partial x} = 2x$, $\frac{\partial v}{\partial y} = 0$, $\frac{\partial u}{\partial y} = 2y$ and $\frac{\partial v}{\partial x} = 0$. Hence the Cauchy–Riemann equations hold if and only if $(x, y) = (0, 0)$. At $z = 0$ we have

$$\frac{f(0+h) - f(0)}{h} = \overline{h} \to 0$$

as $h \to 0$. Hence f is differentiable only at $z = 0$ and thus nowhere holomorphic as there exists no open set G containing 0 on which f is differentiable.

(ii) Let $f(z) = c$, where $c \in \mathbb{C}$ is a constant. Then $f'(z) = 0$ for all $z \in \mathbb{C}$, so $f \in H(\mathbb{C})$. Similarly if $g(z) = z$, then $g'(z) = 1$ for all $z \in \mathbb{C}$, so $g \in H(\mathbb{C})$

(iii) Let $f(z) = 1/z$ on $\mathbb{C} \setminus \{0\}$. Then

$$\frac{f(z+h) - f(z)}{h} = \frac{-1}{z(z+h)} \to \frac{-1}{z^2}$$

for all $z \neq 0$, so that f is holomorphic on $\mathbb{C} \setminus \{0\}$.

DEFINITION 3.5. A function $f : \mathbb{C} \to \mathbb{C}$ is called *entire* if f is holomorphic on $\mathbb{C}$.

The above example shows that $f(z) = c$ and $f(z) = z$ are entire functions. To get additional examples of holomorphic and entire functions we first observe that analogously to the rules of differentiation of real valued functions one can prove the following proposition.

PROPOSITION 3.6. *Let G be a nonempty open subset of $\mathbb{C}$. Then the following holds.*

(1) *If f, g holomorphic on G and $\lambda \in \mathbb{C}$, then so are $f + g$, λf, and fg.*
(2) *If $f(G) \subset G_1$, where G_1 is open and $g \in H(G_1)$, then $h = g \circ f$ is holomorphic on G and $h'(z) = g'(f(z))f'(z)$ for all $z \in G$.*

PROOF. We will only prove 2. Let $z \in G$ and put $w = f(z)$. Then f being holomorphic at z implies that we can write

$$f(z + h) - f(z) = [f'(z) + \epsilon_1(h)]h,$$

where $\epsilon_1(h) \to 0$ as $h \to 0$. Similarly

$$g(w + k) - g(w) = [g'(w) + \epsilon_2(k)]k,$$

where $\epsilon_2(k) \to 0$ as $k \to 0$. Putting $k = f(z + h) - f(z)$ we get

$$\begin{aligned} \frac{g(f(z+h)) - g(f(z))}{h} &= (g'(f(z)) + \epsilon_2(f(z+h) - f(z)))(f'(z) + \epsilon_1(h)) \\ &\to g'(f(z))f'(z) \end{aligned}$$

as $h \to 0$. □

COROLLARY 3.7. (1) *Any polynomial $p(z) = a_0 + a_1 + \ldots + a_n z^n$ is entire.*
(2) *Any rational function $f(z) = \frac{p(z)}{q(z)}$, where p and q are polynomials, is holomorphic on $\mathbb{C} \setminus \{z \in \mathbb{C} : q(z) = 0\}$.*

We will now compare complex differentiability of $f = u + iv$ with the real differentiability of the map $(u, v) : \mathbb{R}^2 \to \mathbb{R}^2$. Recall first the definition of real differentiability of a vector valued mapping.

DEFINITION 3.8. Let $G \subset \mathbb{R}^m$ an open set and $F : G \to \mathbb{R}^n$. Then F is real differentiable at $c \in G$ if there exist a linear mapping $DF(c) : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\lim_{h \to 0} \frac{\|F(c + h) - F(c) - DF(c)h\|}{\|h\|} = 0.$$

Writing $F = (F_1, \cdots, F_n)$, where $F_i : \mathbb{R}^m \to \mathbb{R}$, then real differentiability of F at $c \in G$ is equivalent with the real differentiability of each F_i and $DF_i(c)h = \nabla F_i(c) \cdot h$, where ∇F_i denotes the gradient of F_i and thus $DF(c)$ is the linear map given by the Jacobian matrix of F. We now take $m = n = 2$ to compare complex differentiability of $f = u + iv$ at $z_0 = x_0 + iy_0$ with real differentiability of $F = (u, v)$ at $c = (x_0, y_0)$. We first deal with the special case of a linear map.

LEMMA 3.9. *Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be a real linear map, given by the matrix $[a_{i,j}]$. Then $A = (u, v)$ where $f = u + iv$ is a complex linear map from $\mathbb{C}$ to $\mathbb{C}$ if and only if $a_{1,1} = a_{2,2}$ and $a_{1,2} = -a_{2,1}$.*

Proof. Assume first that $f(z) = Cz$ for some $C = c_1 + ic_2$. Then $u(x, y) = (c_1x - c_2y)$ and $v(x, y) = (c_2x + c_1y)$, which implies immediately that $A = (u, v)$ is a linear map with matrix $[a_{i,j}]$, where $a_{1,1} = a_{2,2} = c_1$ and $a_{1,2} = -a_{2,1} = -c_2$. Conversely, if $a_{1,1} = a_{2,2} = c_1$ and $a_{1,2} = -a_{2,1} = -c_2$, then it is straightforward to check that $f(z) = Cz$ with $C = c_1 + ic_2$. $\square$

Remark 3.10. Note that the condition on the matrix A are the ones imposed by the Cauchy-Riemann equations for $f(z) = Cz = u + iv$. As the real derivative $DF(c)$ of a linear map $F : \mathbb{C} \to \mathbb{C}$ is $F(c)$ this says that a linear map from $\mathbb{R}^2 \to \mathbb{R}^2$ corresponds to a complex differntiable map from $\mathbb{C}$ to $\mathbb{C}$ if and only if it is complex linear.

An immediate consequence of of the Lemma is the following theorem.

Theorem 3.11. *Let $G \subset \mathbb{C}$ be an open set and $f : G \to \mathbb{C}$, where $f(z) = u(x, y) + iv(x, y)$. Let $z_0 = x_0 + iy_0 \in G$. then the following are equivalent.*

1. *f is complex differentiable at z_0.*
2. *$F = (u, v)$ is real differentiable at (x_0, y_0) and the derivative $DF(x_0, y_0)$ is complex linear.*
3. *$F = (u, v)$ is real differentiable at (x_0, y_0) and the Cauchy-Riemann equations hold at (x_0, y_0).*

To prove a theorem about complex differentiability when the Cauchy-Riemann equations hold, we need first a result from vector calculus.

Lemma 3.12. *Let G be an open subset of $\mathbb{R}^2$ and $u : G \to \mathbb{R}^2$ a function which has partial derivatives on G, which are continuous at $(x_0, y_0) \in G$. Then there exist $\epsilon_1(h)$, and $\epsilon_2(h)$ in a neighborhood of $(0, 0)$ with $\epsilon_1(h) \to 0$ and $\epsilon_2(h) \to 0$ as $h = (h_1, h_2) \to (0, 0)$ such that*

$$u(x_0 + h_1, y_0 + h_2) = u(x_0, y_0) + \frac{\partial u}{\partial x}(x_0, y_0)h_1 + \frac{\partial u}{\partial y}(x_0, y_0)h_2 + \epsilon_1(h)h_1 + \epsilon_2(h)h_2.$$

Proof. Let $r > 0$ such that for $h = (h_1, h_2)$ with $\|h\| < r$ we have that $(x_0 + h_1, y_0 + h_2) \in G$. Let $\|h\| < r$. Then by the Mean Value theorem there exist k_1 between x_0 and $x_0 + h_1$ and k_2 between y_0 and $y_0 + h_2$ such that

$$\begin{aligned} u(x_0 + h_1, y_0 + h_2) - u(x_0, y_0) &= u(x_0 + h_1, y_0 + h_2) - u(x_0, y_0 + h_2) \\ &\quad + u(x_0, y_0 + h_2) - u(x_0, y_0) \\ &= \frac{\partial u}{\partial x}(k_1, y_0 + h_2)h_1 + \frac{\partial u}{\partial y}(x_0, k_2)h_2. \end{aligned}$$

The proof now follows if we put $\epsilon_1(h) = \frac{\partial u}{\partial x}(k_1, y_0 + h_2) - \frac{\partial u}{\partial x}(x_0, y_0)$ and $\epsilon_2(h) = \frac{\partial u}{\partial y}(x_0, k_2) - \frac{\partial u}{\partial y}(x_0, y_0)$. $\square$

Theorem 3.13. *Let $G \subset \mathbb{C}$ be an open set and $f : G \to \mathbb{C}$. Let $f(z) = u(x, y) + iv(x, y)$, where u and v are real valued functions on G. Assume that the first order partials $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ exist on G, are continuous at (x, y) and satisfy the Cauchy–Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at the point (x, y). Then f is complex differentiable at $z = x + iy$.

PROOF. Identifying $\mathbb{C}$ with $\mathbb{R}^2$ we can find by the above lemma $\epsilon_j(h)$ with $\epsilon_j(h) \to 0$ as $h = h_1 + ih_2 \to 0$ for $j = 1, \cdots, 4$ such that

$$\begin{aligned}\frac{f(z+h)-f(z)}{h} =& \frac{\partial u}{\partial x}(x,y)\frac{h_1}{h} + \frac{\partial u}{\partial y}(x,y)\frac{h_2}{h} + \epsilon_1(h)\frac{h_1}{h} + \epsilon_2(h)\frac{h_2}{h} \\ &+ i\left(\frac{\partial v}{\partial x}(x,y)\frac{h_1}{h} + \frac{\partial v}{\partial y}(x,y)\frac{h_1}{h} + \epsilon_3(h)\frac{h_1}{h} + \epsilon_4(h)\frac{h_2}{h}\right) \\ &= \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) + \epsilon_1(h)\frac{h_1}{h} + \epsilon_2(h)\frac{h_2}{h} \\ &+ i\epsilon_3(h)\frac{h_1}{h} + i\epsilon_4(h)\frac{h_2}{h} \\ &\to \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y)\end{aligned}$$

as $h \to 0$, since $|\frac{h_1}{h}| \leq 1$ and $|\frac{h_2}{h}| \leq 1$. □

COROLLARY 3.14. *Let $f(z) = e^z$. Then f is entire and $f'(z) = e^z$ for all $z \in \mathbb{C}$.*

PROOF. If $f = u + iv$, then $u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$. Now $\frac{\partial u}{\partial x}(x,y) = e^x \cos y$, $\frac{\partial v}{\partial x}(x,y) = e^x \sin y$, $\frac{\partial u}{\partial y}(x,y) = -e^x \sin y$, and $\frac{\partial v}{\partial y}(x,y) = e^x \cos y$. Hence the Cauchy-Riemann equations hold for all (x,y) and, as the partial are continuous, it follows from the above theorem that f is holomorphic at all $z \in \mathbb{C}$. Moreover $f'(z) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) = e^z$. □

PROPOSITION 3.15. *Let $G_1, G_2 \subset \mathbb{C}$ be open sets and let $f : G_1 \to G_2$, $g : G_2 \to G_1$ be continuous mappings such that $g(f(z)) = z$ for all $z \in G_1$. If g is holomorphic on G_2 and $g'(z) \neq 0$ for all $z \in G_2$, then f is holomorphic on G_1 and $f'(z) = \frac{1}{g'(f(z))}$ for all $z \in G_1$.*

PROOF. Let $z \in G_1$. Then for $h \neq 0$ but small enough we have $z + h \in G_1$ and $f(z+h) \neq f(z)$, since $g(f(z)) = z \neq (z+h) = g(f(z+h))$. Now

$$1 = \frac{g(f(z+h)) - g(f(z))}{f(z+h) - f(z)} \frac{f(z+h) - f(z)}{h}$$

implies that f is differentiable at z and $1 = g'(f(z))f'(z)$. □

COROLLARY 3.16. *Let $G \subset \mathbb{C}$ be an open connected set and $f : G \to \mathbb{C}$ a branch of the logarithm on G. Then f is holomorphic on G and $f'(z) = \frac{1}{z}$ for all $z \in G$.*

PROOF. Take $g(z) = e^z$ in the above proposition. □

We conclude this section with some remarks about harmonic functions. Recall that if $G \subset \mathbb{R}^2$ is open and $u : G \to \mathbb{R}$ satisfies the Laplace equation $\Delta u = \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 v}{\partial y^2}(x,y) = 0$ 0n G. Let now $f \in H(G)$, let $u = \text{Re } f$ and $v = \text{Im } f$. Assume that u and v have continuous second order partials (an assumption which we will show later on to be always true). Then $\Delta u = \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 v}{\partial y^2}(x,y) = \frac{\partial^2 v}{\partial x \partial y}(x,y) + \frac{-\partial^2 v}{\partial y \partial x}(x,y) = 0$. Hence u is harmonic on G. Similarly v is harmonic on G. Two harmonic functions u, and v are called conjugate harmonic functions, when $f = u+iv$ is holomorphic on G. Another consequence of the Cauchy-Riemann equations is that the inner product of the gradients ∇u and ∇v satisfy $\nabla u \cdot \nabla v = 0$, i.e, the level curves $u(x,y) = c_1$ and $v(x,y) = c_2$ intersect orthogonally.

4. Power series

In this section we will see how one can use power series to get a large class of examples of holomorphic functions. In fact, in a later chapter we will see that locally every holomorphic function can be so obtained. We start by recalling some basic facts concerning series. Recall that if $\langle a_n \rangle_{n\geq 0}$ is a sequence of complex numbers, then the series $\sum_{n=0}^{\infty} a_n$ converges to $s \in \mathbb{C}$ if $|s - s_n| \to 0$ as $n \to \infty$, where $s_n = a_0 + \ldots + a_n$. The number s is then called the sum of the series. The series is said to diverge, if it does not converge to any $s \in \mathbb{C}$. As in the real variable case we have:

(1) If $\sum_{n=0}^{\infty} a_n$ converges, then $a_n \to 0$ as $n \to \infty$.
(2) If $\sum_{n=0}^{\infty} |a_n|$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.

A power series is a series of the form $\sum_{n=0}^{\infty} c_n(z-a)^n$. Usually we will treat z as a variable and the c_n's and a as constants in this expression.

Example 4.1. Consider the geometric series $\sum_{n=0}^{\infty} z^n$. The partial sums s_n are in this case given by $s_n = 1 + \ldots + z^n = \frac{1-z^{n+1}}{1-z}$ for all $z \neq 1$. Hence for $|z| < 1$ the series $\sum_{n=0}^{\infty} z^n$ converges and has sum equal to $\frac{1}{1-z}$, while if $|z| \geq 1$ the series diverges, since in that case it is not true that $z^n \to 0$ as $n \to \infty$.

The following simple result turns out to be a useful tool in studying the convergence of power series.

Theorem 4.2. *(Weierstrass M–test) Let $G \subset \mathbb{C}$ and $u_n : G \to \mathbb{C}$ such that $|u_n(z)| \leq M_n$ on G, where $\sum_0^{\infty} M_n < \infty$. Then $\sum_0^{\infty} u_n(z)$ converges uniformly on G.*

Proof. For fixed $z \in G$ we have that $\sum_0^{\infty} |u_n(z)| \leq \sum_0^{\infty} M_n < \infty$. Hence the series $\sum_0^{\infty} u_n(z)$ converges for all $z \in G$. Let $f(z) = \sum_0^{\infty} u_n(z)$ for $z \in G$ denote the sum of the series and let $\epsilon > 0$. Then there exists N such that $\sum_{k=N+1}^{\infty} M_k < \epsilon$. Then we have for all $z \in G$ and all $n \geq N$ that

$$\left| f(z) - \sum_{k=0}^{n} u_n(z) \right| = \left| \sum_{k=n+1}^{\infty} u_n(z) \right| \leq \sum_{k=n+1}^{\infty} |u_n(z)| \leq \sum_{k=n+1}^{\infty} M_k < \epsilon$$

for all $n \geq N$ and all $z \in G$ and thus the series $\sum_0^{\infty} u_n(z)$ converges uniformly to $f(z)$ on G. □

For a given power series $\sum_{n=0}^{\infty} c_n(z-a)^n$ we define the *radius of convergence* R, $0 \leq R \leq \infty$, by $\frac{1}{R} = \overline{\lim}\ \sqrt[n]{|c_n|}$. The circle $\{z \in \mathbb{C} : |z-a| = R\}$ is called the *circle of convergence* of the power series.

Theorem 4.3. *(Cauchy Root test) Let $\sum_{n=0}^{\infty} c_n(z-a)^n$ be a power series with radius of convergence R. Then the following holds.*

(1) *$\sum_{n=0}^{\infty} c_n(z-a)^n$ converges absolutely for $|z-a| < R$.*
(2) *$\sum_{n=0}^{\infty} c_n(z-a)^n$ diverges for $|z-a| > R$.*
(3) *If $0 < r < R$, then $\sum_{n=0}^{\infty} c_n(z-a)^n$ converges uniformly on $|z-a| \leq r$.*

Proof. Let $|z-a| < r < R$. Then $\frac{1}{r} > \frac{1}{R}$ implies that there exists N such that $|c_n|^{\frac{1}{n}} < \frac{1}{r}$ for all $n \geq N$. It follows that $|c_n(z-a)^n| \leq \left(\frac{|z-a|}{r}\right)^n$ for all $n \geq N$.

Since $\frac{|z-a|}{r} < 1$, it follows that $\sum_{n=0}^{\infty} c_n(z-a)^n$ converges absolutely for $|z-a| < r$ for any $r < R$ and thus 1. holds. Let now $|z-a| > r > R$. Then there exist infinitely many n such that $|c_n|^{\frac{1}{n}} > \frac{1}{r}$. Hence $|c_n(z-a)^n| \geq \left(\frac{|z-a|}{r}\right)^n > 1$ for infinitely many n, i.e., the series $\sum_{n=0}^{\infty} c_n(z-a)^n$ diverges for $|z-a| > r$ for any $r > R$ and thus 2. holds. To prove 3. let $0 < r < s < R$. Then as above there exists N such that $|c_n|^{\frac{1}{n}} < \frac{1}{s}$ for all $n \geq N$. It follows that $|c_n(z-a)^n| \leq \left(\frac{r}{s}\right)^n$ for all $n \geq N$ and all $|z-a| < r$. Since $\frac{r}{s} < 1$, it follows that $\sum_{n=0}^{\infty} c_n(z-a)^n$ converges uniformly on $|z-a| \leq r$ by the Weierstrass M–test. □

In dealing with power series with coefficients involving factorials, it is often easier to use the following result.

THEOREM 4.4. *(Ratio test) Let $\sum_{n=0}^{\infty} c_n(z-a)^n$ be a power series with radius of convergence R. Assume $c_n \neq 0$ for all n. Then*

$$\underline{\lim} \left|\frac{c_{n+1}}{c_n}\right| \leq \frac{1}{R} \leq \overline{\lim} \left|\frac{c_{n+1}}{c_n}\right|.$$

In particular, if $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|$ exists, then $\frac{1}{R} = \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|$.

PROOF. Exercise □

A power series can converge or diverge at any point of its circle of convergence as can be seen from the following examples.

EXAMPLE 4.5.

(i) The series $\sum_{n=0}^{\infty} \frac{(z+1)^n}{2^{n+1}}$ has $R = 2$, as $\overline{\lim} \sqrt[n]{\frac{1}{2^{n+1}}} = \frac{1}{2}$. Note that the sum of series equals $\frac{1}{1-z}$ for all $|z+1| < 2$, since $\frac{1}{1-z} = \frac{1}{2-(z+1)} = \frac{1}{2}\frac{1}{1-\frac{z+1}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z+1}{2}\right)^n$ for $\frac{|z+1|}{2} < 1$.
(ii) The series $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ has $R = 1$ (e.g. by the Ratio test), and the series converges absolutely for any z on the circle of convergence as $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$.
(iii) The series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ has $R = 1$ (e.g. by the Ratio test), but it does not converge absolutely for any z on the circle of convergence as $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$. In particular it diverges for $z = 1$. One can show however (but this is not completely trivial) that it converges for any $z \neq 1$ with $|z| = 1$ (for $z = -1$ this follows e.g. from the so-called alternating series test).
(iv) The series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ has $R = \infty$ (e.g. by the Ratio test). We will see after the next theorem that e^z equals the sum of this series.
(v) The series $\sum_{n=1}^{\infty} n!z^n$ has $R = 0$ (e.g. by the Ratio test). Hence it converges only for $z = 0$.

PROPOSITION 4.6. *Let R be the radius of convergence of $\sum_{n=0}^{\infty} c_n(z-a)^n$. Then R is also the radius of convergence of the power series $\sum_{n=1}^{\infty} nc_n(z-a)^{n-1} = \sum_{n=0}^{\infty}(n+1)c_{n+1}(z-a)^n$.*

PROOF. From calculus we know that $\lim_{n\to\infty} \sqrt[n]{n} = 1$. Hence

$$\overline{\lim} \sqrt[n]{(n+1)\,|c_{n+1}|} = \overline{\lim} \left(\sqrt[n+1]{(n+1)\,|c_{n+1}|}\right)^{\frac{n+1}{n}} = \frac{1}{R}.$$

□

Note, if we apply the above proposition twice, we get that $\sum_{n=2}^{\infty} n(n-1)z^{n-2}$ converges absolutely for $|z-a|<R$.

The following theorem says that inside the circle of convergence the sum of the power series is a holomorphic function.

THEOREM 4.7. *Let $\sum_{n=0}^{\infty} c_n(z-a)^n$ have radius of convergence $R \neq 0$ and define $f(z)=\sum_{n=0}^{\infty} c_n(z-a)^n$ for $|z-a|<R$. Then $f \in H(B(a,R))$ and $f'(z)=\sum_{n=1}^{\infty} nc_n(z-a)^{n-1}$ for $|z-a|<R$.*

PROOF. It follows from the above corollary that $g(z)=\sum_{n=1}^{\infty} nc_n(z-a)^{n-1}$ also converges in $B(a,R)$. Remains to show that $f'(z)=g(z)$ on $|z-a|<R$. W.l.o.g. we can assume that $a=0$. In the argument below we will use that $(z+h)^n-z^n = h\sum_{k=1}^{n}(z+h)^{k-1}z^{n-k}$. Let $z, z+h \in B(0,r)$, where $0<r<R$. Then we have

$$
\begin{aligned}
\left|\frac{f(z+h)-f(z)}{h}-g(z)\right| &= \left|\sum_{n=1}^{\infty} c_n\left\{\frac{(z+h)^n-z^n}{h}-nz^{n-1}\right\}\right| \\
&= \left|\sum_{n=2}^{\infty} c_n \sum_{k=1}^{n}\left\{(z+h)^{k-1}z^{n-k}-z^{n-1}\right\}\right| \\
&\leq \sum_{n=2}^{\infty} |c_n| \sum_{k=2}^{n}\left|z^{n-k}\left((z+h)^{k-1}-z^{k-1}\right)\right| \\
&\leq |h| \sum_{n=2}^{\infty} |c_n| \left(\sum_{k=2}^{n}\left|z^{n-k}\right|\left(\sum_{l=1}^{k-1}|z+h|^{l-1}|z|^{k-1-l}\right)\right) \\
&\leq |h| \sum_{n=2}^{\infty} |c_n| \sum_{k=2}^{n}(k-1)r^{n-k}r^{k-2} \\
&= |h| \sum_{n=2}^{\infty} |c_n| \frac{1}{2}n(n-1)r^{n-2} \to 0
\end{aligned}
$$

as $h \to 0$, since, by the above proposition, $\sum_{n=2}^{\infty} |c_n| \frac{1}{2}n(n-1)r^{n-2} < \infty$ as $r<R$. Hence $f'(z)=g(z)$ on $|z|<r$ for any $r<R$ and the proof is complete. ☐

COROLLARY 4.8. *Let $f(z)=\sum_{n=0}^{\infty} c_n(z-a)^n$ have radius of convergence $R \neq 0$. Then $f^{(k)}(z)$ exists on $B(a,R)$ for all $k \geq 1$ and thus $f^{(k)} \in H(B(a,R))$ and*

$$
f^{(k)}(z)=\sum_{n=k}^{\infty} c_n n(n-1)\dots(n-k+1)(z-a)^{n-k}
$$

for all $k \geq 1$ and all $|z-a|<R$. In particular $k!c_k = f^{(k)}(a)$ and thus the coefficients c_k of the power series are unique.

EXAMPLE 4.9. Let $f(z)=\sum_{n=0}^{\infty} \frac{z^n}{n!}$. Then by the above theorem

$$
f'(z)=\sum_{n=1}^{\infty} \frac{nz^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = f(z)
$$

for all $z \in \mathbb{C}$. Let $h(z)=e^{-z}f(z)$. Then $h'(z)=-e^{-z}f(z)+e^{-z}f(z)=0$ for all $z \in \mathbb{C}$. From the next proposition it follows that $h(z)=h(0)=1$ for all z, i.e., $f(z)=e^z$ for all z.

PROPOSITION 4.10. *Let $G \subset \mathbb{C}$ be an open and connected set. Assume $f \in H(G)$ such that $f'(z) = 0$ for all $z \in G$. Then f is constant on G.*

PROOF. Let $z_0 \in G$ and put $A = \{z \in G : f(z) = f(z_0)\}$. Then the continuity of f implies that A is closed. Let now $a \in A$. Then there exists $\epsilon > 0$ such that $B(a, \epsilon) \subset G$. Let $z \in B(a, \epsilon)$ and put $g(t) = f(tz + (1 - t)a)$ for $0 \leq t \leq 1$. Then by the chain rule $g'(t) = f'(tz + (1 - t)a)(z - a) = 0$ for $0 < t < 1$, so g is constant on $0 \leq t \leq 1$. Hence $f(z) = g(0) = g(1) = f(z_0)$, and thus $B(a, \epsilon) \subset A$. It follows that A is nonempty open and closed subset of G, thus $A = G$. $\square$

CHAPTER 2

Integration over contours

1. Curves and Contours

A *curve* is a continuous map $\gamma : [a, b] \to \mathbb{C}$. We call $\gamma(a)$ the initial point and $\gamma(b)$ the end point of the curve γ, and $[a, b]$ is called the parameter interval of γ. If $\gamma(a) = \gamma(b)$, then γ is called a *closed* curve. Denote by γ^* the range of γ. The curve γ induces an orientation of γ^*, namely the direction in which $\gamma(t)$ traces γ^* as t increases from a to b. Often we will specify a curve by its range together with an orientation indicating how (and possibly how often) the range is traversed. Given a curve γ we can find an oriented curve $-\gamma$, with identical range, but with opposite orientation, e.g.,

$$(-\gamma)(t) = \gamma(a + b - t) \text{ where } a \le t \le b$$

as a parametrization of the curve $-\gamma$. If $\gamma_1$and γ_2 are two curves with with parameter intervals $[a_1, b_1]$, $[a_2, b_2]$ respectively such that $\gamma_1(b_1) = \gamma_2(a_2)$, then we can join the two curves to get the curve $\gamma = \gamma_1 \cup \gamma_2$ by taking

$$\gamma(t) = \begin{cases} \gamma_1(t) & a_1 \le t \le b_1 \\ \gamma_2(t + a_2 - b_1) & b_1 \le t \le b_1 + b_2 - a_2. \end{cases}$$

A curve γ is called *smooth*, if $\gamma'(t)$ exists and is continuous for all $a \le t \le b$ (with one-sided derivatives at a and b). Note if we write $\gamma(t) = x(t) + iy(t)$, then $\gamma'(t)$ exists if and only if $x'(t)$ and $y'(t)$ exist. From multi-variable calculus we know that $\gamma'(t)$ represents a tangent vector to the curve γ.

A *path or contour* γ is a piecewise smooth curve, i.e., $\gamma : [a, b] \to \mathbb{C}$ such that there exist $a = t_0 < t_1 < \ldots < t_n = b$ where γ restricted to $[t_{i-1}, t_i]$ is smooth for $i = 1, \ldots, n$. Note that γ can have corners at the points $\gamma(t_i)$, i.e., the right and left hand derivatives of $\gamma(t)$ at t_i can differ.

A path γ is called *simple* if $\gamma : [a, b] \to \mathbb{C}$ is such that $\gamma(s) \ne \gamma(t)$ for all $a \le s < t \le b$, except possibly for $s = a$ and $t = b$. The path γ is *closed* if $\gamma(a) = \gamma(b)$.

Example 1.1.

(i) The directed line segment C from z_1 to z_2 is the range of a smooth curve. A parametrization of C is given $\gamma : [0, 1] \to \mathbb{C}$ defined by $\gamma(t) = (1-t)z_1+tz_2$. We will denote this curve by $[z_1, z_2]$.

(ii) A circular arc oriented counterclockwise is the range of an curve. Suppose the arc is part of the circle with center z_0 and radius r, then $\gamma(t) = z_0+re^{it}$ with $\theta_1 \le t \le \theta_2$ will trace a circular arc counterclockwise. If $\theta_2 - \theta_1 = 2\pi$ the curve will be the complete circle. Note the curve is simple if and only if $\theta_2 - \theta_1 \le 2\pi$.

1.1. Conformal mappings. Let f be a holomorphic function on an open set $G \subset \mathbb{C}$. Let $z_0 \in G$ be a fixed point and let $\gamma : [a, b] \to \mathbb{C}$ be a smooth curve in G passing through z_0 with non-zero tangent, i.e., $\gamma(t_0) = z_0$ for some $t_0 \in (a, b)$ and $\gamma'(t_0) \neq 0$. Then $\gamma_1 = f \circ \gamma$ is a curve passing through $f(z_0)$ and $\gamma_1'(t_0) = f'(z_0)\gamma'(t_0)$. If now $f'(z_0) \neq 0$, we see that $\arg \gamma_1'(t_0) = \arg f'(z_0) + \arg \gamma'(t_0)$ and $|\gamma_1'(t_0)| = |f'(z_0)||\gamma'(t_0)|$. Thus the tangent vector $\gamma'(t_0)$ to the curve γ at z_0 is under the mapping f rotated over an angle $\theta \in \arg f'(z_0)$ and stretched by a factor $|f'(z_0)|$. Applying this to two curves passing through z_0 we see that under the mapping f the angle between the two curves is preserved (including the direction they are measured), while their tangent vectors are stretched by the same amount. Mappings which preserve angles (including the direction they are measured) between smooth curves are called *conformal*. Thus we have proved:

THEOREM 1.2. *Let f be a holomorphic function on an open set $G \subset \mathbb{C}$.Assume $f'(z) \neq 0$ for all $z \in G$. Then f is conformal on G.*

We will now see that in fact the converse is true too, To do so we will introduce some additional notation. Let $f = u + iv$ as usual. Then we define $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial \overline{z}} = \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}$. It is now a routine calculation to show that u and v satisfy the Cauchy-Riemann equations if and only if $\frac{\partial f}{\partial \overline{z}} = 0$.

THEOREM 1.3. *Let $f = u + iv$ be a function on an open set $G \subset \mathbb{C}$ with continuous partials. Assume f is conformal on G. Then f is holomorphic on G and $f'(z) \neq 0$ for all $z \in G$.*

PROOF. Let γ be a smooth curve with non-zero tangent passing through $z_0 \in G$. Let $\gamma_1(t) = f(\gamma(t))$. Write $\gamma(t) = x(t) + iy(t)$. Then $\gamma_1' = \frac{\partial u}{\partial x}x' + \frac{\partial u}{\partial y}y' + i\frac{\partial v}{\partial v}v' + i\frac{\partial v}{\partial y}y' = \frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' = \frac{\partial f}{\partial z}\gamma' + \frac{\partial f}{\partial \overline{z}}\overline{\gamma'}$. Let $\gamma(t_0) = z_0$. Then

$$\frac{\gamma_1'(t_0)}{\gamma'(t_0)} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \overline{z}}\frac{\overline{\gamma'(t_0)}}{\gamma'(t_0)}.$$

Now f conformal implies that the argument of the left hand side of this equation is constant modulo 2π. This implies that $\frac{\partial f}{\partial \overline{z}}(z_0) = 0$, since the argument of $\frac{\overline{\gamma'(t_0)}}{\gamma'(t_0)}$ is not constant modulo 2π, when we take e.g. $\gamma(t) = z_0 + te^{i\theta}$. Hence u and v satisfy the Cauchy-Riemann equations at z_0 and thus $f'(z_0)$ exists and $f'(z_0) = \frac{\partial f}{\partial z}(z_0) = \frac{\gamma_1'(t_0)}{\gamma'(t_0)} \neq 0$.

□

2. Contour integrals

DEFINITION 2.1. A curve $\gamma : [a, b] \to \mathbb{C}$ is called *rectifiable* if γ is of bounded variation, i.e., if

$$\ell(\gamma) = \sup\left\{\sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i-1})| : a = t_0 < \ldots < t_n = b\right\} < \infty.$$

In this case the *length* of γ is defined to be $\ell(\gamma)$. Given a continuous $\gamma : [a, b] \to \mathbb{C}$ we define $\int_a^b \gamma(t)\,dt = \int_a^b \mathrm{Re}\,\gamma(t)\,dt + i\int_a^b \mathrm{Im}\,\gamma(t)\,dt$. In case γ is (piecewise) smooth we have by the Fundamental Theorem of Calculus for real integrals that $\int_a^b \gamma'(t)\,dt = \gamma(b) - \gamma(a)$.

LEMMA 2.2. *Let $f : [a,b] \to \mathbb{C}$ be a continuous function. Then*

$$\left| \int_a^b f(t)\,dt \right| \leq \int_a^b |f(t)|\,dt.$$

PROOF. Let $\alpha = \int_a^b f(t)\,dt$. If $\alpha = 0$, then the inequality is trivial. Assume $\alpha \neq 0$. Then we can write $\alpha = re^{i\theta}$, where $r = |\int_a^b f(t)\,dt|$. Now we have

$$r = \text{Re }(e^{-i\theta}\alpha) = \int_a^b \text{Re }(e^{-i\theta} f(t))\,dt \leq \int_a^b |f(t)|\,dt.$$

□

THEOREM 2.3. *Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise smooth curve. Then γ is rectifiable and*

$$\ell(\gamma) = \int_a^b |\gamma'(t)|\,dt.$$

PROOF. Without loss of generality we can assume that γ is smooth. Let $a = t_0 < \ldots < t_n = b$ be a partition of $[a,b]$. Then by the Fundamental Theorem of Calculus and the above lemma we have

$$|\gamma(t_i) - \gamma(t_{i-1})| = \left| \int_{t_{i-1}}^{t_i} \gamma'(t)\,dt \right| \leq \int_{t_{i-1}}^{t_i} |\gamma'(t)|\,dt.$$

This implies that γ is rectifiable and $\ell(\gamma) \leq \int_a^b |\gamma'(t)|\,dt$. For the reverse inequality, let $\epsilon > 0$. Then γ' is uniformly continuous on $[a,b]$, so there exists $\delta > 0$ such that $|\gamma'(t) - \gamma'(s)| < \epsilon$ whenever $|t - s| < \delta$. Now there exists a partition $a = t_0 < \ldots < t_n = b$ with $\Delta t_i = t_i - t_{i-1} < \delta$ such that

$$\left| \int_a^b |\gamma'(t)|\,dt - \sum_{i=1}^n |\gamma'(t_i)|\Delta t_i \right| < \epsilon.$$

For $1 \leq i \leq n$ we have now that

$$\begin{aligned} ||\gamma(t_i) - \gamma(t_{i-1})| - |\gamma'(t_i)|\Delta t_i| &\leq |\gamma(t_i) - \gamma(t_{i-1}) - \gamma'(t_i)\Delta t_i| \\ &= \left| \int_{t_{i-1}}^{t_i} \gamma'(t) - \gamma'(t_i)\,dt \right| \\ &\leq \int_{t_{i-1}}^{t_i} |\gamma'(t) - \gamma'(t_i)|\,dt < \epsilon \Delta t_i. \end{aligned}$$

Combining the last two estimates we get

$$\begin{aligned}\int_a^b |\gamma'(t)|\,dt &\le \sum_{i=1}^n |\gamma'(t_i)|\Delta t_i + \epsilon \\ &\le \sum_{i=1}^n (|\gamma(t_i) - \gamma(t_{i-1})| + \epsilon \Delta t_i) + \epsilon \\ &\le \ell(\gamma) + \epsilon(b-a) + \epsilon\end{aligned}$$

for all $\epsilon > 0$. Hence $\int_a^b |\gamma'(t)|\,dt \le \ell(\gamma)$. $\square$

Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise smooth curve and let $f : \gamma^* \to \mathbb{C}$ be continuous. Then we define $\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt$.

Example 2.4. Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise smooth curve. Then $\int_\gamma 1\,dz = \gamma(b) - \gamma(a)$. This is immediate from the definition and the Fundamental Theorem of Calculus.

Proposition 2.5. *Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise smooth curve and let $f : \gamma^* \to \mathbb{C}$ be a continuous function. Then the following hold.*

(i) $\int_{-\gamma} f(z)\,dz = -\int_\gamma f(z)\,dz$, *where* $-\gamma(t) = \gamma(a+b-t)$.

(ii) *If* $\gamma = \gamma_1 \cup \gamma_2$, *then*
$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz.$$

(iii) *If* $|f(z)| \le M$ *on* γ^*, *then* $|\int_\gamma f(z)\,dz| \le M\ell(\gamma)$.

(iv) *("Independence of parametrization") Let* $\tau : [a_1, b_1] \to [a,b]$ *be a smooth onto function with* $\tau' > 0$. *Then for* $\gamma_1 = \gamma \circ \tau$ *we have*
$$\int_{\gamma_1} f(z)\,dz = \int_\gamma f(z)\,dz.$$

(v) *If also* $g : \gamma^* \to \mathbb{C}$ *continuous and* $\alpha, \beta \in \mathbb{C}$, *then* $\int_\gamma \alpha f(z) + \beta g(z)\,dz = \alpha \int_\gamma f(z)\,dz + \beta \int_\gamma g(z)\,dz$.

Proof. Let $-\gamma(t) = \gamma(a+b-t)$. Then $-\gamma : [a,b] \to \mathbb{C}$ is piecewise smooth and $(-\gamma)'(t) = -\gamma'(a+b-t)$ except possibly finitely many points, from which (i) follows directly. Part (ii) is an immediate consequence of the definition. Part (iii) follows from

$$\left|\int_\gamma f(z)\,dz\right| \le \int_a^b |f(\gamma(t))\gamma'(t)|\,dt \le M \int_a^b |\gamma'(t)|\,dt = M\ell(\gamma).$$

Part (iv) follows from the chain rule $\gamma_1'(t) = \gamma'(\tau(t))\tau'(t)$ and the change of variable rules for real integrals

$$\begin{aligned}\int_{\gamma_1} f(z)\,dz &= \int_{a_1}^{b_1} f(\gamma_1(t))\gamma_1'(t)\,dt = \int_{a_1}^{b_1} f(\gamma(\tau(t)))\gamma'(\tau(t))\tau'(t)\,dt \\ &= \int_{\tau(a_1)}^{\tau(b_1)} f(\gamma(s))\gamma'(s)\,ds = \int_\gamma f(z)\,dz.\end{aligned}$$

Part (v) is immediate from the definition and the corresponding property of real integrals.

□

COROLLARY 2.6. *Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth curve and let $f_n : \gamma^* \to \mathbb{C}$ be continuous functions which converge uniformly to f on γ^*. Then*

$$\int_\gamma f_n(z)\, dz \to \int_\gamma f(z)\, dz$$

as $n \to \infty$.

PROOF. Note first that f is also continuous on γ^* as it is the uniform limit of a sequence of continuous functions. Let $M_n = \sup_{z\in\gamma^*} |f_n(z) - f(z)|$. Then by assumption $M_n \to 0$ as $n \to \infty$. From (iii) and (v) above we have now

$$\left|\int_\gamma f_n(z)\, dz - \int_\gamma f(z)\, dz\right| = \left|\int_\gamma f_n(z) - f(z)\, dz\right| \le M_n \ell(\gamma) \to 0$$

as $n \to \infty$. □

The following example is important for the development of the theory.

EXAMPLE 2.7. Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be given by $\gamma(t) = a + re^{it}$, i.e., γ is the circle with center a and radius r traversed counterclockwise. We will show that

$$\int_\gamma (z-a)^n\, dz = \begin{cases} 0 & \text{if } n \in \mathbb{Z}\setminus\{-1\} \\ 2\pi i & \text{if } n = -1 \end{cases} \tag{2.1}$$

Since γ is smooth we can write

$$\begin{aligned} \int_\gamma (z-a)^n\, dz &= \int_0^{2\pi} (re^{it})^n ire^{it}\, dt \\ &= ir^{n+1}\int_0^{2\pi} e^{i(n+1)t}\, dt \\ &\begin{cases} = ir^{n+1}\left(\frac{1}{i(n+1)} e^{i(n+1)t}|_0^{2\pi}\right) = 0 & \text{if } n \in \mathbb{Z}\setminus\{-1\} \\ = 2\pi i & \text{if } n = -1, \end{cases} \end{aligned}$$

which proves the formula. Note that this integral does not depend on r.

The following Theorem will allow us to extend this example, in case $n \ne -1$, to arbitrary closed contours γ with $a \notin \gamma^*$.

THEOREM 2.8. *Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth curve and assume F is holomorphic on (an open set containing) γ^* with F' continuous on γ^*. Then*

$$\int_\gamma F'(z)\, dz = F(\gamma(b)) - F(\gamma(a)).$$

In particular, if γ is a closed contour, then $\int_\gamma F'(z)\, dz = 0$.

PROOF. Assume first that γ is smooth. Then by the chainrule $(F \circ \gamma)'(t) = F'(\gamma(t))\gamma'(t)$ for all $a \le t \le b$. Hence $\int_\gamma F'(z)\, dz = \int_a^b F'(\gamma(t))\gamma'(t)\, dt = \int_\gamma (F \circ \gamma)'(t)\, dz = F(\gamma(b)) - F(\gamma(a))$, which proves the theorem for the special case of a smooth curve. In the general case, choose $a = s_0 < s_1 < \cdots < s_n = b$ such that

$\gamma_i = \gamma|_{[s_{i-1},s_i]}$ is smooth. Then $\int_\gamma F'(z)\,dz = \sum_{i=1}^n \int_{\gamma_i} F'(z)\,dz = \sum_{i=1}^n F(\gamma(s_i)) - F(\gamma(s_{i-1})) = F(\gamma(b)) - F(\gamma(a))$. ☐

COROLLARY 2.9. *Let γ be any closed contour. Then $\int_\gamma (z-a)^n\,dz = 0$ for all $n \geq 0$ and if in addition $a \notin \gamma^*$, then also $\int_\gamma (z-a)^n\,dz = 0$ for all $n \leq -2$.*

PROOF. Take $F(z) = \frac{1}{n+1}(z-a)^{n+1}$ in the above theorem. ☐

Let now $\{a, b, c\}$ be an ordered triple of complex numbers. Then $\Delta = \Delta(a, b, c)$ denotes the triangle with vertices a, b, and c. By $\partial\Delta$ we denote curve obtained by joining the line segments $[a, b]$, $[b, c]$ and $[c, a]$, i.e., $\partial\Delta$ denotes the boundary of $\Delta(a, b, c)$ traversed counterclockwise. Hence

$$\int_{\partial\Delta} f(z)\,dz = \int_{[a,b]} f(z)\,dz + \int_{[b,c]} f(z)\,dz + \int_{[c,a]} f(z)\,dz$$

for any continuous f on $\partial\Delta^*$.

THEOREM 2.10. *(Cauchy's Theorem for a Triangle) Let $G \subset \mathbb{C}$ be an open set and assume $\Delta = \Delta(a, b, c) \subset G$. Let $p \in G$ and $f : G \to \mathbb{C}$ such that f is continuous on G and holomorphic on $G \setminus \{p\}$. Then*

$$\int_{\partial\Delta} f(z)\,dz = 0.$$

REMARK. If f satisfies the above hypotheses, then we shall see later that f is actually holomorphic on G.

PROOF. Assume first that $p \notin \Delta = \Delta(a, b, c)$. Let $\{a_1, b_1, c_1\}$ be the midpoints of $[b, c]$, $[c, a]$, and $[a, b]$ respectively. Consider the four triangles Δ_1, Δ_2, Δ_3, and Δ_4 formed by the triples $\{a, c_1, b_1\}$, $\{c_1, b, a_1\}$, $\{a_1, b_1, c_1\}$ and $\{a_1, c, b_1\}$ (see Figure 1). Put $I = \int_{\partial\Delta} f(z)\,dz$. Then

$$I = \sum_{j=1}^{4} \int_{\partial\Delta_j} f(z)\,dz.$$

Now $|\int_{\partial\Delta_j} f(z)\,dz| \geq \frac{|I|}{4}$ for at least one j. By relabeling we can assume that

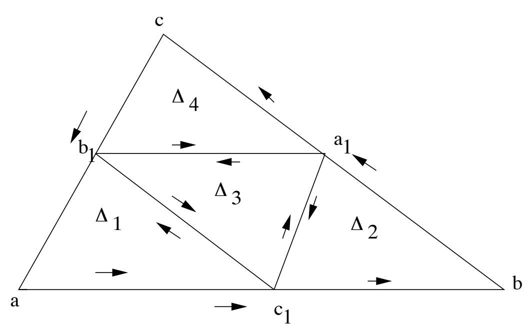

FIGURE 1. $\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$

$$\left|\int_{\partial\Delta_1} f(z)\,dz\right| \geq \frac{|I|}{4}.$$

Dividing similarly Δ_1 into four triangles by means of the midpoints of the edges and repeating this process, we get a sequence of triangles $\Delta \supset \Delta_1 \supset \Delta_2 \supset \cdots$ such that $\ell(\partial\Delta_n) = \frac{1}{2^n}L$, where $L = \ell(\partial\Delta)$, and such that

$$\left|\int_{\partial\Delta_n} f(z)\,dz\right| \geq \frac{|I|}{4^n}. \tag{2.2}$$

Since Δ is compact and $\{\Delta_n\}$ has the finite intersection property, it follows that there exists $z_0 \in \cap_n \Delta_n$. As $p \notin \Delta$, we have that $z_0 \neq p$ and thus f is differentiable at z_0. Let $\epsilon > 0$. Then there exists $r > 0$ such that

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \leq \epsilon|z - z_0|$$

for all z with $|z - z_0| < r$. Now $\ell(\partial\Delta_n) \to 0$ implies that there exists N such that $\Delta_N \subset B(z_0, r)$. This implies that $|z - z_0| < \ell(\partial\Delta_N) = \frac{1}{2^N}L$ for all $z \in \Delta_N$. By Corollary 2.9 we know that

$$\int_{\partial\Delta_N} f(z)\,dz = \int_{\partial\Delta_N} f(z) - f(z_0) - f'(z_0)(z - z_0)\,dz.$$

This implies that

$$\left|\int_{\partial\Delta_N} f(z)\,dz\right| \leq \left(\epsilon 2^{-N} L\right)\left(2^{-N} L\right) = \epsilon\left(2^{-N}\right)^2 L^2.$$

From the inequality 2.2 it follows that $|I| \leq \epsilon L^2$ for all $\epsilon > 0$ and thus $I = 0$. This completes the proof in case $p \notin \Delta$. Assume next that p is a vertex of the

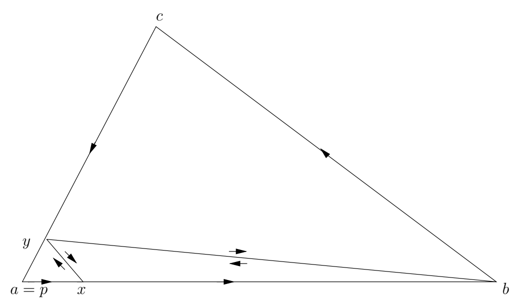

FIGURE 2. The case $a = p$

triangle $\Delta(a, b, c)$, say $p = a$. Then pick $x \in [a, b]$ and $y \in [a, c]$. Then by the above $\int_{\Delta(x,b,y)} f(z)\,dz = \int_{\Delta(y,b,c)} f(z)\,dz = 0$ and thus

$$\int_{\partial\Delta} f(z)\,dz = \int_{\partial\Delta(a,x,y)} f(z)\,dz \to 0$$

as $x, y \to a$, since $\ell(\partial\Delta(a, x, y)) \to 0$ and f is bounded on $\Delta(a, x, y)$. Hence $\int_{\partial\Delta} f(z)\,dz = 0$ also in the case that p is a vertex of Δ. It remains the case that $p \in \Delta \setminus \{a, b, c\}$. In that case apply the above to the triangles $\Delta(a, b, p)$, $\Delta(b, c, p)$ and $\Delta(c, a, p)$ to get the desired result. $\square$

DEFINITION 2.11. A set $S \subset \mathbb{C}$ is called *starlike* if the exists $a \in S$ such that the line segment $[a, z] \subset S$ for all $z \in S$. The point a is called a star center of S in this case.

Recall that a set $S \subset \mathbb{C}$ is called *convex* if for $z_1, z_2 \in S$ we have that $[z_1, z_2] \subset S$, i.e., a convex set is a starlike set such that every point of S is a star center of S.

THEOREM 2.12. *(Cauchy's Theorem for starlike sets) Let $G \subset \mathbb{C}$ be an open starlike set. Let $p \in G$ and $f : G \to \mathbb{C}$ such that f is continuous on G and holomorphic on $G \setminus \{p\}$. Then $f = F'$ for some holomorphic F on G. In particular*

$$\int_\gamma f(z)\, dz = 0 \tag{2.3}$$

for every closed contour γ in G.

PROOF. Let $a \in G$ be a star center of G. Then the line segment $[a, z] \subset G$ for all $z \in G$. Now define

$$F(z) = \int_{[a,z]} f(w)\, dw.$$

Let $z_0 \in G$. Then there exists $r > 0$ such that $B(z_0, r) \subset G$. Now for any $z \in B(z_0, r)$ the triangle $\Delta(a, z_0, z) \subset G$, so by Theorem 2.10 we have

$$\int_{\partial\Delta(a,z_0,z)} f(w)\, dw = 0,$$

and thus

$$F(z) - F(z_0) = \int_{[a,z]} f(w)\, dw - \int_{[a,z_0]} f(w)\, dw = \int_{[z_0,z]} f(w)\, dw.$$

Fixing z_0 we get for all $z \neq z_0$ in G, since $\int_{[z_0,z]} 1\, dw = z - z_0$, that

$$\begin{aligned} \left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| &= \left| \frac{1}{z - z_0} \int_{[z_0,z]} f(w) - f(z_0)\, dw \right| \\ &\le \frac{1}{|z - z_0|} \left(\sup_{w \in [z_0,z]} |f(w) - f(z_0)| \right) |z - z_0| \\ &= \sup_{w \in [z_0,z]} |f(w) - f(z_0)| \to 0 \end{aligned}$$

as $z \to z_0$, by the continuity of f at z_0. This proves that $f(z_0) = F'(z_0)$ for all $z_0 \in G$ and thus F is holomorphic on G. Now equation 2.3 follows from Theorem 2.8. □

DEFINITION 2.13. Let γ be a closed piecewise smooth curve in $\mathbb{C}$ and let $a \in G = \mathbb{C} \setminus \gamma^*$. Then

$$\mathrm{Ind}_\gamma(a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - a}$$

is called the *index* of γ with respect to a or *winding number* of γ around a.

THEOREM 2.14. *(Cauchy's Integral Formula for starlike sets) Let $G \subset \mathbb{C}$ be an open starlike set and let γ be a closed contour in G. Let f be holomorphic on G and $z_0 \in G \setminus \gamma^*$. Then*

$$f(z_0) \cdot \ Ind_\gamma(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} \, dz.$$

PROOF. Let $z \in G \setminus \gamma^*$ and define

$$g(z) = \begin{cases} \frac{f(z)-f(z_0)}{z-z_0} & \text{if } z \in G \setminus \{z_0\} \\ f'(z_0) & \text{if } z = z_0 \end{cases}$$

Then g satisfies the hypotheses of Theorem 2.12, so

$$\frac{1}{2\pi i} \int_\gamma g(z) \, dz = 0.$$

Hence

$$\begin{aligned} \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} \, dz &= \frac{1}{2\pi i} \int_\gamma \frac{f(z_0)}{z - z_0} \, dz \\ &= f(z_0) \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0} \, dz \\ &= f(z_0) \cdot \mathrm{Ind}_\gamma(z_0), \end{aligned}$$

and thus the proof of the theorem is complete. □

REMARK 2.15. The above theorem is used most often for the case that $\mathrm{Ind}_\gamma(a) = 1$. We will see e.g. that $\mathrm{Ind}_\gamma(z_0) = 1$, when γ is a circle containing z_0, traversed counter clockwise once.

THEOREM 2.16. *(Fundamental Theorem of Algebra) Let $p(z)$ be a polynomial of degree $m \geq 1$. Then p has exactly m zeros in $\mathbb{C}$, counting each zero according to its multiplicity.*

PROOF. Assume $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then $f(z) = \frac{1}{p(z)}$ is an entire function. We can assume that $p(z) = z^m + \cdots + a_1 z + a_0$. Now

$$\begin{aligned} |p(z)| &= |z|^m \left| 1 + \cdots + \frac{a_1}{z^{m-1}} + \frac{a_0}{z^m} \right| \\ &\geq |z|^m \left| |1 - \cdots - \frac{|a_1|}{|z|^{m-1}} - \frac{|a_0|}{|z|^m} \right| \geq \frac{1}{2} |z|^m \geq \frac{1}{2} R^m \end{aligned}$$

for $|z| \geq R$ for R large enough. Now applying Cauchy's Integral formula to $f(z)$ and $\gamma_R = Re^{it}$ with $0 \leq t \leq 2\pi$, we get

$$\int_{\gamma_R} \frac{f(z)}{z} \, dz = 2\pi i f(0) = \frac{2\pi i}{p(0)} \neq 0,$$

while

$$\left| \int_{\gamma_R} \frac{f(z)}{z} \, dz \right| \leq 2\pi \max_{|z|=R} |\frac{1}{p(z)}| \leq 2\pi \frac{2}{R^m} \to 0,$$

as $R \to \infty$, which is a contradiction. Hence there exists $z_1 \in \mathbb{C}$ such that $p(z_1) = 0$. Now factor $p(z) = (z - z_1) p_1(z)$ and repeat the above argument.

□

To apply the Cauchy's Integral formula, we need to be able to compute the index of a curve. We will derive a number of properties of the index, which will facilitate this.

PROPOSITION 2.17. *Let γ be a closed contour and let $G = \mathbb{C} \setminus \gamma^*$. Then $Ind_\gamma(a)$ is an integer for all $a \in G$.*

PROOF. Let $\gamma : [b, c] \to \mathbb{C}$ be piecewise smooth such that $\gamma(b) = \gamma(c)$. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = \frac{1}{2\pi i}\int_b^c \frac{\gamma'(s)}{\gamma(s)-a}\, ds.$$

Let

$$g(t) = \int_b^t \frac{\gamma'(s)}{\gamma(s)-a}\, ds.$$

Then $g(b) = 0$ and $g'(t) = \frac{\gamma'(t)}{\gamma(t)-a}$, except possibly on the finite set S where γ is not differentiable. Now $e^{-g(t)}(\gamma(t) - a)$ is a continuous function such that

$$\begin{aligned}\frac{d}{dt}\, e^{-g(t)}(\gamma(t)-a) &= e^{-g(t)}\gamma'(t) - g'(t)e^{-g(t)}(\gamma(t)-a)\\ &= e^{-g(t)}\{\gamma'(t) - g'(t)(\gamma(t)-a)\} = 0,\end{aligned}$$

except on the finite set S. This implies that $e^{-g(t)}(\gamma(t) - a)$ is constant on $[b, c]$. Evaluating this function at $t = b$ and $t = c$ gives then

$$e^{-g(b)}(\gamma(b)-a) = \gamma(b) - a = e^{-g(c)}(\gamma(c)-a),$$

which implies $e^{-g(c)} = 1$, since $\gamma(b) = \gamma(c)$. Hence $g(c) = 2\pi i m$ for some integer m, and thus $\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = m$ which completes the proof of the theorem. □

By the above proposition the index of a closed contour is an integer m. Intuitively this integer measures how many times the contour γ winds around the point a and in what direction. From the properties of contour integrals we have immediately that the following properties hold.

(1) $\mathrm{Ind}_{-\gamma}(a) = -\mathrm{Ind}_\gamma(a)$.
(2) If γ is obtained by joining the closed contours γ_1 and γ_2, then

$$\mathrm{Ind}_\gamma(a) = \mathrm{Ind}_{\gamma_1}(a) + \mathrm{Ind}_{\gamma_2}(a).$$

We shall prove that the index of a closed contour depends continuously on the point a and that therefore the index is constant on each connected component of $\mathbb{C} \setminus \gamma^*$. We recall first the relevant definitions. Let $S \subset \mathbb{C}$. Then S_1 is called a (connected) *component* of S, if S_1 is a maximal connected subset of S. One can show that if S_1 is a connected subset of S, then so is the relative closure of S_1. Hence connected components of a set S are always relatively closed.

PROPOSITION 2.18. *Let G be an open set in $\mathbb{C}$. Then every connected component of G is also open and thus G is a countable disjoint union of open and relatively closed components.*

PROOF. Let C denote a component of G and let $z_0 \in C$. Let $\epsilon > 0$ such that $B(z_0, \epsilon) \subset G$. Then $C \cup B(z_0, \epsilon)$ is a connected subset of G and thus $C = C \cup B(z_0, \epsilon)$, i.e., $B(z_0, \epsilon) \subset C$. Hence C is open. In each component we can pick a different $a + bi$ with $a, b \in \mathbb{Q}$, so there are countably many components. □

REMARK 2.19. If $G = \mathbb{C} \setminus K$, where K is a compact set, then G has exactly one unbounded component. In particular, when $G = \mathbb{C} \setminus \gamma^*$ for a closed contour γ, then G has one unbounded component.

THEOREM 2.20. *Let γ be a closed contour and let $G = \mathbb{C} \setminus \gamma^*$. Then Ind_γ is constant on each component of G and $Ind_\gamma(a) = 0$ for all a in the unbounded component of G.*

PROOF. Define $f(w) = \mathrm{Ind}_\gamma(w)$ for $w \in G$. We first show that $f : G \to \mathbb{C}$ is continuous. Let $w \in G$. Then $r = \mathrm{dist}(w, \gamma^*) > 0$, since γ^* is compact. Let $\epsilon > 0$ and then take $0 < \delta < \min\{\frac{r}{2}, \frac{\epsilon\pi r^2}{L}.\}$, where $L = \ell(\gamma)$. Then for $|w_1 - w| < \delta$ we have

$$|f(w) - f(w_1)| = \frac{1}{2\pi}\left|\int_\gamma \frac{(w - w_1)}{(z - w)(z - w_1)}\, dz\right|.$$

For $z \in \gamma^*$ we have $|z - w| \geq r$ and $|z - w_1| \geq |z - w| - |w - w_1| > \frac{r}{2}$. Hence

$$|f(w) - f(w_1)| < \frac{\delta}{\pi r^2} L < \epsilon.$$

It follows that f is continuous. If now $C \subset G$ is a component, then $f(C)$ is a connected subset of $\mathbb{C}$. On the other hand $f(C) \subset \mathbb{Z}$ and thus $f(C)$ consists of a single point. To see that $\mathrm{Ind}_\gamma(a) = 0$ for all a in the unbounded component of G, let $R > 0$ such that $\{z : |z| > R\}$ is contained in the unbounded component of G. Then find $a \in \mathbb{C}$ with $|a| > R$ such that $|z - a| > \frac{L}{\pi}$ for all $z \in \gamma^*$. Then

$$|\mathrm{Ind}_\gamma(a)| \leq \frac{1}{2\pi}\frac{\pi}{L}L = \frac{1}{2},$$

and thus $\mathrm{Ind}_\gamma(a) = 0$. As $\mathrm{Ind}_\gamma(a)$ is constant on the unbounded component it follows that this holds for all a in the unbounded component of G. □

EXAMPLE 2.21.

(i) Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be defined by $\gamma(t) = z_0 + Re^{it}$. Then γ traces the circle $|z - z_0| = R$ once counterclockwise. In this case $\mathrm{Ind}_\gamma(a) = 1$ for $|a - z_0| < R$ and $\mathrm{Ind}_\gamma(a) = 0$ for $|a - z_0| > R$, since $\mathrm{Ind}_\gamma(z_0) = 0$ and the component of $G \setminus \gamma$ containing z_0 equals $|z - z_0| < R$.

(ii) Let $\gamma : [0, 4\pi] \to \mathbb{C}$ be defined by $\gamma(t) = z_0 + Re^{-it}$. Then γ traces the circle $|z - z_0| = R$ twice clockwise. In this case $\mathrm{Ind}_\gamma(a) = -2$ for $|a - z_0| < R$ and $\mathrm{Ind}_\gamma(a) = 0$ for $|a - z_0| > R$.

The following proposition provides the index for practically every curve encountered in applications.

PROPOSITION 2.22. *Let $\gamma : [a, b] \to \mathbb{C}$ a closed curve. Assume there exists $z_0 \in \mathbb{C} \setminus \gamma^*$, $t_0 \in (a, b)$ and $\epsilon > 0$ so that the rays $R_t = \{z_0 + s(\gamma(t) - z_0) : s \geq 0\}$ have the following properties.*

(1) *$R_t \cap \gamma^* = \{\gamma(t)\}$ for all $t \in (t_0 - \epsilon, t_0 + \epsilon)$*

(2) *The part of R_t with $s > 1$ lies in the unbounded component of $\mathbb{C} \setminus \gamma^*$ and the part with $0 < s < 1$ lies in a bounded component of $\mathbb{C} \setminus \gamma^*$.*

(3) *γ traces $\gamma^* \cap \{\gamma(t) : t \in (t_0 - \epsilon, t_0 + \epsilon)\}$ once counter clockwise.*

Then $Ind_\gamma(z_0) = 1$.

PROOF. Let $f(z) = \log|z - z_0| + i\arg^*(z - z_0)$ be a branch of $\log(z - z_0)$ with domain $\mathbb{C} \setminus R_{t_0}$. Denote by γ_ϵ the part of the curve γ in $\mathbb{C} \setminus R_{t_0}$ with initial point $\gamma(t_0 + \epsilon)$ and terminal point $\gamma(t_0 - \epsilon)$. Then

$$\int_{\gamma_\epsilon} \frac{1}{z - z_0}\,dz = f(\gamma(t_0 - \epsilon)) - f(\gamma(t_0 + \epsilon)) \to 2\pi i$$

as $\epsilon \to 0$. On the other hand

$$\int_{\gamma_\epsilon} \frac{1}{z - z_0}\,dz \to \int_\gamma \frac{1}{z - z_0}\,dz$$

as $\epsilon \to 0$ and thus $\mathrm{Ind}_\gamma(z_0) = 1$.

□

THEOREM 2.23. *(Power series expansion of holomorphic functions) Let $G \subset \mathbb{C}$ and let f be holomorphic on G. Then for all $a \in G$ and all $R > 0$ such that $B(a, R) \subset G$ there exists (unique) c_n such that*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n$$

for all $z \in B(a, R)$.

PROOF. Let $0 < r < R$ and define $\gamma : [0, 2\pi] \to B(a, R)$ by $\gamma(t) = a + re^{it}$. Then $\mathrm{Ind}_\gamma(z) = 1$ for all $z \in B(a, r)$. Hence by the Cauchy's Integral formula (applied to the open set $B(a, R)$) we have

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

Now $\left|\frac{z-a}{\zeta-a}\right| = \frac{|z-a|}{r} < 1$ for all $z \in B(a, r)$ and all $\zeta \in \gamma^*$. Hence the geometric series

$$\sum_{n=0}^{\infty} \frac{(z - a)^n}{(\zeta - a)^{n+1}} = \frac{1}{\zeta - a}\left\{\frac{1}{1 - \frac{z-a}{\zeta-a}}\right\} = \frac{1}{\zeta - z}$$

converges uniformly in ζ on γ^* for each $z \in B(a, r)$. Hence

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_\gamma f(\zeta) \sum_{n=0}^{\infty} \frac{(z - a)^n}{(\zeta - a)^{n+1}}\,d\zeta \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - a)^{n+1}}\,d\zeta\right)(z - a)^n \\ &= \sum_{n=0}^{\infty} c_n (z - a)^n \end{aligned}$$

where

$$c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - a)^{n+1}}\,d\zeta.$$

The uniqueness follows from Corollary 4.8 in Chapter1, where it was shown that $c_n = \frac{f^{(n)}(a)}{n!}$. □

COROLLARY 2.24. *Let $G \subset \mathbb{C}$ be an open set and assume $f : G \to \mathbb{C}$ is holomorphic. Then f' is holomorphic on G and thus $f^{(n)}$ exists for all $n \geq 1$ on G. Moreover, if $B(a, R) \subset G$ and $|f(z)| \leq M$ on $B(a, R)$, then*

$$|f^{(n)}(a)| \leq \frac{n!M}{R^n} (\textit{Cauchy Estimates}).$$

PROOF. The fact that f' is holomorphic on G follows immediately from the above theorem and Theorem 4.7. From Corollary 4.8 we get

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - a)^{n+1}} \, d\zeta,$$

where $\gamma(t) = a + re^{it}, 0 \leq t \leq 2\pi, 0 < r < R$ and thus

$$|f^{(n)}(a)| \leq \frac{n!}{2\pi} 2\pi r \frac{M}{r^{n+1}} = \frac{n!M}{r^n}.$$

As this holds for all $0 < r < R$ the proof is complete. □

THEOREM 2.25. *(Morera's Theorem) Let $G \subset \mathbb{C}$ be an open set and $f : G \to \mathbb{C}$ a continuous function such that*

$$\int_{\partial \Delta} f(z) \, dz = 0$$

for all triangles $\Delta \subset G$. Then f is holomorphic on G.

PROOF. Let $B(a, R) \subset G$ for $a \in G$. Then as in the proof of Theorem 2.12 we can find F holomorphic on $B(a, R)$ such that $F' = f$ on $B(a, R)$. From the above corollary we now conclude that f is holomorphic on $B(a, R)$. As this holds for all $B(a, R) \subset G$ we conclude that f is holomorphic on G. □

THEOREM 2.26. *(Liouville's Theorem) Let f be an entire function. Assume that f is bounded on $\mathbb{C}$. Then f is constant.*

PROOF. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be the power series expansion around $z = 0$. Since f is entire, this series has radius of convergence equal to ∞. Let M be such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Then for all $R > 0$ we have for $n \geq 1$ that $|f^{(n)}(0)| \leq \frac{n!M}{R^n} \to 0$ as $R \to \infty$. Hence $f^{(n)}(0) = 0$ for all $n \geq 1$, and thus also $a_n = 0$ for all $n \geq 1$. Therefore $f(z) = a_0$ for all $z \in \mathbb{C}$. □

CHAPTER 3

Zeros and singularities of holomorphic functions

1. Zeros of holomorphic functions

A subset $G \subset \mathbb{C}$ is called a *region* if G is open and connected. If $f : G \to \mathbb{G}$, then $z_0 \in G$ is called a *zero* of f if $f(z_0) = 0$.

THEOREM 1.1. *Let f be a holomorphic function on a region G. Then either every zero of f is isolated or f is identically zero on G. For each isolated zero $a \in G$ there exists a unique $m \in \mathbb{N}$ such that $f(z) = (z-a)^m g(z)$, where g is holomorphic on G and $g(a) \neq 0$. Moreover if f is not identically zero on G, then f has countably many zeros in G.*

PROOF. Let $a \in G$ such that $f(a) = 0$. Then there exists $r > 0$ such that $B(a,r) \subset G$. Then by Theorem 2.23 we can expand f in a power series

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

on $B(a,r)$. Note that $c_0 = 0$ as $f(a) = 0$. There are now two possibilities: either $c_n = 0$ for all n in which case f vanishes identically on $B(a,r)$ and thus on G, or there exists a smallest $m \geq 1$ such that $c_m \neq 0$. In the latter case we define

$$g(z) = \begin{cases} \frac{f(z)}{(z-a)^m} & \text{if } z \in G \setminus \{a\} \\ c_m & \text{if } z = a \end{cases}$$

Then clearly $f(z) = g(z)(z-a)^m$ for all $z \in G$ and thus g is holomorphic on $G \setminus \{a\}$. But the power series of f shows that g has a power series expansion on $B(a,r)$ and is thus holomorphic on $B(a,r)$. This shows that g is holomorphic on G and $g(a) = c_m \neq 0$. Since g is continuous it follows that $g(z) \neq 0$ in a open disk around a and thus a is an isolated zero of f in this case. We have thus shown that a zero of f is either isolated or f is identically zero on a disk around the zero. We show next that if f has a non-isolated zero, then f is identically zero on G. Assume f has a non isolated zero on G. Then the interior U of $f^{-1}(0) = \{z \in G : f(z) = 0$ is non-empty. We next observe that U is relatively closed in G. Let $z_n \in U$ such that $z_n \to z_0$. Then continuity of f implies that also $f(z_0) = 0$. Since the z_n's are non-isolated zeros, we can assume that $z_n \neq z_m$. Then z_0 is a non-isolated zero of f and thus $z_0 \in U$. It follows that U is open and closed in G and thus $U = G$ by connectedness of G. It remains to show that $f^{-1}(0)$ is countable, in case f is not identically zero on G. In this case every $a \in f^{-1}(0)$ is isolated, so for every $a \in f^{-1}(0)$ there exists $r > 0$ such that $B(a,r) \cap f^{-1}(0) = \{a\}$. Hence there exists $a_1 \in \mathbb{Q} + i\mathbb{Q}$ and $r_a > 0$ such that $a \in D_a = B(a_1, r_a) \subset B(a,r)$. The collection $\{D_a : a \in f^{-1}(0)\}$ is countable and if $a, b \in f^{-1}(0)$ with $a \neq b$, then

$D_a \neq D_b$. Hence the mapping $a \mapsto D_a$ is a one-to-one mapping and thus $f^{-1}(0)$ is countable. □

REMARK 1.2. The number m associated with the zero a, as in the above theorem, is called the *order* of the zero a.

COROLLARY 1.3. *Let f and g be a holomorphic functions on a region G and assume there exists a subset $S \subset G$ with limit point in G such that $f(z) = g(z)$ on S. Then $f(z) = g(z)$ for all $z \in G$.*

PROOF. Let $h = f - g$. Then h has a non-isolated zero in G (namely the limit point of S) and thus by the above theorem h is identically zero on G. □

EXAMPLE 1.4.

(i) Let f be an entire function such that $f(\frac{1}{n}) = \sin(\frac{1}{n})$ for all $n \in \mathbb{N}$. Then by the above corollary $f(z) = \sin z$ for all $z \in \mathbb{C}$.

(ii) Let f be a holomorphic fuction on $\mathbb{C} \setminus \{0\}$ such that $f(z) = \sin(\frac{1}{z})$ for all $z = \frac{1}{n\pi}$, $n = 1, 2, \cdots$, i.e., $f(\frac{1}{n\pi}) = 0$ for all $n \geq 1$. It does not follow in this case that $f(z) = \sin \frac{1}{z}$ for all $z \neq 0$, since $f(z) \equiv 0$ also satisfies $f(\frac{1}{n\pi}) = 0$.

THEOREM 1.5. *(Maximum Modulus Theorem) Let $G \subset \mathbb{C}$ be open and connected and $f : G \to \mathbb{G}$ holomorphic. Assume $|f|$ attains a maximum at a point $a \in G$, i.e., $|f(z)| \leq |f(a)|$ for all $z \in G$. Then f is constant on G.*

PROOF. Let $a \in G$ such that $|f(z)| \leq |f(a)|$ for all $z \in G$. Then there exists $R > 0$ such that $B(a, R) \subset G$. Take $0 < r < R$ and let $\gamma(t) = a + re^{it}$ for $0 \leq t \leq 2\pi$. Then Cauchy's Integral Formula 2.14, applied to $B(a, R)$, gives

$$\begin{aligned} f(a) &= \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - a}\, dz \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a + re^{it})}{re^{it}} rie^{it}\, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{it})\, dt. \end{aligned}$$

Hence $|f(a)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(a + re^{it})|\, dt \leq \frac{1}{2\pi} \int_0^{2\pi} |f(a)|\, dt = |f(a)|$. It follows that

$$\int_0^{2\pi} |f(a)| - |f(a + re^{it})|\, dt = 0.$$

The integrand is non-negative and continuous, so it must be identically zero. Hence $|f(a)| = |f(a+re^{it})|$ for all $0 < r < R$ and it follows that $|f|$ is constant on $B(a, R)$. It is now an an exercise to show, using the Cauchy-Riemann equations, that f is constant on $B(a, R)$. Hence $f(z) - f(a) = 0$ on $B(a, R)$ and from the connectedness of G it follows that $f(z) - f(a) = 0$ on G. □

REMARK 1.6. If G is a bounded region and f is holomorphic on G and continuous on $\overline{G}$, then by the above theorem $|f|$ must attains its maximum on the boundary ∂G of G.

A corollary of the maximum modulus theorem in the form of the remark is the Minimum Modulus Theorem.

COROLLARY 1.7. *(Minimum Modulus Theorem) Let $G \subset \mathbb{C}$ be a bounded region and let $f : \overline{G} \to \mathbb{C}$ a non-constant continuous function which is holomorphic on G. If there exists $z_0 \in G$ such that $|f(z_0)| \leq \inf\{|f(z)| : z \in \partial G\}$, then f has a zero on G.*

PROOF. Assume f has no zero on G. Then $g = \frac{1}{f}$ is holomorphic on G and has an interior maximum on $\overline{G}$. Hence g is constant, which contradicts that f is non-constant. ☐

We now note that if G is region and $f : G \to \mathbb{C}$ is non-constant, then for $a \in \mathbb{C}$ there exists a closed disk $\overline{B(a,r)}$ such that $f(z) \neq f(a)$ for all $z \in \overline{B(a,r)} \setminus \{a\}$. This follows from the fact that $z = a$ is an isolated zero of $f(z) - f(a)$. Combined withe minimum modulus theorem we can use this observation to prove the Open Mapping Theorem. Recall first that a mapping f from a metroic space X into a metric space Y is called open, if $f(U)$ is open in Y for all open $U \subset X$.

THEOREM 1.8. *(Open Mapping Theorem) Let f be a non-constant holomorphic function on a region G. Then f is an open mapping*

PROOF. Let $U \subset G$ be open and $a \in U$. We need to prove that $f(a)$ is an interior point of $f(U)$. By the above remark there exists $r > 0$ such that $\overline{B(a,r)} \subset U$ and $f(z) \neq f(a)$ for all z with $|z - a| = r$. This implies that $\delta = \frac{1}{2} \min_{\{z:|z-a|=r\}} |f(z) - f(a)| > 0$. We claim that $B(f(a), \delta) \subset f(U)$. To see this let $w \in B(f(a), \delta)$. Then $|f(a) - w| < \delta$. For z with $|z - a| = r$ we have that

$$|f(z) - w| \geq |f(z) - f(a)| - |f(a) - w| \geq 2\delta - \delta = \delta.$$

This implies that $|f(a) - w| < \min_{\{z:|z-a|=r\}} |f(z) - w|$. By the minimum modulus theorem there exists $z \in B(a,r)$ such that $f(z) - w = 0$. This shows $B(f(a), \delta) \subset f(U)$ and the proof is complete. ☐

2. Singularities of holomorphic functions

Let $G \subset \mathbb{C}$ be an open set and let $a \in G$. Assume f is holomorphic on $G \setminus \{a\}$, then we say that f has an *isolated singularity* at a. If we can define $f(a)$ in such a way that f becomes differentiable at a, then a is called a *removable singularity* of f.

THEOREM 2.1. *Let $G \subset \mathbb{C}$ be an open set and let $a \in G$. Assume f is holomorphic on $G \setminus \{a\}$ and that f is bounded on $B(a,r) \setminus \{a\}$ for some $r > 0$. Then f has a removable singularity at $z = a$.*

PROOF. Define

$$h(x) = \begin{cases} (z-a)^2 f(z) & z \neq a, z \in G \\ 0 & z = a. \end{cases}$$

Then f bounded on $B(a,r)\setminus\{a\}$ implies that $h'(a)=0$. Hence h is holomorphic on G. It follows that h has a power series expansion

$$h(z)=\sum_{n=2}^{\infty} c_n(z-a)^n$$

for all $z\in B(a,r)$. Now define $f(a)=c_2$, then f has the power series expansion

$$f(z)=\sum_{n=0}^{\infty} c_{n+2}(z-a)^n$$

for all $z\in B(a,r)$, which implies that f is holomorphic on $B(a,r)$. Thus f is holomorphic on G.

□

REMARK 2.2. Note that if f is holomorphic on $G\setminus\{a\}$ and if $\lim_{z\to a} f(z)$ exists in $\mathbb{C}$, then f is bounded on $B(a,r)\setminus\{a\}$ for some $r>0$ and thus f has a removable singularity at $z=a$ in that case. In particular, if f is holomorphic on $G\setminus\{a\}$ and if f is continuous at a, then f is holomorphic on G. Conversely, if f has a removable singularity at $z=a$, then $\lim_{z\to a} f(z)$ exists in $\mathbb{C}$.

EXAMPLE 2.3. Let $f(z)=\frac{\sin z}{z}$ on $\mathbb{C}\setminus\{0\}$. For $z\neq 0$ we find by using the power series of $\sin z$ that

$$f(z)=1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots.$$

Now the series on the right hand side converges for all $z\in\mathbb{C}$ to a holomorphic function g, which agrees with f on $\mathbb{C}\setminus\{0\}$. Thus 0 is a removable singularity of f and by defining $f(0)=g(0)=1$ we extend f to an entire function.

An isolated singularity a of f is called a *pole* of f if $\lim_{z\to a}|f(z)|=\infty$. An isolated singularity a of f which is neither a removable singularity or a a pole is called an *essential singularity* of f. We first characterize poles.

THEOREM 2.4. *Let $G\subset\mathbb{C}$ be an open set and let $a\in G$. Assume f is holomorphic on $G\setminus\{a\}$. Then the following are equivalent.*

(i) *f has a pole at a.*

(ii) *There exist a unique $m\in\mathbb{N}$ and a holomorphic function g on G with $g(a)\neq 0$ such that*

$$f(z)=\frac{g(z)}{(z-a)^m}$$

for all $z\in G\setminus\{a\}$.

(iii) *There exist a unique $m\in\mathbb{N}$ and $c_{-1},c_{-2},\cdots,c_{-m}\in\mathbb{C}$ with $c_{-m}\neq 0$ such that*

$$f(z)-\sum_{k=1}^{m}\frac{c_{-k}}{(z-a)^k}$$

has a removable singularity at a.

PROOF. Assume first that (i) holds, i.e., f has a pole at a. Then $\lim_{z\to a}|f(z)|=\infty$ implies that there exists $r>0$ such that $f(z)\neq 0$ on $B(a,r)\setminus\{a\}$. Then define $h(z)=\frac{1}{f(z)}$ for $z\in B(a,r)\setminus\{a\}$. Then h is holomorphic on $B(a,r)\setminus\{a\}$ and $\lim_{z\to a} h(z)=0$, so a is a removable singularity of h and by defining $h(a)=0$

we have a holomorphic function on $B(a,r)$ with its only zero zero at a. Hence by Theorem 1.1 we know that there exist $m \in \mathbb{N}$ and a holomorphic function g_1 on $B(a,r)$ with $g_1(z) \neq 0$ such that $h(z) = (z-a)^m g_1(z)$, so (ii) holds on $B(a,r)$ with $g = \frac{1}{g_1}$. Now $(z-a)^m f(z)$ is holomorphic on $G \setminus \{a\}$ and agrees with g on $B(a,r)$, so we can extend g to a holomorphic function on G so that (ii) holds. If (ii) holds then there exists $r > 0$ such that $g(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ for all $z \in B(a,r)$ and $g(a) = a_0 \neq 0$. Now

$$f(z) - \sum_{k=1}^{m} \frac{a_{m-k}}{(z-a)^k} = \sum_{k=0}^{\infty} a_{k+m}(z-a)^k$$

for all $z \in B(a,r) \setminus \{a\}$, which shows that (iii) holds if we take $c_{-k} = a_{m-k}$ for $k = 1, \cdots, m$. If (iii) holds, then $(z-a)^m f(z)$ defines a holomorphic function g with $g(a) \neq 0$ on an open disk $B(a,r)$ for some $r > 0$. Hence

$$\lim_{z \to a} |f(z)| = \lim_{z \to a} \frac{|g(z)|}{|z-a|^m} = \infty,$$

which completes the proof of the theorem. □

Remark 2.5. If f has a pole at a, then the number m as in the above theorem is called the *order* of the pole and $\sum_{k=1}^{m} \frac{c_{-k}}{(z-a)^k}$ is called the *principal part* of f at the pole a. Note also that (iii) above implies that if f has a pole of order m at a, then there exist $r > 0$ and $c_{-m}, \cdots, c_{-1}, c_0, c_1, \cdots$ such that we have

$$f(z) = \frac{c_{-m}}{(z-a)^m} + \cdots + \frac{c_{-1}}{(z-a)} + \sum_{k=0}^{\infty} c_k (z-a)^k$$

for all $z \in B(a,r) \setminus \{a\}$.

We now present a limit characterization of essential singularities.

Theorem 2.6. *(Casorati-Weierstrass Theorem) Let $G \subset \mathbb{C}$ be an open set and let $a \in G$. Assume f is holomorphic on $G \setminus \{a\}$. Then the following are equivalent.*

(i) *f has an essential singularity at a.*
(ii) *If $r > 0$ such that $B(a,r) \subset G$, then $f(B(a,r))$ is dense in $\mathbb{C}$, i.e., for all $w \in \mathbb{C}$ there exist $z_n \in G \setminus \{a\}$ with $z_n \to a$ such that $\lim_{n\to\infty} f(z_n) = w$.*
(iii) *There exist $z_n \to a$ and $z'_n \to a$ in $G \setminus \{a\}$ such that $\lim_{n\to\infty} f(z_n)$ and $\lim_{n\to\infty} f(z'_n)$ exist, but are unequal.*

Proof. Assume a is an essential singularity of f. If (ii) does not hold, then there exist a $w \in \mathbb{C}$ such that

$$g(z) = \frac{1}{f(z) - w}$$

is bounded in a neighborhood of a. Hence g has a removable singularity at a. This implies that f has either a removable singularity at a (in case $g(a) \neq 0$) or a pole at a, which contradicts our assumption. hence (ii) holds. Clearly (ii) implies (iii). If (iii) holds, then a can not be a removable singularity of f by Remark 2.2 and a can not a pole either, so it must be an essential singularity of f. □

Example 2.7. Let $f(z) = e^{\frac{1}{z}}$ on $G = \mathbb{C} \setminus \{0\}$. Then $z = 0$ is an essential singularity of f, since $\lim_{n\to\infty} f(-\frac{1}{n}) = 0$ and $\lim_{n\to\infty} f(\frac{1}{2\pi n i}) = 1$.

THEOREM 2.8. *(Laurent Series expansion) Let $G \subset \mathbb{C}$ be an open set and let $a \in G$. Assume f is holomorphic on $G \setminus \{a\}$. Then there exists $R > 0$ and c_n $(n = 0, \pm 1, \pm 2, \dots)$ such that for all $z \in B(a, R) \setminus \{a\}$ we have*

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z-a)^k,$$

where

$$c_k = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\zeta)}{(\zeta - a)^{k+1}}\, d\zeta$$

and $\gamma_r(t) = a + re^{it}, 0 \le t \le 2\pi$ with $0 < r < R$. Moreover the series converges uniformly on any annulus $0 < r_1 \le |z-a| \le r_2 < R$.

PROOF. Let $R > 0$ such that $B(a, R) \subset G$ and let $0 < r_1 < |z-a| < r_2 < R$. Define $\gamma_{r_k}(t) = a + r_k e^{it}, 0 \le t \le 2\pi$ for $k = 1, 2$. Now write $\gamma_{r_1} \cup \gamma_{r_2}$ as a join of two curves, each one lying in a starlike open set contained in $0 < |z-a| < R$ and such that z is inside exactly one of the two curves (see figure 1 below). Than $g(\zeta) = \frac{f(\zeta)}{\zeta - z}$ is holomorphic inside the other curve. We see by Theorems 2.12 and

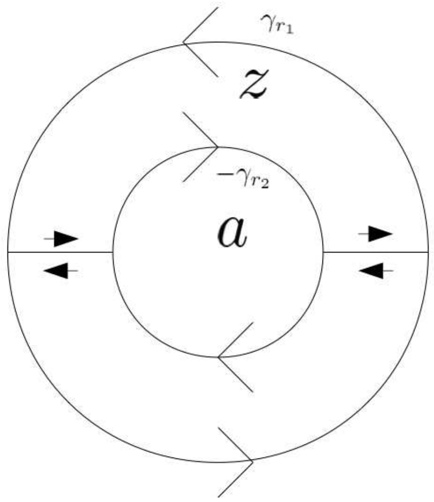

FIGURE 1. $-\gamma_{r_1 \cup} \gamma_{r_2}$ as a join of two curves.

2.14 that

$$\int_{\gamma_{r_2}} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \int_{\gamma_{r_1}} \frac{f(\zeta)}{\zeta - z}\, d\zeta = 2\pi i f(z). \tag{3.1}$$

Now we have

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - a) - (z - a)} = \frac{1}{\zeta - a} \sum_{n=0}^{\infty} \left(\frac{z-a}{\zeta - a} \right)^n$$

for $\zeta \in \gamma_{r_2}^*$ for all $|z-a| < r_2 = |\zeta - a|$ and

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - a) - (z - a)} = -\frac{1}{z - a} \sum_{n=0}^{\infty} \left(\frac{\zeta - a}{z - a} \right)^n$$

for $\zeta \in \gamma_{r_1}^*$ for all $|z-a| > r_1 = |\zeta - a|$. Inserting these expansions in 3.1 we get

$$f(z) = \sum_{k=0}^{\infty} a_k (z-a)^k + \sum_{k=-\infty}^{-1} b_k (z-a)^k, \tag{3.2}$$

where

$$a_k = \frac{1}{2\pi i} \int_{\gamma_{r_2}} \frac{f(\zeta)}{(\zeta - a)^{k+1}} \, d\zeta$$

and

$$b_k = \frac{1}{2\pi i} \int_{\gamma_{r_1}} \frac{f(\zeta)}{(\zeta - a)^{k+1}} \, d\zeta.$$

The two series for $\frac{1}{\zeta - z}$ converge uniformly in $z - a$ for $r_1' \leq |z-a| \leq r_2'$, where $r_1 < r_1'$ and $r_2' < r_2$. Thus the series (3.2) converges uniformly on $r_1' \leq |z - a| \leq r_2'$. Similarly to how we established the equation (3.1), we can see that $c_k = \int_{\gamma_r} \frac{f(\zeta)}{(\zeta - a)^{k+1}} \, d\zeta$ does not depend on r, so that $c_k = a_k$ for $k \geq 0$ and $c_k = b_k$ for $k \leq -1$, which completes the proof of the theorem. $\square$

The following corollary follows now immediately from the previous characterizations of removable singularities and poles.

Corollary 2.9. *Let $G \subset \mathbb{C}$ be an open set and let $a \in G$. Assume f is holomorphic on $G \setminus \{a\}$. Let $f(z) = \sum_{k=-\infty}^{\infty} c_k (z-a)^k$ be the Laurent series expansion of f around a. Then the following hold.*

(i) *f has a removable singularity at $z = a$ if and only if $c_k = 0$ for all $k \leq -1$.*
(ii) *f has a pole at $z = a$ of order m if and only if $c_k = 0$ for all $k \leq -(m+1)$ and $c_{-m} \neq 0$.*
(iii) *f has an essential singularity at a if and only $c_k \neq 0$ for infinitely many $k < 0$.*

Remark 2.10. Let f be holomorphic on $G \setminus \{a\}$ and let

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z-a)^k$$

be the Laurent series expansion of f around a. Then the coefficient c_{-1} is called the *residue* of f at a and denoted by $\mathrm{Res}(f, a)$. Its importance derives from the fact that if $\gamma(t) = a + re^{it}$ with $0 \leq t \leq 2\pi$ is a curve in G, then $\int_\gamma f(z) \, dz = 2\pi i \mathrm{Res}(f, a)$. This follows immediately from the uniform convergence of the series, which allows us to integrate the series term by term. In case f has a pole of order m at a, we can compute $\mathrm{Res}(f, a)$ without using the Laurent series as follows:

$$\mathrm{Res}(f, a) = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z-a)^m f(z)].$$

3. The Residue Theorem and Applications

We start with an application of the Laurent expansion.

THEOREM 3.1. *(Residue Theorem) Let G be a starlike region. Let $p_1, \cdots, p_n$ in G and $f : G \setminus \{p_1, \cdots, p_n\} \to \mathbb{C}$ be holomorphic. Let γ be a piecewise smooth closed curve in $G \setminus \{p_1, \cdots, p_n\}$. Then*

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} [Res(f, p_k)] Ind_\gamma(p_k).$$

PROOF. For each p_k there exists $R_k > 0$ such that $B(p_k, R_k) \subset G$. Then for $z \in B(p_k, R_k) \setminus \{p_k\}$ we have a Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} c_n (z - p_k)^n$. Denote by $S_k(z)$ the singular part $\sum_{n=-\infty}^{-1} c_n (z-p_k)^n$. Then there exist $\epsilon > 0$ such that S_k converges uniformly on $|z - p_k| \geq \epsilon$ and such that γ^* is a subset of each $|z - p_k| \geq \epsilon$. In particular each S_k is holomorphic on $\mathbb{C} \setminus \{p_k\}$. Define $g(z) = f(z) - \sum_{k=1}^{n} S_k(z)$ on $G \setminus \{p_1, \cdots, p_n\}$. Clearly g is holomorphic on $G \setminus \{p_1, \cdots, p_n\}$. We claim that each p_k is a removable singularity of g. To see this, note that on $B(p_k, R_k)$ we have that

$$g(z) = -\sum_{j \neq k}^{n} S_j(z) + \sum_{m=0}^{\infty} c_m (z - p_k)^m.$$

Both terms on the right hand side are holomorphic on $B(p_k, R_k)$ so that $z = p_k$ is a removable singularity for g. Hence we can extend g to a holomorphic function on G. It follows now from Cauchy's theorem 2.12 that $\int_\gamma f(z)\,dz = 0$. Hence

$$\int_\gamma f(z)\,dz = \sum_{k=1}^{n} \int_\gamma S_k(z)\,dz.$$

Now each S_k converges uniformly on γ^*, so that

$$\int_\gamma S_k(z)\,dz = \sum_{m=-\infty}^{-1} c_m \int_\gamma (z - p_k)^m\,dz = 2\pi i [\mathrm{Res}(f, p_k)] \mathrm{Ind}_\gamma(p_k),$$

from which the conclusion follows. □

To apply the above theorem to the evaluation of improper teal integrals, we first recall some definitions. Assume $f : \mathbb{R} \to \mathbb{R}$ is function which is Riemann integrable over $[-R_1, R_2]$ for all $R_1, R_2 > 0$ (this holds e.g. when f is continuous). Then f is (improper) Riemann integrable over $\mathbb{R}$ if both limits $\lim_{R_1 \to \infty} \int_{-R_1}^{0} f(x)\,dx$ and $\lim_{R_2 \to \infty} \int_0^{R_2} f(x)\,dx$ exist and $\int_{-\infty}^{\infty} f(x)\,dx$ is by definition the sum of these two limits. One can also define the Cauchy principal value integral of f by

$$(PV) \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R \to \infty} \int_{-R}^{R} f(x)\,dx.$$

It is easy to see that if f is improper Riemann integrable, then the Cauchy principal value of the integral of f exists and equals $\int_{-\infty}^{\infty} f(x)\,dx$, but that the converse is false in general (take e.g. $f(x) = x$). There are two cases, where the two integrals coincide. The first case is when f is an even function, i.e. $f(-x) = f(x)$ for all x. In this case we have

$$\int_0^R f(x)\,dx = \frac{1}{2} \int_{-R}^{R} f(x)\,dx.$$

The other case is when the integral of $|f(x)|$ has a finite Cauchy principal value. We present now some examples of applications of the residue theorem.

EXAMPLE 3.2. Let $f(x) = \frac{1}{(1+x^2)^2}$. We want to compute $\int_0^\infty f(x)\,dx$. Observe first that f is an even function, so that by the above

$$\int_0^\infty f(x)\,dx = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^R f(x)\,dx.$$

First extend f to $\mathbb{C}\setminus\{i,-i\}$ by $f(z) = \frac{1}{(1+z^2)^2}$. Now define $\gamma_R(t) = Re^{it}$ for $0 \le t \le \pi$ and define the closed curve $C_R = \gamma_R \cup [-R, R]$. Then for $R > 1$ we have $\operatorname{Ind}_{C_R}(i) = 1$ and $\operatorname{Ind}_{C_R}(-i) = 0$. By the residue theorem we have for $R > 1$ that

$$\int_{C_R} f(z)\,dz = 2\pi i\ \operatorname{Res}(f(z), i).$$

Now $z = i$ is a pole of order 2, so $\operatorname{Res}(f(z), i) = \frac{d}{dz}(\frac{1}{(z+i)^2})|_{z=i} = \frac{1}{4i}$. It follows that for $R > 1$

$$\int_{C_R} f(z)\,dz = 2\pi i\cdot\frac{1}{4i} = \frac{\pi}{2}.$$

Now $\int_{C_R} f(z)\,dz = \int_{\gamma_R} f(z)\,dz + \int_{-R}^R f(x)\,dx$ and

$$\left|\int_{\gamma_R} f(z)\,dz\right| \le \pi R \max_{z\in\gamma_R^*}\frac{1}{|(1+z^2)^2|} \le \frac{\pi R}{(R^2-1)^2} \to 0$$

as $R\to\infty$. It follows that

$$\int_0^\infty \frac{1}{(1+x^2)^2}\,dx = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^R f(x)\,dx = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}.$$

EXAMPLE 3.3. Suppose we want to compute $\int_0^\infty \frac{1}{1+x^4}\,dx$. We first observe that $f(x) = \frac{1}{1+x^4}$ is even, so that

$$\int_0^\infty \frac{1}{1+x^4}\,dx = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^R f(x)\,dx.$$

Let $f(z) = \frac{1}{1+z^4}$. Then f is holomorphic on $\mathbb{C}\setminus\{z_1, z_2, z_3, z_4\}$, where z_j are the solutions of $z^4 = -1$, i.e., $z_1 = e^{\frac{\pi i}{4}} = \frac{1}{2}\sqrt{2} + \frac{i}{2}\sqrt{2}$, $z_2 = -\frac{1}{2}\sqrt{2} + \frac{i}{2}\sqrt{2}$, $z_3 = -z_1$, and $z_4 = -z_2$. Let $\Gamma_R = \gamma_R \cup [-R, R]$, where $\gamma_R(t) = Re^{it}$ with $0 \le t \le \pi$. Then for $R > 1$ we have

$$\int_{\Gamma_R} f(z)\,dz = 2\pi i\left(\operatorname{Res}(f, z_1) + \operatorname{Res}(f, z_2)\right).$$

As the pole at z_1 is simple we compute the residue by

$$\operatorname{Res}(f, z_1) = \lim_{z\to z_1}(z - z_1)f(z) = \frac{1}{(z_1-z_2)(z_1-z_3)(z_1-z_4)} = \frac{1}{8}(-\sqrt{2} - i\sqrt{2}).$$

Similarly $\operatorname{Res}(f, z_2) = \frac{1}{8}(\sqrt{2} - i\sqrt{2})$. Thus $\int_{\Gamma_R} f(z)\,dz = \frac{\pi\sqrt{2}}{2}$. Now

$$\left|\int_{\gamma_R} f(z)\,dz\right| \le \pi R\max_{|z|=R}|f(z)| \le \frac{\pi R}{R^4-1} \to 0$$

as $R\to\infty$. Hence

$$\lim_{R\to\infty}\int_{-R}^R f(x)\,dx = \frac{\pi\sqrt{2}}{2},$$

which implies that

$$\int_0^\infty \frac{1}{1+x^4}\,dx = \frac{\pi\sqrt{2}}{4}.$$

We now apply the residue theorem to location and counting of zeros and poles of a holomorphic function.

THEOREM 3.4. *(Principle of the Argument) Let $G \subset \mathbb{C}$ be a starlike region and γ a closed contour in G. Let f be holomorphic on G, except for poles of order l_k at $z = p_k \in G \setminus \gamma^*$ $(1 \le k \le m)$. Assume f has zeros of order m_j at $z = a_j \in G \setminus \gamma^*$ $(1 \le j \le n)$. Then*

$$\begin{aligned} Ind_{\gamma_1}(0) &= \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz \\ &= \sum_{j=1}^n m_j Ind_\gamma(a_j) - \sum_{k=1}^m l_k Ind_\gamma(p_k) \end{aligned}$$

where $\gamma_1 = f \circ \gamma$.

PROOF. Let $h(z) = \frac{f'(z)}{f(z)}$. Then h is holomorphic at all $z \in G$ where $f(z) \neq 0$ and has a pole at the zeros of f. If $z = a_j$ is a zero of order m_j, then $f(z) = (z-a_j)^{m_j} g(z)$, where g is holomorphic on $G \setminus \{p_i, \cdots, p_m\}$ and $g(a) \neq 0$. Then there exists $r > 0$ such that $h(z) = \frac{f'(z)}{f(z)} = \frac{m_j}{z-a_j} + \frac{g'(z)}{g(z)}$ on $B(a,r)\setminus\{a\}$, where $\frac{g'(z)}{g(z)}$ is holomorphic on $B(a,r)$. Hence h has a simple pole at $z = a_j$ and $\mathrm{Res}(h, a_j) = m_j$. Similarly at the point $z = p_k$ we can write $f(z) = (z-p_k)^{-l_k} g(z)$, where g is holomorphic near p_k and $g(p_k) \neq 0$. As above this implies that h has a simple pole at $z = p_k$ and $\mathrm{Res}(h, p_k) = -l_k$. Hence we have by the residue theorem that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^n m_j \mathrm{Ind}_\gamma(a_j) - \sum_{k=1}^m l_k \mathrm{Ind}_\gamma(p_k).$$

Let $\gamma : [a,b] \to \mathbb{C}$. Then

$$\begin{aligned} \mathrm{Ind}_{\gamma_1}(0) &= \frac{1}{2\pi i}\int_{\gamma_1} \frac{1}{z}\,dz = \frac{1}{2\pi i}\int_a^b \frac{\gamma_1'(s)}{\gamma_1(s)}\,ds \\ &= \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(s))}{f(\gamma(s))}\gamma'(s)\,ds = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz. \end{aligned}$$

□

THEOREM 3.5. *(General Rouché's Theorem) Let $G \subset \mathbb{C}$ be a starlike region and γ a closed contour in G. Let f, g be holomorphic on G, except for poles of order l_k at $z = p_k \in G \setminus \gamma^*$ $(1 \le k \le m)$ for f and poles of order n_i at $z = q_i \in G \setminus \gamma^*$ for g $(1 \le j \le r)$. Let f have zeros of order m_j at $z = a_j \in G$ $(1 \le j \le n)$ and g have zeros of order s_j at $z = b_j \in G$ $(1 \le j \le t)$. Assume $|f(z) + g(z)| < |f(z)| + |g(z)|$ on γ^*. Then*

$$\sum_{j=1}^n m_j Ind_\gamma(a_j) - \sum_{k=1}^m l_k Ind_\gamma(p_k) = \sum_{j=1}^t s_j Ind_\gamma(b_j) - \sum_{k=1}^r l_k Ind_\gamma(q_k).$$

PROOF. Observe first that the strict inequality $|f(z)+g(z)| < |f(z)| + |g(z)|$ implies that f and g can't have zeros on γ^*. By hypothesis

$$\left|\frac{f(z)}{g(z)} + 1\right| < \left|\frac{f(z)}{g(z)}\right| + 1$$

on γ^*. Observe that the strict inequality prevents that $\frac{f(z)}{g(z)}$ is a non-negative real number for $z \in \gamma^*$. Hence $\frac{f}{g}$ maps γ^* into $\mathbb{C} \setminus [0, \infty)$. Let $\gamma_1 = (\frac{f}{g}) \circ \gamma$, then 0 is in the unbounded component of $\mathbb{C} \setminus \gamma_1^*$. Hence $\operatorname{Ind}_{\gamma_1^*}(0) = 0$. Hence by the lefthand part of the equality in the above theorem we have

$$0 = \frac{1}{2\pi i}\int_\gamma (\frac{f}{g})'(\frac{f}{g})^{-1}\,dz = \frac{1}{2\pi i}\int_\gamma (\frac{f'}{f} - \frac{g'}{g})\,dz = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz - \frac{1}{2\pi i}\int_\gamma \frac{g'}{g}\,dz,$$

and the conclusion now follows from the righthand part of the equality in the above theorem. $\square$

REMARK 3.6. Note most often the above theorems are applied to curves γ with $\mathbb{C} \setminus \gamma^*$ having exactly two components, namely the "exterior" of γ where $\operatorname{Ind}_\gamma = 0$ and the "interior" with $\operatorname{Ind}_\gamma = 1$. In that case $\frac{1}{2\pi i}\int_{\gamma_1} \frac{f'(z)}{f(z)}\,dz = N_f - P_f$, where N_f is the number of zeros of f inside γ (counting with their orders) and P_f is the number of poles of f inside γ (also counted according to their order). With this notation and with these hypotheses we have then as conclusion in the above theorem that $N_f - P_f = N_g - P_g$.

The following corollary is the classical Rouché's Theorem, which has a slightly stronger hypothesis than the above theorem.

COROLLARY 3.7. *(Rouché's Theorem) Assume f and g are holomorphic in a neighborhood of $\overline{B(a,R)}$. Assume also that $|f(z)+g(z)| < |f(z)|$ on $\{z : |z-a| = R\}$. Let N_f, N_g denote the number of zeros of f, respectively g inside γ (with orders). Then $N_f = N_g$.*

EXAMPLE 3.8. Let $g(z) = z^5 - 12z^3 + 14$. Let first $R = 1$. Then with the choice of $f(z) = -14$ we get $|g(z) + f(z)| \le |z|^5 + 12|z|^2 = 13 < |f(z)|$ on $|z| = 1$. Hence g has the same number of zeros on $B(0,1)$ as f, namely zero. Now take $R = 2$. Now take $f(z) = 12z^3$. Then $|g(z) + f(z)| \le 2^5 + 14 = 46 < 96 = |f(z)|$ on $|z| = 2$. Hence g has three zeros inside $|z| = 2$, since f has three zeros inside $|z| = 2$. For $R = 4$ we can take $f(z) = -z^5$ and see that all five zeros of g lie in the disk $B(0,4)$.

4. The Global Cauchy Theorem

The goal of this section is to obtain versions of Theorems 2.12 and 2.14 of chapter 2 without starlike assumptions.

PROPOSITION 4.1. *Let $G \subset \mathbb{C}$ be an open set, f be a holomorphic function on G, and let $g : G \times G \to \mathbb{C}$ be defined by*

$$g(\zeta, z) = \begin{cases} \frac{f(\zeta)-f(z)}{\zeta - z} & \zeta \ne z \\ f'(z) & \zeta = z. \end{cases}$$

Then for fixed $\zeta \in G$, the function $g(\zeta, \cdot)$ is holomorphic on G and if γ is a contour in G, the function h defined by $h(z) = \int_\gamma g(\zeta, z)\,d\zeta$ is holomorphic on G.

PROOF. For fixed $\zeta \in G$, the function $g(\zeta, z)$ is holomorphic on $G \setminus \{\zeta\}$ and continuous at ζ, and thus by Remark 2.2 we see that $g(\zeta, z)$ is holomorphic on G. We claim now that $g(\zeta, z)$ is continuous on $G \times G$. To prove continuity of g we only need to consider points $(a, a) \in G \times G$. Let $a \in G$ and $\epsilon > 0$. Then there exists $r > 0$ such that $B(a : r) \subset G$ and $|f'(w) - f'(a)| < \epsilon$ for all $w \in B(a, r)$. Let now $\zeta, z \in B(a, r)$. Then

$$f(\zeta) - f(z) - f'(a)(\zeta - z) = \int_{[z,\zeta]} f'(w) - f'(a)\, dw$$

implies that $|g(\zeta, z) - g(a, a)| \leq \max_{w \in [z.\zeta]} |f'(w) - f'(a)| < \epsilon$. Hence g is continuous at (a, a) for all $a \in G$. Next we prove that $h(z) = \int_\gamma g(\zeta, z)\, d\zeta$ is continuous on G, let $z_n \to z$ in G. Then $g(\zeta, z_n) \to g(\zeta, z)$ uniformly on γ^*. This implies that $h(z_n) \to h(z)$. Now let $\Delta \subset G$ be a triangle. Then

$$\int_{\partial\Delta} h(z)\, dz = \int_\gamma \left(\int_{\partial\Delta} g(\zeta, z)\, dz \right) d\zeta = 0.$$

Hence h is holomorphic on G by Morera's Theorem. $\square$

THEOREM 4.2. *(Global Cauchy's Theorem) Let $G \subset \mathbb{C}$ be an open set and let f be a holomorphic function on G. If $\gamma_1, \ldots, \gamma_m$ are closed contours in G such that $Ind_{\gamma_1}(z) + \cdots + Ind_{\gamma_m}(z) = 0$ for all $z \in \mathbb{C} \setminus G$, then for all $z \in G \setminus \cup_{k=1}^m \gamma_k^*$ we have Cauchy's Integral Formula*

$$f(z) \cdot \sum_{k=1}^m Ind_{\gamma_k}(z) = \frac{1}{2\pi i} \sum_{k=1}^m \int_{\gamma_k} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

and Cauchy's Theorem

$$\sum_{k=1}^m \int_{\gamma_k} f(\zeta)\, d\zeta = 0.$$

PROOF. Let g be as in the previous proposition and let h be defined as $h(z) = \frac{1}{2\pi i} \sum_{k=1}^m \int_{\gamma_k} g(\zeta, z)\, d\zeta$. Then h is holomorphic on G and Cauchy's Integral Formula is equivalent to proving that $h(z) = 0$ for all $z \in G \setminus \cup_{k=1}^m \gamma_k^*$. Define $H = \{z \in \mathbb{C} \setminus \cup_{k=1}^m \gamma_k^* : \sum_{k=1}^m \mathrm{Ind}_{\gamma_k}(z) = 0\}$. Then H is open (since the index is a continuous integer valued function) and $\mathbb{C} = G \cup H$. Now define h_1 on H by

$$h_1(z) = \frac{1}{2\pi i} \sum_{k=1}^m \int_{\gamma_k} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

If $z \in G \cap H$, then $h_1(z) = h(z)$ by the definition of h and H. As in the proof of Theorem 2.23 we can expand each integral $\int_{\gamma_k} \frac{f(\zeta)}{\zeta - z}\, d\zeta$ in a power series around each point $z \in H$ and thus h_1 is holomorphic on H. Therefore we can extend h to an entire function by defining $h = h_1$ on H. Now $\lim_{|z|\to\infty} |h(z)| \leq \lim_{|z|\to\infty} \frac{1}{2\pi} \sum_{k=1}^m |\int_{\gamma_k} \frac{f(\zeta)}{\zeta - z}\, d\zeta| \leq \frac{1}{2\pi} \sum_{k=1}^m \ell(\gamma_k) \lim_{|z|\to\infty} \max_{\zeta \in \gamma_k^*} \frac{|f(\zeta)|}{|\zeta - z|} = 0$. It follows from Liouville's Theorem that $h(z) = 0$ for all z. For $z \in G \setminus \cup_{k=1}^m \gamma_k^*$ we get

$$f(z) \cdot \sum_{k=1}^m \mathrm{Ind}_{\gamma_k}(z) = \frac{1}{2\pi i} \sum_{k=1}^m \int_{\gamma_k} \frac{f(z)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \sum_{k=1}^m \int_{\gamma_k} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

To prove Cauchy's Theorem, take $a \in G \setminus \cup_{k=1}^m \gamma_k^*$. Then

$$\sum_{k=1}^m \int_{\gamma_k} f(\zeta)\, d\zeta = \sum_{k=1}^m \int_{\gamma_k} \frac{f(\zeta)(\zeta - a)}{\zeta - a}\, d\zeta = 2\pi i \left(\sum_{k=1}^m \mathrm{Ind}_{\gamma_k}(a) \right) f(a)(a-a) = 0.$$

□

COROLLARY 4.3. *Let $G \subset \mathbb{C}$ be an open set and let f be a holomorphic function on G. If $\gamma_1, \dots, \gamma_m$ and $\sigma_1, \dots, \sigma_n$ are closed contours in G such that $Ind_{\gamma_1}(z) + \cdots + Ind_{\gamma_m}(z) = Ind_{\sigma_1}(z) + \cdots + Ind_{\sigma_n}(z)$ for all $z \in \mathbb{C} \setminus G$, then*

$$\sum_{k=1}^m \int_{\gamma_k} f(\zeta)\, d\zeta = \sum_{k=1}^n \int_{\sigma_k} f(\zeta)\, d\zeta.$$

PROOF. Apply Cauchy's Theorem to the closed contours $\gamma_1, \dots, \gamma_m, -\sigma_1, \dots, -\sigma_n$. □

Next we prove a theorem generalizing Theorem 3.1.

THEOREM 4.4. *(General Residue Theorem) Let $G \subset \mathbb{C}$ be an open set and let f be a holomorphic function on G except for a subset A of G of isolated singularities. If $\gamma_1, \dots, \gamma_m$ are closed contours in $G \setminus A$ such that $Ind_{\gamma_1}(z) + \cdots + Ind_{\gamma_m}(z) = 0$ for all $z \in \mathbb{C} \setminus G$, then*

$$\sum_{k=1}^m \int_{\gamma_k} f(\zeta)\, d\zeta = 2\pi i \sum_{a \in A} \left\{ Res(f, a) \sum_{k=1}^m Ind_{\gamma_k}(a) \right\}.$$

PROOF. Let $B = \{a \in A : \sum_{k=1}^m \mathrm{Ind}_{\gamma_k}(a) \neq 0\}$. The unbounded component of $\mathbb{C} \setminus \cup \gamma_k^*$ and $\widetilde{G}$ are both contained in the set $\{z \in \mathbb{C} : \sum_{k=1}^m \mathrm{Ind}_{\gamma_k}(z) = 0\}$ and thus B is the intersection of A with a compact subset of G. This implies that B is finite, as every point of A is isolated. Let $B = \{a_1, \dots, a_n\}$ and define $l_j = \sum_{k=1}^m \mathrm{Ind}_{\gamma_k}(a_j)$ for $1 \le j \le n$. Then pick $r_j > 0$ for $j = 1, \dots, n$ such that $B(a_j, r_j)$ are mutually disjoint, none of them intersects any γ_k^*, and are contained in $G \setminus (A \setminus B)$. Then define σ_j to be the boundary of $B(a_j, r_j)$ traversed l_j times (clockwise when $l_j < 0$). Let $G_1 = (G \setminus A) \cup B$. Then f is holomorphic on $G_1 \setminus B$ and $\sum_{k=1}^m \mathrm{Ind}_{\gamma_k}(z) = 0$ on $\mathbb{C} \setminus G_1$. We also have $\sum_{j=1}^n \mathrm{Ind}_{\sigma_j}(z) = 0$ on $\mathbb{C} \setminus G_1$ as $A \setminus B$ is outside each disk $B(a_j, r_j)$ for $j = 1, \dots, n$. For $a_i \in B$ we have $\sum_{j=1}^n \mathrm{Ind}_{\sigma_j}(a_i) = l_i = \sum_{k=1}^m \mathrm{Ind}_{\gamma_k}(a_i)$. Hence by Corollary 4.3 applied to $G_1 \setminus B$ we have

$$\begin{aligned}
\sum_{k=1}^m \int_{\gamma_k} f(\zeta)\, d\zeta &= \sum_{j=1}^n \int_{\sigma_j} f(\zeta)\, d\zeta \\
&= 2\pi i \sum_{j=1}^n \mathrm{Res}(f, a_j)\, l_j \\
&= 2\pi i \sum_{a \in A} \left\{ \mathrm{Res}(f, a) \sum_{k=1}^m \mathrm{Ind}_{\gamma_k}(a) \right\}.
\end{aligned}$$

□