INTRODUCTION TO REAL ANALYSIS
Third Edition

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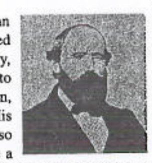
CHAPTER 7

THE RIEMANN INTEGRAL

We have already mentioned the developments, during the 1630s, by Fermat and Descartes leading to analytic geometry and the theory of the derivative. However, the subject we know as calculus did not begin to take shape until the late 1660s when Isaac Newton created his theory of "fluxions" and invented the method of "inverse tangents" to find areas under curves. The reversal of the process for finding tangent lines to find areas was also discovered in the 1680s by Gottfried Leibniz, who was unaware of Newton's unpublished work and who arrived at the discovery by a very different route. Leibniz introduced the terminology "calculus differentialis" and "calculus integralis", since finding tangent lines involved differences and finding areas involved summations. Thus, they had discovered that integration, being a process of summation, was inverse to the operation of differentiation.

During a century and a half of development and refinement of techniques, calculus consisted of these paired operations and their applications, primarily to physical problems. In the 1850s, Bernhard Riemann adopted a new and different viewpoint. He separated the concept of integration from its companion, differentiation, and examined the motivating summation and limit process of finding areas by itself. He broadened the scope by considering all functions on an interval for which this process of "integration" could be defined: the class of "integrable" functions. The Fundamental Theorem of Calculus became a result that held only for a restricted set of integrable functions. The viewpoint of Riemann led others to invent other integration theories, the most significant being Lebesgue's theory of integration. But there have been some advances made in more recent times that extend even

Bernard Riemann



(Georg Friedrich) Bernard Riemann (1826–1866), the son of a poor Lutheran minister, was born near Hanover, Germany. To please his father, he enrolled (1846) at the University of Göttingen as a student of theology and philosophy, but soon switched to mathemtics. He interrupted his studies at Göttingen to study at Berlin under C. G. J. Jacobi, P. G. J. Dirichlet, and F. G. Eisenstein, but returned to Göttingen in 1849 to complete his thesis under Gauss. His thesis dealt with what are now called "Riemann surfaces". Gauss was so enthusiastic about Riemann's work that he arranged for him to become a *privatdozent* at Göttingen in 1854. On admission as a *privatdozent*, Riemann was required to prove himself by delivering a probationary lecture before the entire faculty. As tradition dictated, he submitted three topics, the first two of which he was well prepared to discuss. To Riemann's surprise, Gauss chose that he should lecture on the third topic: "On the hypotheses that underlie the foundations of geometry". After its publication, this lecture had a profound effect on modern geometry.

Despite the fact that Riemann contracted tuberculosis and died at the age of 39, he made major contributions in many areas: the foundations of geometry, number theory, real and complex analysis, topology, and mathematical physics.

the Lebesgue theory to a considerable extent. We will give a brief introduction to these results in Chapter 10.

We begin by defining the concept of Riemann integrability of real-valued functions defined on a closed bounded interval of $\mathbb{R}$, using the Riemann sums familiar to the reader from calculus. This method has the advantage that it extends immediately to the case of functions whose values are complex numbers, or vectors in the space $\mathbb{R}^n$. In Section 7.2, we will establish the Riemann integrability of several important classes of functions: step functions, continuous functions, and monotone functions. However, we will also see that there are functions that are *not* Riemann integrable. The Fundamental Theorem of Calculus is the principal result in Section 7.3. We will present it in a form that is slightly more general than is customary and does not require the function to be a derivative at every point of the interval. A number of important consequences of the Fundamental Theorem are also given. In Section 7.3 we also give a statement of the definitive Lebesgue Criterion for Riemann integrability. This famous result is usually not given in books at this level, since its proof (given in Appendix C) is somewhat complicated. However, its statement is well within the reach of students, who will also comprehend the power of this result. The final section presents several methods of approximating integrals, a subject that has become increasingly important during this era of high-speed computers. While the proofs of these results are not particularly difficult, we defer them to Appendix D.

An interesting history of integration theory, including a chapter on the Riemann integral, is given in the book by Hawkins cited in the References.

Section 7.1 Riemann Integral

We will follow the procedure commonly used in calculus courses and define the Riemann integral as a kind of limit of the Riemann sums as the norm of the partitions tend to 0. Since we assume that the reader is familiar—at least informally—with the integral from a calculus course, we will not provide a motivation of the integral, or disuss its interpretation as the "area under the graph", or its many applications to physics, engineering, economics, etc. Instead, we will focus on the purely mathematical aspects of the integral.

However, we first recall some basic terms that will be frequently used.

Partitions and Tagged Partitions

If $I := [a, b]$ is a closed bounded interval in $\mathbb{R}$, then a **partition** of I is a finite, ordered set $\mathcal{P} := (x_0, x_1, \cdots, x_{n-1}, x_n)$ of points in I such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

(See Figure 7.1.1.) The points of $\mathcal{P}$ are used to divide $I = [a, b]$ into non-overlapping subintervals

$$I_1 := [x_0, x_1], \quad I_2 := [x_1, x_2], \cdots, \quad I_n := [x_{n-1}, x_n].$$

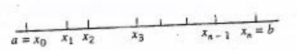


Figure 7.1.1 A partition of $[a, b]$.

Often we will denote the partition $\mathcal{P}$ by the notation $\mathcal{P} = \{[x_{i-1}, x_i]\}_{i=1}^n$. We define the **norm** (or **mesh**) of $\mathcal{P}$ to be the number

$$\|\mathcal{P}\| := \max\{x_1 - x_0, x_2 - x_1, \cdots, x_n - x_{n-1}\}. \tag{1}$$

Thus the norm of a partition is merely the length of the largest subinterval into which the partition divides $[a, b]$. Clearly, many partitions have the same norm, so the partition is *not* a function of the norm.

If a point t_i has been selected from each subinterval $I_i = [x_{i-1}, x_i]$, for $i = 1, 2, \cdots, n$, then the points are called **tags** of the subintervals I_i. A set of ordered pairs

$$\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$$

of subintervals and corresponding tags is called a **tagged partition** of I; see Figure 7.1.2. (The dot over the $\mathcal{P}$ indicates that a tag has been chosen for each subinterval.) The tags can be chosen in a wholly arbitrary fashion; for example, we can choose the tags to be the left endpoints, or the right endpoints, or the midpoints of the subintervals, etc. Note that an endpoint of a subinterval can be used as a tag for two consecutive subintervals. Since each tag can be chosen in infinitely many ways, each partition can be tagged in infinitely many ways. The norm of a tagged partition is defined as for an ordinary partition and does not depend on the choice of tags.

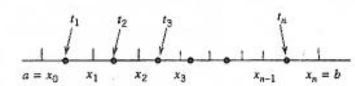


Figure 7.1.2 A tagged partition of $[a, b]$

If $\dot{\mathcal{P}}$ is the tagged partition given above, we define the **Riemann sum** of a function $f : [a, b] \to \mathbb{R}$ corresponding to $\dot{\mathcal{P}}$ to be the number

$$S(f; \dot{\mathcal{P}}) := \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}). \tag{2}$$

We will also use this notation when $\dot{\mathcal{P}}$ denotes a *subset* of a partition, and not the entire partition.

The reader will perceive that if the function f is positive on $[a, b]$, then the Riemann sum (2) is the sum of the areas of n rectangles whose bases are the subintervals $I_i = [x_{i-1}, x_i]$ and whose heights are $f(t_i)$. (See Figure 7.1.3.)

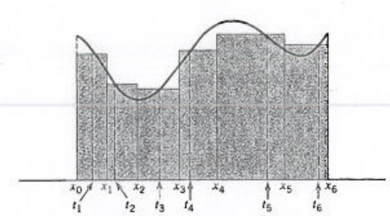


Figure 7.1.3 A Riemann sum.

Definition of the Riemann Integral

We now define the Riemann integral of a function f on an interval $[a, b]$.

7.1.1 Definition A function $f : [a, b] \to \mathbb{R}$ is said to be **Riemann integrable** on $[a, b]$ if there exists a number $L \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ is any tagged partition of $[a, b]$ with $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$, then

$$|S(f; \dot{\mathcal{P}}) - L| < \varepsilon.$$

The set of all Riemann integrable functions on $[a, b]$ will be denoted by $\mathcal{R}[a, b]$.

Remark It is sometimes said that the integral L is "the limit" of the Riemann sums $S(f; \dot{\mathcal{P}})$ as the norm $\|\dot{\mathcal{P}}\| \to 0$. However, since $S(f; \dot{\mathcal{P}})$ is not a function of $\|\dot{\mathcal{P}}\|$, this limit is not of the type that we have studied before.

First we will show that if $f \in \mathcal{R}[a, b]$, then the number L is uniquely determined. It will be called the **Riemann integral of** f over $[a, b]$. Instead of L, we will usually write

$$L = \int_a^b f \qquad \text{or} \qquad \int_a^b f(x)\, dx.$$

It should be understood that any letter other than x can be used in the latter expression, so long as it does not cause any ambiguity.

7.1.2 Theorem *If $f \in \mathcal{R}[a, b]$, then the value of the integral is uniquely determined.*

Proof. Assume that L' and L'' both satisfy the definition and let $\varepsilon > 0$. Then there exists $\delta'_{\varepsilon/2} > 0$ such that if $\dot{\mathcal{P}}_1$ is any tagged partion with $\|\dot{\mathcal{P}}_1\| < \delta'_{\varepsilon/2}$, then

$$|S(f; \dot{\mathcal{P}}_1) - L'| < \varepsilon/2.$$

Also there exists $\delta''_{\varepsilon/2} > 0$ such that if $\dot{\mathcal{P}}_2$ is any tagged partition with $\|\dot{\mathcal{P}}_2\| < \delta''_{\varepsilon/2}$, then

$$|S(f; \dot{\mathcal{P}}_2) - L''| < \varepsilon/2.$$

Now let $\delta_\varepsilon := \min\{\delta'_{\varepsilon/2}, \delta''_{\varepsilon/2}\} > 0$ and let $\dot{\mathcal{P}}$ be a tagged partition with $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$. Since both $\|\dot{\mathcal{P}}\| < \delta'_{\varepsilon/2}$ and $\|\dot{\mathcal{P}}\| < \delta''_{\varepsilon/2}$, then

$$|S(f; \dot{\mathcal{P}}) - L'| < \varepsilon/2 \qquad \text{and} \qquad |S(f; \dot{\mathcal{P}}) - L''| < \varepsilon/2,$$

whence it follows from the Triangle Inequality that

$$\begin{aligned} |L' - L''| &= |L' - S(f; \dot{\mathcal{P}}) + S(f; \dot{\mathcal{P}}) - L''| \\ &\le |L' - S(f; \dot{\mathcal{P}})| + |S(f; \dot{\mathcal{P}}) - L''| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that $L' = L''$. Q.E.D.

Some Examples

If we use only the definition, in order to show that a function f is Riemann integrable we must (i) know (or guess correctly) the value L of the integral, and (ii) construct a δ_ε that will suffice for an arbitrary $\varepsilon > 0$. The determination of L is sometimes done by

calculating Riemann sums and guessing what L must be. The determination of δ_ε is likely to be difficult.

In actual practice, we usually show that $f \in \mathcal{R}[a, b]$ by making use of some of the theorems that will be given later.

7.1.3 Examples **(a)** Every constant function on $[a, b]$ is in $\mathcal{R}[a, b]$.

Let $f(x) := k$ for all $x \in [a, b]$. If $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is any tagged partition of $[a, b]$, then it is clear that

$$S(f; \dot{\mathcal{P}}) = \sum_{i=1}^n k(x_i - x_{i-1}) = k(b - a).$$

Hence, for any $\varepsilon > 0$, we can choose $\delta_\varepsilon := 1$ so that if $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$, then

$$|S(f; \dot{\mathcal{P}}) - k(b - a)| = 0 < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $f \in \mathcal{R}[a, b]$ and $\int_a^b f = k(b - a)$.

(b) Let $g : [0, 3] \to \mathbb{R}$ be defined by $g(x) := 2$ for $0 \le x \le 1$, and $g(x) := 3$ for $1 < x \le 3$. A preliminary investigation, based on the graph of g (see Figure 7.1.4), suggests that we might expect that $\int_0^3 g = 8$.

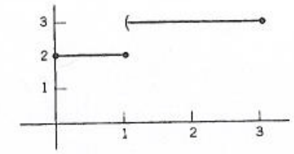


Figure 7.1.4 Graph of g.

Let $\dot{\mathcal{P}}$ be a tagged partition of $[0, 3]$ with norm $< \delta$; we will show how to determine δ in order to ensure that $|S(g; \dot{\mathcal{P}}) - 8| < \varepsilon$. Let $\dot{\mathcal{P}}_1$ be the subset of $\dot{\mathcal{P}}$ having its tags in $[0, 1]$ where $g(x) = 2$, and let $\dot{\mathcal{P}}_2$ be the subset of $\dot{\mathcal{P}}$ with its tags in $(1, 3]$ where $g(x) = 3$. It is obvious that we have

$$(3) \qquad S(g; \dot{\mathcal{P}}) = S(g; \dot{\mathcal{P}}_1) + S(g; \dot{\mathcal{P}}_2).$$

Since $\|\dot{\mathcal{P}}\| < \delta$, if $u \in [0, 1 - \delta]$ and $u \in [x_{i-1}, x_i]$, then $x_{i-1} \le 1 - \delta$ so that $x_i < x_{i-1} + \delta \le 1$, whence the tag $t_i \in [0, 1]$. Therefore, the interval $[0, 1 - \delta]$ is contained in the union of all subintervals in $\dot{\mathcal{P}}$ with tags $t_i \in [0, 1]$. Similarly, this union is contained in $[0, 1 + \delta]$. (Why?) Since $g(t_i) = 2$ for these tags, we have

$$2(1 - \delta) \le S(g; \dot{\mathcal{P}}_1) \le 2(1 + \delta).$$

A similar argument shows that the union of all subintervals with tags $t_i \in (1, 3]$ contains the interval $[1 + \delta, 3]$ of length $2 - \delta$, and is contained in $[1 - \delta, 3]$ of length $2 + \delta$. Therefore,

$$3(2 - \delta) \le S(g; \dot{\mathcal{P}}_2) \le 3(2 + \delta).$$

Adding these inequalities and using equation (3), we have

$$8 - 5\delta \le S(g; \dot{\mathcal{P}}) = S(g; \dot{\mathcal{P}}_1) + S(g; \dot{\mathcal{P}}_2) \le 8 + 5\delta,$$

whence it follows that

$$|S(g;\dot{\mathcal{P}}) - 8| \le 5\delta.$$

To have this final term $< \varepsilon$, we are led to take $\delta_\varepsilon < \varepsilon/5$.

Making such a choice (for example, if we take $\delta_\varepsilon := \varepsilon/10$), we can retrace the argument and see that $|S(g;\dot{\mathcal{P}}) - 8| < \varepsilon$ when $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have proved that $g \in \mathcal{R}[0,3]$ and that $\int_0^3 g = 8$, as predicted.

(c) Let $h(x) := x$ for $x \in [0,1]$; we will show that $h \in \mathcal{R}[0,1]$.

We will employ a "trick" that enables us to guess the value of the integral by considering a particular choice of the tag points. Indeed, if $\{I_i\}_{i=1}^n$ is any partition of $[0,1]$ and we choose the tag of the interval $I_i = [x_{i-1}, x_i]$ to be the midpoint $q_i := \frac{1}{2}(x_{i-1} + x_i)$, then the contribution of this term to the Riemann sum corresponding to the tagged partition $\dot{\mathcal{Q}} := \{(I_i, q_i)\}_{i=1}^n$ is

$$h(q_i)(x_i - x_{i-1}) = \tfrac{1}{2}(x_i + x_{i-1})(x_i - x_{i-1}) = \tfrac{1}{2}(x_i^2 - x_{i-1}^2).$$

If we add these terms and note that the sum telescopes, we obtain

$$S(h;\dot{\mathcal{Q}}) = \sum_{i=1}^n \tfrac{1}{2}(x_i^2 - x_{i-1}^2) = \tfrac{1}{2}(1^2 - 0^2) = \tfrac{1}{2}.$$

Now let $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^n$ be an arbitrary tagged partition of $[0,1]$ with $\|\dot{\mathcal{P}}\| < \delta$ so that $x_i - x_{i-1} < \delta$ for $i = 1, \cdots, n$. Also let $\dot{\mathcal{Q}}$ have the same partition points, but where we choose the tag q_i to be the midpoint of the interval I_i. Since both t_i and q_i belong to this interval, we have $|t_i - q_i| < \delta$. Using the Triangle Inequality, we deduce

$$\begin{aligned}|S(h;\dot{\mathcal{P}}) - S(h;\dot{\mathcal{Q}})| &= \left|\sum_{i=1}^n t_i(x_i - x_{i-1}) - \sum_{i=1}^n q_i(x_i - x_{i-1})\right| \\ &\le \sum_{i=1}^n |t_i - q_i|(x_i - x_{i-1}) < \delta \sum_{i=1}^n (x_i - x_{i-1}) = \delta(x_n - x_0) = \delta.\end{aligned}$$

Since $S(h;\dot{\mathcal{Q}}) = \frac{1}{2}$, we infer that if $\dot{\mathcal{P}}$ is any tagged partition with $\|\dot{\mathcal{P}}\| < \delta$, then

$$|S(h;\dot{\mathcal{P}}) - \tfrac{1}{2}| < \delta.$$

Therefore we are led to take $\delta_\varepsilon \le \varepsilon$. If we choose $\delta_\varepsilon := \varepsilon$, we can retrace the argument to conclude that $h \in \mathcal{R}[0,1]$ and $\int_0^1 h = \int_0^1 x\,dx = \frac{1}{2}$.

(d) Let $F(x) := 1$ for $x = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$, and $F(x) := 0$ elsewhere on $[0,1]$. We will show that $F \in \mathcal{R}[0,1]$ and that $\int_0^1 F = 0$.

Here there are four points where F is not 0, each of which can belong to two subintervals in a given tagged partition $\dot{\mathcal{P}}$. Only these terms will make a nonzero contribution to $S(F;\dot{\mathcal{P}})$. Therefore we choose $\delta_\varepsilon := \varepsilon/8$.

If $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$, let $\dot{\mathcal{P}}_0$ be the subset of $\dot{\mathcal{P}}$ with tags different from $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$, and let $\dot{\mathcal{P}}_1$ be the subset of $\dot{\mathcal{P}}$ with tags at these points. Since $S(F;\dot{\mathcal{P}}_0) = 0$, it is seen that $S(F;\dot{\mathcal{P}}) = S(F;\dot{\mathcal{P}}_0) + S(F;\dot{\mathcal{P}}_1) = S(F;\dot{\mathcal{P}}_1)$. Since there are at most 8 terms in $S(F;\dot{\mathcal{P}}_1)$ and each term is $< 1 \cdot \delta_\varepsilon$, we conclude that $0 \le S(F;\dot{\mathcal{P}}) = S(F;\dot{\mathcal{P}}_1) < 8\delta_\varepsilon = \varepsilon$. Thus $F \in \mathcal{R}[0,1]$ and $\int_0^1 F = 0$.

(e) Let $G(x) := 1/n$ for $x = 1/n$ ($n \in \mathbb{N}$), and $G(x) := 0$ elsewhere on $[0,1]$.

Given $\varepsilon > 0$, let E_ε be the (finite) set of points where $G(x) \ge \varepsilon$, let n_ε be the number of points in E_ε, and let $\delta_\varepsilon := \varepsilon/(2n_\varepsilon)$. Let $\dot{\mathcal{P}}$ be a tagged partition such that $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$. Let

$\dot{\mathcal{P}}_0$ be the subset of $\dot{\mathcal{P}}$ with tags outside of E_ε and let $\dot{\mathcal{P}}_1$ be the subset of $\dot{\mathcal{P}}$ with tags in E_ε. As in (d), we have

$$0 \le S(G;\dot{\mathcal{P}}) = S(G;\dot{\mathcal{P}}_1) < (2n_\varepsilon)\delta_\varepsilon = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $G \in \mathcal{R}[0,1]$ and $\int_0^1 G = 0$. □

Some Properties of the Integral

The difficulties involved in determining the value of the integral and of δ_ε suggest that it would be very useful to have some general theorems. The first result in this direction enables us to form certain algebraic combinations of integrable functions.

7.1.4 Theorem *Suppose that f and g are in $\mathcal{R}[a,b]$. Then:*

(a) *If $k \in \mathbb{R}$, the function kf is in $\mathcal{R}[a,b]$ and*

$$\int_a^b kf = k\int_a^b f.$$

(b) *The function $f + g$ is in $\mathcal{R}[a,b]$ and*

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

(c) *If $f(x) \le g(x)$ for all $x \in [a,b]$, then*

$$\int_a^b f \le \int_a^b g.$$

Proof. If $\dot{\mathcal{P}} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is a tagged partition of $[a,b]$, then it is an easy exercise to show that

$$S(kf;\dot{\mathcal{P}}) = kS(f;\dot{\mathcal{P}}), \qquad S(f+g;\dot{\mathcal{P}}) = S(f;\dot{\mathcal{P}}) + S(g;\dot{\mathcal{P}}),$$

$$S(f;\dot{\mathcal{P}}) \le S(g;\dot{\mathcal{P}}).$$

We leave it to the reader to show that the assertion (a) follows from the first equality. As an example, we will complete the proofs of (b) and (c).

Given $\varepsilon > 0$, we can use the argument in the proof of the Uniqueness Theorem 7.1.2 to construct a number $\delta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ is any tagged partition with $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$, then both

$$\left|S(f;\dot{\mathcal{P}}) - \int_a^b f\right| < \varepsilon/2 \quad \text{and} \quad \left|S(g;\dot{\mathcal{P}}) - \int_a^b g\right| < \varepsilon/2. \tag{4}$$

To prove (b), we note that

$$\begin{aligned}\left|S(f+g;\dot{\mathcal{P}}) - \left(\int_a^b f + \int_a^b g\right)\right| &= \left|S(f;\dot{\mathcal{P}}) + S(g;\dot{\mathcal{P}}) - \int_a^b f - \int_a^b g\right| \\ &\le \left|S(f;\dot{\mathcal{P}}) - \int_a^b f\right| + \left|S(g;\dot{\mathcal{P}}) - \int_a^b g\right| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $f + g \in \mathcal{R}[a,b]$ and that its integral is the sum of the integrals of f and g.

To prove (c), we note that the Triangle Inequality applied to (4) implies

$$\int_a^b f - \varepsilon/2 < S(f;\dot{\mathcal{P}}) \quad \text{and} \quad S(g;\dot{\mathcal{P}}) < \int_a^b g + \varepsilon/2.$$

If we use the fact that $S(f;\dot{\mathcal{P}}) \le S(g;\dot{\mathcal{P}})$, we have

$$\int_a^b f \le \int_a^b g + \varepsilon.$$

But, since $\varepsilon > 0$ is arbitrary, we conclude that $\int_a^b f \le \int_a^b g$. Q.E.D.

Boundedness Theorem

We now show that an unbounded function cannot be Riemann integrable.

7.1.5 Theorem *If $f \in \mathcal{R}[a,b]$, then f is bounded on $[a,b]$.*

Proof. Assume that f is an unbounded function in $\mathcal{R}[a,b]$ with integral L. Then there exists $\delta > 0$ such that if $\dot{\mathcal{P}}$ is any tagged partition of $[a,b]$ with $\|\dot{\mathcal{P}}\| < \delta$, then we have $|S(f;\dot{\mathcal{P}}) - L| < 1$, which implies that

$$(5) \qquad |S(f;\dot{\mathcal{P}})| < |L| + 1.$$

Now let $\mathcal{Q} = \{[x_{i-1}, x_i]\}_{n=1}^n$ be a partition of $[a,b]$ with $\|\mathcal{Q}\| < \delta$. Since $|f|$ is not bounded on $[a,b]$, then there exists at least one subinterval in $\mathcal{Q}$, say $[x_{k-1}, x_k]$, on which $|f|$ is not bounded—for, if $|f|$ is bounded on each subinterval $[x_{i-1}, x_i]$ by M_i, then it is bounded on $[a,b]$ by $\max\{M_1, \cdots, M_n\}$.

We will now pick tags for $\mathcal{Q}$ that will provide a contradiction to (5). We tag $\mathcal{Q}$ by $t_i := x_i$ for $i \ne k$ and we pick $t_k \in [x_{k-1}, x_k]$ such that

$$\left|f(t_k)(x_k - x_{k-1})\right| > |L| + 1 + \left|\sum_{i\ne k} f(t_i)(x_i - x_{i-1})\right|.$$

From the Triangle Inequality (in the form $|A + B| \ge |A| - |B|$), we have

$$|S(f;\dot{\mathcal{Q}})| \ge |f(t_k)(x_k - x_{k-1})| - \left|\sum_{i\ne k} f(t_i)(x_i - x_{i-1})\right| > |L| + 1,$$

which contradicts (5). Q.E.D.

We will close this section with an example of a function that is discontinuous at every rational number and is not monotone, but is Riemann integrable nevertheless.

7.1.6 Example We consider Thomae's function $h : [0,1] \to \mathbb{R}$ defined, as in Example 5.1.6(h), by $h(x) := 0$ if $x \in [0,1]$ is irrational, $h(0) := 1$ and by $h(x) := 1/n$ if $x \in [0,1]$ is the rational number $x = m/n$ where $m, n \in \mathbb{N}$ have no common integer factors except 1. It was seen in 5.1.6(h) that h is continuous at every irrational number and discontinuous at every rational number in $[0,1]$. We will now show that $h \in \mathcal{R}[0,1]$.

Let $\varepsilon > 0$; then the set $E_\varepsilon := \{x \in [0,1] : h(x) \ge \varepsilon/2\}$ is a finite set. We let n_ε be the number of elements in E_ε and let $\delta_\varepsilon := \varepsilon/(4n_\varepsilon)$. If $\dot{\mathcal{P}}$ is a tagged partition with $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$, let $\dot{\mathcal{P}}_1$ be the subset of $\dot{\mathcal{P}}$ having tags in E_ε and $\dot{\mathcal{P}}_2$ be the subset of $\dot{\mathcal{P}}$ having tags elsewhere in $[0,1]$. We observe that $\dot{\mathcal{P}}_1$ has at most $2n_\varepsilon$ intervals whose total length is $< 2n_\varepsilon\delta_\varepsilon = \varepsilon/2$

and that $0 < h(t_i) \le 1$ for every tag in $\dot{\mathcal{P}}_1$. Also the total lengths of the subintervals in $\dot{\mathcal{P}}_2$ is ≤ 1 and $h(t_i) < \varepsilon/2$ for every tag in $\dot{\mathcal{P}}_2$. Therefore we have

$$|S(h;\dot{\mathcal{P}})| = S(h;\dot{\mathcal{P}}_1) + S(h;\dot{\mathcal{P}}_2) < 1 \cdot 2n_\varepsilon\delta_\varepsilon + (\varepsilon/2)\cdot 1 = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we infer that $h \in \mathcal{R}[0,1]$ with integral 0. □

Exercises for Section 7.1

1. If $I := [0,4]$, calculate the norms of the following partitions:
   (a) $\mathcal{P}_1 := (0,1,2,4)$, (b) $\mathcal{P}_2 := (0,2,3,4)$,
   (c) $\mathcal{P}_3 := (0,1,1.5,2,3.4,4)$, (d) $\mathcal{P}_4 := (0,.5,2.5,3.5,4)$.
2. If $f(x) := x^2$ for $x \in [0,4]$, calculate the following Riemann sums, where $\dot{\mathcal{P}}_i$ has the same partition points as in Exercise 1, and the tags are selected as indicated.
   (a) $\mathcal{P}_1$ with the tags at the left endpoints of the subintervals.
   (b) $\mathcal{P}_1$ with the tags at the right endpoints of the subintervals.
   (c) $\mathcal{P}_2$ with the tags at the left endpoints of the subintervals.
   (d) $\mathcal{P}_2$ with the tags at the right endpoints of the subintervals.
3. Show that $f : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ if and only if there exists $L \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ is any tagged partition with norm $\|\dot{\mathcal{P}}\| \le \delta_\varepsilon$, then $|S(f;\dot{\mathcal{P}}) - L| \le \varepsilon$.
4. Let $\dot{\mathcal{P}}$ be a tagged parition of $[0,3]$.
   (a) Show that the union U_1 of all subintervals in $\dot{\mathcal{P}}$ with tags in $[0,1]$ satisfies $[0, 1 - \|\dot{\mathcal{P}}\|] \subseteq U_1 \subseteq [0, 1 + \|\dot{\mathcal{P}}\|]$.
   (b) Show that the union U_2 of all subintervals in $\dot{\mathcal{P}}$ with tags in $[1,2]$ satisfies $[1 + \|\dot{\mathcal{P}}\|, 2 - \|\dot{\mathcal{P}}\|] \subseteq U_2 \subseteq [1 - \|\dot{\mathcal{P}}\|, 2 + \|\dot{\mathcal{P}}\|]$.
5. Let $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^n$ be a tagged partition of $[a,b]$ and let $c_1 < c_2$.
   (a) If u belongs to a subinterval I_i whose tag satisfies $c_1 \le t_i \le c_2$, show that $c_1 - \|\dot{\mathcal{P}}\| \le u \le c_2 + \|\dot{\mathcal{P}}\|$.
   (b) If $v \in [a,b]$ and satisfies $c_1 + \|\dot{\mathcal{P}}\| \le v \le c_2 - \|\dot{\mathcal{P}}\|$, then the tag t_i of any subinterval I_i that contains v satisfies $t_i \in [c_1, c_2]$.
6. (a) Let $f(x) := 2$ if $0 \le x < 1$ and $f(x) := 1$ if $1 \le x \le 2$. Show that $f \in \mathcal{R}[0,2]$ and evaluate its integral.
   (b) Let $h(x) := 2$ if $0 \le x < 1$, $h(1) := 3$ and $h(x) := 1$ if $1 < x \le 2$. Show that $h \in \mathcal{R}[0,2]$ and evaluate its integral.
7. Use Mathematical Induction and Theorem 7.1.4 to show that if $f_1, \cdots, f_n$ are in $\mathcal{R}[a,b]$ and if $k_1, \cdots, k_n \in \mathbb{R}$, then the linear combination $f = \sum_{i=1}^n k_i f_i$ belongs to $\mathcal{R}[a,b]$ and $\int_a^b f = \sum_{i=1}^n k_i \int_a^b f_i$.
8. If $f \in \mathcal{R}[a,b]$ and $|f(x)| \le M$ for all $x \in [a,b]$, show that $|\int_a^b f| \le M(b-a)$.
9. If $f \in \mathcal{R}[a,b]$ and if $(\dot{\mathcal{P}}_n)$ is any sequence of tagged partitions of $[a,b]$ such that $\|\dot{\mathcal{P}}_n\| \to 0$, prove that $\int_a^b f = \lim_n S(f;\dot{\mathcal{P}}_n)$.
10. Let $g(x) := 0$ if $x \in [0,1]$ is rational and $g(x) := 1/x$ if $x \in [0,1]$ is irrational. Explain why $g \notin \mathcal{R}[0,1]$. However, show that there exists a sequence $(\dot{\mathcal{P}}_n)$ of tagged partitions of $[a,b]$ such that $\|\dot{\mathcal{P}}_n\| \to 0$ and $\lim_n S(g;\dot{\mathcal{P}}_n)$ exists.
11. Suppose that f is bounded on $[a,b]$ and that there exists two sequences of tagged partitions of $[a,b]$ such that $\|\dot{\mathcal{P}}_n\| \to 0$ and $\|\dot{\mathcal{Q}}_n\| \to 0$, but such that $\lim_n S(f;\dot{\mathcal{P}}_n) \ne \lim_n S(f;\dot{\mathcal{Q}}_n)$. Show that f is not in $\mathcal{R}[a,b]$.

12. Consider the Dirichlet function, introduced in Example 5.1.5(g), defined by $f(x) := 1$ for $x \in [0,1]$ rational and $f(x) := 0$ for $x \in [0,1]$ irrational. Use the preceding exercise to show that f is *not* Riemann integrable on $[0,1]$.

13. Suppose that $f : [a,b] \to \mathbb{R}$ and that $f(x) = 0$ except for a finite number of points $c_1, \cdots, c_n$ in $[a,b]$. Prove that $f \in \mathcal{R}[a,b]$ and that $\int_a^b f = 0$.

14. If $g \in \mathcal{R}[a,b]$ and if $f(x) = g(x)$ except for a finite number of points in $[a,b]$, prove that $f \in \mathcal{R}[a,b]$ and that $\int_a^b f = \int_a^b g$.

HS 15. Suppose that $c \le d$ are points in $[a,b]$. If $\varphi : [a,b] \to \mathbb{R}$ satisfies $\varphi(x) = \alpha > 0$ for $x \in [c,d]$ and $\varphi(x) = 0$ elsewhere in $[a,b]$, prove that $\varphi \in \mathcal{R}[a,b]$ and that $\int_a^b \varphi = \alpha(d-c)$. [*Hint:* Given $\varepsilon > 0$ let $\delta_\varepsilon := \varepsilon/4\alpha$ and show that if $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$ then we have $\alpha(d-c-2\delta_\varepsilon) \le S(\varphi; \dot{\mathcal{P}}) \le \alpha(d-c+2\delta_\varepsilon)$.]

16. Let $0 \le a < b$, let $Q(x) := x^2$ for $x \in [a,b]$ and let $\mathcal{P} := \{[x_{i-1}, x_i]\}_{i=1}^n$ be a partition of $[a,b]$. For each i, let q_i be the positive *square root* of

$$\tfrac{1}{3}(x_i^2 + x_i x_{i-1} + x_{i-1}^2).$$

(a) Show that q_i satisfies $0 \le x_{i-1} \le q_i \le x_i$.
(b) Show that $Q(q_i)(x_i - x_{i-1}) = \frac{1}{3}(x_i^3 - x_{i-1}^3)$.
(c) If $\dot{\mathcal{Q}}$ is the tagged partition with the same subintervals as $\mathcal{P}$ and the tags q_i, show that $S(Q; \dot{\mathcal{Q}}) = \frac{1}{3}(b^3 - a^3)$.
(d) Use the argument in Example 7.1.3(c) to show that $Q \in \mathcal{R}[a,b]$ and

$$\int_a^b Q = \int_a^b x^2\,dx = \tfrac{1}{3}(b^3 - a^3).$$

17. Let $0 \le a < b$ and $m \in \mathbb{N}$, let $M(x) := x^m$ for $x \in [a,b]$ and let $\mathcal{P} := \{[x_{i-1}, x_i]\}_{i=1}^n$ be a partition of $[a,b]$. For each i, let q_i be the positive mth *root* of

$$\frac{1}{m+1}\left(x_i^m + x_i^{m-1}x_{i-1} + \cdots + x_i x_{i-1}^{m-1} + x_{i-1}^m\right).$$

(a) Show that q_i satisfies $0 \le x_{i-1} \le q_i \le x_i$.
(b) Show that $M(q_i)(x_i - x_{i-1}) = \frac{1}{m+1}\left(x_i^{m+1} - x_{i-1}^{m+1}\right)$.
(c) If $\dot{\mathcal{Q}}$ is the tagged partition with the same subintervals as $\mathcal{P}$ and the tags q_i, show that $S(M; \dot{\mathcal{Q}}) = \frac{1}{m+1}(b^{m+1} - a^{m+1})$.
(d) Use the argument in Example 7.1.3(c) to show that $M \in \mathcal{R}[a,b]$ and

$$\int_a^b M = \int_a^b x^m\,dx = \frac{1}{m+1}(b^{m+1} - a^{m+1}).$$

18. If $f \in \mathcal{R}[a,b]$ and $c \in \mathbb{R}$, we define g on $[a+c, b+c]$ by $g(y) := f(y-c)$. Prove that $g \in \mathcal{R}[a+c, b+c]$ and that $\int_{a+c}^{b+c} g = \int_a^b f$. The function g is called the c-**translate** of f.

Section 7.2 Riemann Integrable Functions

We begin with a proof of the important Cauchy Criterion. We will then prove the Squeeze Theorem, which will be used to establish the Riemann integrability of several classes of functions (step functions, continuous functions, and monotone functions). Finally we will establish the Additivity Theorem.

We have already noted that direct use of the definition requires that we know the value of the integral. The Cauchy Criterion removes this need, but at the cost of considering two Riemann sums, instead of just one.

δ_ε].

7.2.1 Cauchy Criterion *A function $f : [a,b] \to \mathbb{R}$ belongs to $\mathcal{R}[a,b]$ if and only if for every $\varepsilon > 0$ there exists $\eta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ are any tagged partitions of $[a,b]$ with $\|\dot{\mathcal{P}}\| < \eta_\varepsilon$ and $\|\dot{\mathcal{Q}}\| < \eta_\varepsilon$, then*

$$|S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}})| < \varepsilon.$$

Proof. ($\Rightarrow$) If $f \in \mathcal{R}[a,b]$ with integral L, let $\eta_\varepsilon := \delta_\varepsilon/2 > 0$ be such that if $\dot{\mathcal{P}}, \dot{\mathcal{Q}}$ are tagged partitions such that $\|\dot{\mathcal{P}}\| < \eta_\varepsilon$ and $\|\dot{\mathcal{Q}}\| < \eta_\varepsilon$, then

$$|S(f;\dot{\mathcal{P}}) - L| < \varepsilon/2 \quad \text{and} \quad |S(f;\dot{\mathcal{Q}}) - L| < \varepsilon/2.$$

Therefore we have

$$\begin{aligned} |S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}})| &\le |S(f;\dot{\mathcal{P}}) - L + L - S(f;\dot{\mathcal{Q}})| \\ &\le |S(f;\dot{\mathcal{P}}) - L| + |L - S(f;\dot{\mathcal{Q}})| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

($\Leftarrow$) For each $n \in \mathbb{N}$, let $\delta_n > 0$ be such that if $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ are tagged partitions with norms $< \delta_n$, then

$$|S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}})| < 1/n.$$

Evidently we may assume that $\delta_n \ge \delta_{n+1}$ for $n \in \mathbb{N}$; otherwise, we replace δ_n by $\delta_n' := \min\{\delta_1, \cdots, \delta_n\}$.

For each $n \in \mathbb{N}$, let $\dot{\mathcal{P}}_n$ be a tagged partition with $\|\dot{\mathcal{P}}_n\| < \delta_n$. Clearly, if $m > n$ then both $\dot{\mathcal{P}}_m$ and $\dot{\mathcal{P}}_n$ have norms $< \delta_n$, so that

$$(1) \qquad |S(f;\dot{\mathcal{P}}_n) - S(f;\dot{\mathcal{P}}_m)| < 1/n \qquad \text{for} \quad m > n.$$

Consequently, the sequence $(S(f;\dot{\mathcal{P}}_m))_{m=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$. Therefore (by Theorem 3.5.5) this sequence converges in $\mathbb{R}$ and we let $A := \lim_m S(f;\dot{\mathcal{P}}_m)$.

Passing to the limit in (1) as $m \to \infty$, we have

$$|S(f;\dot{\mathcal{P}}_n) - A| \le 1/n \qquad \text{for all} \quad n \in \mathbb{N}.$$

To see that A is the Riemann integral of f, given $\varepsilon > 0$, let $K \in \mathbb{N}$ satisfy $K > 2/\varepsilon$. If $\dot{\mathcal{Q}}$ is any tagged partition with $\|\dot{\mathcal{Q}}\| < \delta_K$, then

$$\begin{aligned} |S(f;\dot{\mathcal{Q}}) - A| &\le |S(f;\dot{\mathcal{Q}}) - S(f;\dot{\mathcal{P}}_K)| + |S(f;\dot{\mathcal{P}}_K) - A| \\ &\le 1/K + 1/K < \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, then $f \in \mathcal{R}[a,b]$ with integral A. Q.E.D.

We will now give two examples of the use of the Cauchy Criterion.

7.2.2 Examples **(a)** Let $g : [0,3] \to \mathbb{R}$ be the function considered in Example 7.1.3(b). In that example we saw that if $\dot{\mathcal{P}}$ is a tagged partition of $[0,3]$ with norm $\|\dot{\mathcal{P}}\| < \delta$, then

$$8 - 5\delta \le S(g;\dot{\mathcal{P}}) \le 8 + 5\delta.$$

Hence if $\dot{\mathcal{Q}}$ is another tagged partition with $\|\dot{\mathcal{Q}}\| < \delta$, then

$$8 - 5\delta \le S(g;\dot{\mathcal{Q}}) \le 8 + 5\delta.$$

If we subtract these two inequalities, we obtain

$$|S(g;\dot{\mathcal{P}}) - S(g;\dot{\mathcal{Q}})| \le 10\delta.$$

In order to make this final term $< \varepsilon$, we are led to employ the Cauchy Criterion with $\eta_\varepsilon := \varepsilon/20$. (We leave the details to the reader.)

(b) The Cauchy Criterion can be used to show that a function $f : [a,b] \to \mathbb{R}$ is *not* Riemann integrable. To do this we need to show that: *There exists $\varepsilon_0 > 0$ such that for any $\eta > 0$ there exists tagged partitions $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ with $\|\dot{\mathcal{P}}\| < \eta$ and $\|\dot{\mathcal{Q}}\| < \eta$ such that* $|S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}})| \geq \varepsilon_0$.

We will apply these remarks to the Dirichlet function, considered in 5.1.6(g), defined by $f(x) := 1$ if $x \in [0,1]$ is rational and $f(x) := 0$ if $x \in [0,1]$ is irrational.

Here we take $\varepsilon_0 := \frac{1}{2}$. If $\dot{\mathcal{P}}$ is any partition all of whose tags are rational numbers then $S(f;\dot{\mathcal{P}}) = 1$, while if $\dot{\mathcal{Q}}$ is any tagged partition all of whose tags are irrational numbers then $S(f;\dot{\mathcal{Q}}) = 0$. Since we are able to take such tagged partitions with arbitrarily small norms, we conclude that the Dirichlet function is *not* Riemann integrable. □

The Squeeze Theorem

The next result will be used to establish the Riemann integrability of some important classes of functions.

7.2.3 Squeeze Theorem *Let $f : [a,b] \to \mathbb{R}$. Then $f \in \mathcal{R}[a,b]$ if and only if for every $\varepsilon > 0$ there exist functions α_ε and ω_ε in $\mathcal{R}[a,b]$ with*

$$(2) \qquad \alpha_\varepsilon(x) \leq f(x) \leq \omega_\varepsilon(x) \quad \text{for all} \quad x \in [a,b],$$

and such that

$$(3) \qquad \int_a^b (\omega_\varepsilon - \alpha_\varepsilon) < \varepsilon.$$

Proof. ($\Rightarrow$) Take $\alpha_\varepsilon = \omega_\varepsilon = f$ for all $\varepsilon > 0$.

($\Leftarrow$) Let $\varepsilon > 0$. Since α_ε and ω_ε belong to $\mathcal{R}[a,b]$, there exists $\delta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ is any tagged partition with $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$ then

$$\left|S(\alpha_\varepsilon;\dot{\mathcal{P}}) - \int_a^b \alpha_\varepsilon\right| < \varepsilon \quad \text{and} \quad \left|S(\omega_\varepsilon;\dot{\mathcal{P}}) - \int_a^b \omega_\varepsilon\right| < \varepsilon.$$

It follows from these inequalities that

$$\int_a^b \alpha_\varepsilon - \varepsilon < S(\alpha_\varepsilon;\dot{\mathcal{P}}) \quad \text{and} \quad S(\omega_\varepsilon;\dot{\mathcal{P}}) < \int_a^b \omega_\varepsilon + \varepsilon.$$

In view of inequality (2), we have $S(\alpha_\varepsilon;\dot{\mathcal{P}}) \leq S(f;\dot{\mathcal{P}}) \leq S(\omega_\varepsilon;\dot{\mathcal{P}})$, whence

$$\int_a^b \alpha_\varepsilon - \varepsilon < S(f;\dot{\mathcal{P}}) < \int_a^b \omega_\varepsilon + \varepsilon.$$

If $\dot{\mathcal{Q}}$ is another tagged partition with $\|\dot{\mathcal{Q}}\| < \delta_\varepsilon$, then we also have

$$\int_a^b \alpha_\varepsilon - \varepsilon < S(f;\dot{\mathcal{Q}}) < \int_a^b \omega_\varepsilon + \varepsilon.$$

If we subtract these two inequalities and use (3), we conclude that

$$\begin{aligned} |S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}})| &< \int_a^b \omega_\varepsilon - \int_a^b \alpha_\varepsilon + 2\varepsilon \\ &= \int_a^b (\omega_\varepsilon - \alpha_\varepsilon) + 2\varepsilon < 3\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the Cauchy Criterion implies that $f \in \mathcal{R}[a,b]$. Q.E.D.

Classes of Riemann Integrable Functions

The Squeeze Theorem is often used in connection with the class of step functions. It will be recalled from Definition 5.4.9 that a function $\varphi : [a,b] \to \mathbb{R}$ is a **step function** if it has only a finite number of distinct values, each value being assumed on one or more subintervals of $[a,b]$. For illustrations of step functions, see Figures 5.4.3 or 7.1.4.

7.2.4 Lemma *If J is a subinterval of $[a,b]$ having endpoints $c < d$ and if $\varphi_J(x) := 1$ for $x \in J$ and $\varphi_J(x) := 0$ elsewhere in $[a,b]$, then $\varphi_J \in \mathcal{R}[a,b]$ and $\int_a^b \varphi_J = d - c$.*

Proof. If $J = [c,d]$ with $c \leq d$, this is Exercise 7.1.15 and we can choose $\delta_\varepsilon := \varepsilon/4$. A similar proof can be given for the three other subintervals having these endpoints. Alternatively, we observe that we can write

$$\varphi_{[c,d)} = \varphi_{[c,d]} - \varphi_{[d,d]}, \quad \varphi_{(c,d]} = \varphi_{[c,d]} - \varphi_{[c,c]} \quad \text{and} \quad \varphi_{(c,d)} = \varphi_{[c,d)} - \varphi_{[c,c]}.$$

Since $\int_a^b \varphi_{[c,c]} = 0$, all four of these functions have integral equal to $d - c$. Q.E.D.

It is an important fact that any step function is Riemann integrable.

7.2.5 Theorem *If $\varphi : [a,b] \to \mathbb{R}$ is a step function, then $\varphi \in \mathcal{R}[a,b]$.*

Proof. Step functions of the type appearing in 7.2.4 are called "elementary step functions". In Exercise 5 it is shown that an arbitrary step function φ can be expressed as a linear combination of such elementary step functions:

$$(4) \qquad \varphi = \sum_{j=1}^m k_j \varphi_{J_j},$$

where J_j has endpoints $c_j < d_j$. The lemma and Theorem 7.1.4(a,b) imply that $\varphi \in \mathcal{R}[a,b]$ and that

$$(5) \qquad \int_a^b \varphi = \sum_{j=1}^m k_j(d_j - c_j).$$

Q.E.D.

We will now use the Squeeze Theorem to show that an arbitrary continuous function is Riemann integrable.

7.2.6 Theorem *If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then $f \in \mathcal{R}[a,b]$.*

Proof. It follows from Theorem 5.4.3 that f is uniformly continuous on $[a,b]$. Therefore, given $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $u, v \in [a,b]$ and $|u - v| < \delta_\varepsilon$, then we have $|f(u) - f(v)| < \varepsilon/(b-a)$.

Let $\mathcal{P} = \{I_i\}_{i=1}^n$ be a partition such that $\|\mathcal{P}\| < \delta_\varepsilon$, let $u_i \in I_i$ be a point where f attains its minimum value on I_i, and let $v_i \in I_i$ be a point where f attains its maximum value on I_i.

Let α_ε be the step function defined by $\alpha_\varepsilon(x) := f(u_i)$ for $x \in [x_{i-1}, x_i)$ $(i = 1, \cdots, n-1)$ and $\alpha_\varepsilon(x) := f(u_n)$ for $x \in [x_{n-1}, x_n]$. Let ω_ε be defined similarly using the points v_i instead of the u_i. Then one has

$$\alpha_\varepsilon(x) \leq f(x) \leq \omega_\varepsilon(x) \quad \text{for all} \quad x \in [a,b].$$

Moreover, it is clear that

$$0 \le \int_a^b (\omega_\varepsilon - \alpha_\varepsilon) = \sum_{i=1}^{n} (f(v_i) - f(u_i))(x_i - x_{i-1})$$
$$< \sum_{i=1}^{n} \left(\frac{\varepsilon}{b-a}\right)(x_i - x_{i-1}) = \varepsilon.$$

Therefore it follows from the Squeeze Theorem that $f \in \mathcal{R}[a, b]$. Q.E.D.

Monotone functions are not necessarily continuous at every point, but they are also Riemann integrable.

7.2.7 Theorem *If $f : [a, b] \to \mathbb{R}$ is monotone on $[a, b]$, then $f \in \mathcal{R}[a, b]$.*

Proof. Suppose that f is increasing on the interval $[a, b]$, $a < b$. If $\varepsilon > 0$ is given, we let $q \in \mathbb{N}$ be such that

$$h := \frac{f(b) - f(a)}{q} < \frac{\varepsilon}{b-a}.$$

Let $y_k := f(a) + kh$ for $k = 0, 1, \cdots, q$ and consider sets $A_k := f^{-1}([y_{k-1}, y_k))$ for $k = 1, \cdots, q-1$ and $A_q := f^{-1}([y_{q-1}, y_q])$. The sets $\{A_k\}$ are pairwise disjoint and have union $[a, b]$. The Characterization Theorem 2.5.1 implies that each A_k is either (i) empty, (ii) contains a single point, or (iii) is a nondegenerate interval (not necessarily closed) in $[a, b]$. We discard the sets for which (i) holds and relabel the remaining ones. If we adjoin the endpoints to the remaining intervals $\{A_k\}$, we obtain closed intervals $\{I_k\}$. It is an exercise to show that the relabeled intervals $\{A_k\}_{k=1}^{q}$ are pairwise disjoint, satisfy $[a, b] = \bigcup_{k=1}^{q} A_k$ and that $f(x) \in [y_{k-1}, y_k]$ for $x \in A_k$.

We now define step functions α_ε and ω_ε on $[a, b]$ by setting

$$\alpha_\varepsilon(x) := y_{k-1} \quad \text{and} \quad \omega_\varepsilon(x) := y_k \quad \text{for} \quad x \in A_k.$$

It is clear that $\alpha_\varepsilon(x) \le f(x) \le \omega_\varepsilon(x)$ for all $x \in [a, b]$ and that

$$\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) = \sum_{k=1}^{q} (y_k - y_{k-1})(x_k - x_{k-1})$$
$$= \sum_{k=1}^{q} h \cdot (x_k - x_{k-1}) = h \cdot (b - a) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the Squeeze Theorem implies that $f \in \mathcal{R}[a, b]$. Q.E.D.

The Additivity Theorem

We now return to arbitrary Riemann integrable functions. Our next result shows that the integral is an "additive function" of the interval over which the function is integrated. This property is no surprise, but its proof is a bit delicate and may be omitted on a first reading.

7.2.8 Additivity Theorem *Let $f : [a, b] \to \mathbb{R}$ and let $c \in (a, b)$. Then $f \in \mathcal{R}[a, b]$ if and only if its restrictions to $[a, c]$ and $[c, b]$ are both Riemann integrable. In this case*

$$(6) \qquad \int_a^b f = \int_a^c f + \int_c^b f.$$

Proof. ($\Leftarrow$) Suppose that the restriction f_1 of f to $[a, c]$, and the restriction f_2 of f to $[c, b]$ are Riemann integrable to L_1 and L_2, respectively. Then, given $\varepsilon > 0$ there exists $\delta' > 0$ such that if $\dot{\mathcal{P}}_1$ is a tagged partition of $[a, c]$ with $\|\dot{\mathcal{P}}_1\| < \delta'$, then $|S(f_1; \dot{\mathcal{P}}_1) - L_1| < \varepsilon/3$. Also there exists $\delta'' > 0$ such that if $\dot{\mathcal{P}}_2$ is a tagged partition of $[c, b]$ with $\|\dot{\mathcal{P}}_2\| < \delta''$ then $|S(f_2; \dot{\mathcal{P}}_2) - L_2| < \varepsilon/3$. If M is a bound for $|f|$, we define $\delta_\varepsilon := \min\{\delta', \delta'', \varepsilon/6M\}$ and let $\dot{\mathcal{P}}$ be a tagged partition of $[a, b]$ with $\|\dot{\mathcal{Q}}\| < \delta$. We will prove that

$$(7) \qquad \left|S(f; \dot{\mathcal{Q}}) - (L_1 + L_2)\right| < \varepsilon.$$

(i) If c is a partition point of $\dot{\mathcal{Q}}$, we split $\dot{\mathcal{Q}}$ into a partition $\dot{\mathcal{Q}}_1$ of $[a, c]$ and a partition $\dot{\mathcal{Q}}_2$ of $[c, b]$. Since $S(f; \dot{\mathcal{Q}}) = S(f; \dot{\mathcal{Q}}_1) + S(f; \dot{\mathcal{Q}}_2)$, and since $\dot{\mathcal{Q}}_1$ has norm $< \delta'$ and $\dot{\mathcal{Q}}_2$ has norm $< \delta''$, the inequality (7) is clear.

(ii) If c is not a partition point in $\dot{\mathcal{Q}} = \{(I_k, t_k)\}_{k=1}^{m}$, there exists $k \le m$ such that $c \in (x_{k-1}, x_k)$. We let $\dot{\mathcal{Q}}_1$ be the tagged partition of $[a, c]$ defined by

$$\dot{\mathcal{Q}}_1 := \{(I_1, t_1), \cdots, (I_{k-1}, t_{k-1}), ([x_{k-1}, c], c)\},$$

and $\dot{\mathcal{Q}}_2$ be the tagged partition of $[c, b]$ defined by

$$\dot{\mathcal{Q}}_2 := \{([c, x_k], c), (I_{k+1}, t_{k+1}), \cdots, (I_m, t_m)\}.$$

A straightforward calculation shows that

$$S(f; \dot{\mathcal{Q}}) - S(f; \dot{\mathcal{Q}}_1) - S(f; \dot{\mathcal{Q}}_2) = f(t_k)(x_k - x_{k-1}) - f(c)(x_k - x_{k-1})$$
$$= (f(t_k) - f(c)) \cdot (x_k - x_{k-1}),$$

whence it follows that

$$|S(f; \dot{\mathcal{Q}}) - S(f; \dot{\mathcal{Q}}_1) - S(f; \dot{\mathcal{Q}}_2)| \le 2M(x_k - x_{k-1}) < \varepsilon/3.$$

But since $\|\dot{\mathcal{Q}}_1\| < \delta \le \delta'$ and $\|\dot{\mathcal{Q}}_2\| < \delta \le \delta''$, it follows that

$$|S(f; \dot{\mathcal{Q}}_1) - L_1| < \varepsilon/3 \quad \text{and} \quad |S(f; \dot{\mathcal{Q}}_2) - L_2| < \varepsilon/3,$$

from which we obtain (7). Since $\varepsilon > 0$ is arbitrary, we infer that $f \in \mathcal{R}[a, b]$ and that (6) holds.

($\Rightarrow$) We suppose that $f \in \mathcal{R}[a, b]$ and, given $\varepsilon > 0$, we let $\eta_\varepsilon > 0$ satisfy the Cauchy Criterion 7.2.1. Let f_1 be the restriction of f to $[a, c]$ and let $\dot{\mathcal{P}}_1$, $\dot{\mathcal{Q}}_1$ be tagged partitions of $[a, c]$ with $\|\dot{\mathcal{P}}_1\| < \eta_\varepsilon$ and $\|\dot{\mathcal{Q}}_1\| < \eta_\varepsilon$. By adding additional partition points and tags from $[c, b]$, we can extend $\dot{\mathcal{P}}_1$ and $\dot{\mathcal{Q}}_1$ to tagged partitions $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ of $[a, b]$ that satisfy $\|\dot{\mathcal{P}}\| < \eta_\varepsilon$ and $\|\dot{\mathcal{Q}}\| < \eta_\varepsilon$. If we use the *same* additional points and tags in $[c, b]$ for both $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$, then

$$S(f_1; \dot{\mathcal{P}}_1) - S(f_1; \dot{\mathcal{Q}}_1) = S(f; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{Q}}).$$

Since both $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ have norm $< \eta_\varepsilon$, then $|S(f_1; \dot{\mathcal{P}}_1) - S(f_1; \dot{\mathcal{Q}}_1)| < \varepsilon$. Therefore the Cauchy Condition shows that the restriction f_1 of f to $[a, c]$ is in $\mathcal{R}[a, c]$. In the same way, we see that the restriction f_2 of f to $[c, b]$ is in $\mathcal{R}[c, d]$.

The equality (6) now follows from the first part of the theorem. Q.E.D.

7.2.9 Corollary *If $f \in \mathcal{R}[a, b]$, and if $[c, d] \subseteq [a, b]$, then the restriction of f to $[c, d]$ is in $\mathcal{R}[c, d]$.*

Proof. Since $f \in \mathcal{R}[a, b]$ and $c \in [a, b]$, it follows from the theorem that its restriction to $[c, b]$ is in $\mathcal{R}[c, b]$. But if $d \in [c, b]$, then another application of the theorem shows that the restriction of f to $[c, d]$ is in $\mathcal{R}[c, d]$. Q.E.D.

7.2.10 Corollary *If $f \in \mathcal{R}[a, b]$ and if $a = c_0 < c_1 < \cdots < c_m = b$, then the restrictions of f to each of the subintervals $[c_{i-1}, c_i]$ are Riemann integrable and*

$$\int_a^b f = \sum_{i=1}^{m} \int_{c_{i-1}}^{c_i} f.$$

Until now, we have considered the Riemann integral over an interval $[a, b]$ where $a < b$. It is convenient to have the integral defined more generally.

7.2.11 Definition If $f \in \mathcal{R}[a, b]$ and if $\alpha, \beta \in [a, b]$ with $\alpha < \beta$, we define

$$\int_\beta^\alpha f := -\int_\alpha^\beta f \qquad \text{and} \qquad \int_\alpha^\alpha f := 0.$$

7.2.12 Theorem *If $f \in \mathcal{R}[a, b]$ and if α, β, γ are any numbers in $[a, b]$, then*

$$(8) \qquad \int_\alpha^\beta f = \int_\alpha^\gamma f + \int_\gamma^\beta f,$$

in the sense that the existence of any two of these integrals implies the existence of the third integral and the equality (8).

Proof. If any two of the numbers α, β, γ are equal, then (8) holds. Thus we may suppose that all three of these numbers are distinct.

For the sake of symmetry, we introduce the expression

$$L(\alpha, \beta, \gamma) := \int_\alpha^\beta f + \int_\beta^\gamma f + \int_\gamma^\alpha f.$$

It is clear that (8) holds if and only if $L(\alpha, \beta, \gamma) = 0$. Therefore, to establish the assertion, we need to show that $L = 0$ for all six permutations of the arguments α, β and γ.

We note that the Additivity Theorem 7.2.8 implies that $L(\alpha, \beta, \gamma) = 0$ when $\alpha < \gamma < \beta$. But it is easily seen that both $L(\beta, \gamma, \alpha)$ and $L(\gamma, \alpha, \beta)$ equal $L(\alpha, \beta, \gamma)$. Moreover, the numbers

$$L(\beta, \alpha, \gamma), \qquad L(\alpha, \gamma, \beta), \qquad \text{and} \qquad L(\gamma, \beta, \alpha)$$

are all equal to $-L(\alpha, \beta, \gamma)$. Therefore, L vanishes for all possible configurations of these three points. Q.E.D.

Exercises for Section 7.2

1. Let $f : [a, b] \to \mathbb{R}$. Show that $f \notin \mathcal{R}[a, b]$ if and only if there exists $\varepsilon_0 > 0$ such that for every $n \in \mathbb{N}$ there exist tagged partitions $\dot{\mathcal{P}}_n$ and $\dot{\mathcal{Q}}_n$ with $\|\dot{\mathcal{P}}_n\| < 1/n$ and $\|\dot{\mathcal{Q}}_n\| < 1/n$ such that $|S(f; \dot{\mathcal{P}}_n) - S(f; \dot{\mathcal{Q}}_n)| \ge \varepsilon_0$.
2. Consider the function h defined by $h(x) := x + 1$ for $x \in [0, 1]$ rational, and $h(x) := 0$ for $x \in [0, 1]$ irrational. Show that h is not Riemann integrable.

3. Let $H(x) := k$ for $x = 1/k$ ($k \in \mathbb{N}$) and $H(x) := 0$ elsewhere on $[0, 1]$. Use Exercise 1, or the argument in 7.2.2(b), to show that H is not Riemann integrable.
4. If $\alpha(x) := -x$ and $\omega(x) := x$ and if $\alpha(x) \le f(x) \le \omega(x)$ for all $x \in [0, 1]$, does it follow from the Squeeze Theorem 7.2.3 that $f \in \mathcal{R}[0, 1]$?
5. If J is any subinterval of $[a, b]$ and if $\varphi_J(x) := 1$ for $x \in J$ and $\varphi_J(x) := 0$ elsewhere on $[a, b]$, we say that φ_J is an *elementary step function* on $[a, b]$. Show that every step function is a linear combination of elementary step functions.
6. If $\psi : [a, b] \to \mathbb{R}$ takes on only a finite number of distinct values, is ψ a step function?
7. If $S(f; \dot{\mathcal{P}})$ is any Riemann sum of $f : [a, b] \to \mathbb{R}$, show that there exists a step function $\varphi : [a, b] \to \mathbb{R}$ such that $\int_a^b \varphi = S(f; \dot{\mathcal{P}})$.
8. Suppose that f is continuous on $[a, b]$, that $f(x) \ge 0$ for all $x \in [a, b]$ and that $\int_a^b f = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.
9. Show that the continuity hypothesis in the preceding exercise cannot be dropped.
10. If f and g are continuous on $[a, b]$ and if $\int_a^b f = \int_a^b g$, prove that there exists $c \in [a, b]$ such that $f(c) = g(c)$.
11. If f is bounded by M on $[a, b]$ and if the restriction of f to every interval $[c, b]$ where $c \in (a, b)$ is Riemann integrable, show that $f \in \mathcal{R}[a, b]$ and that $\int_c^b f \to \int_a^b f$ as $c \to a+$. [*Hint:* Let $\alpha_c(x) := -M$ and $\omega_c(x) := M$ for $x \in [a, c)$ and $\alpha_c(x) := \omega_c(x) := f(x)$ for $x \in [c, b]$. Apply the Squeeze Theorem 7.2.3 for c sufficiently near a.]
12. Show that $g(x) := \sin(1/x)$ for $x \in (0, 1]$ and $g(0) := 0$ belongs to $\mathcal{R}[0, 1]$.
13. Give an example of a function $f : [a, b] \to \mathbb{R}$ that is in $\mathcal{R}[c, b]$ for every $c \in (a, b)$ but which is not in $\mathcal{R}[a, b]$. see 11 - $f(x) = 1/x$
14. Suppose that $f : [a, b] \to \mathbb{R}$, that $a = c_0 < c_1 < \cdots < c_m = b$ and that the restrictions of f to $[c_{i-1}, c_i]$ belong to $\mathcal{R}[c_{i-1}, c_i]$ for $i = 1, \cdots, m$. Prove that $f \in \mathcal{R}[a, b]$ and that the formula in Corollary 7.2.10 holds.
15. If f is bounded and there is a finite set E such that f is continuous at every point of $[a, b] \backslash E$, show that $f \in \mathcal{R}[a, b]$.
16. If f is continuous on $[a, b]$, $a < b$, show that there exists $c \in [a, b]$ such that we have $\int_a^b f = f(c)(b - a)$. This result is sometimes called the *Mean Value Theorem for Integrals*.
17. HI If f and g are continuous on $[a, b]$ and $g(x) > 0$ for all $x \in [a, b]$, show that there exists $c \in [a, b]$ such that $\int_a^b fg = f(c) \int_a^b g$. Show that this conclusion fails if we do not have $g(x) > 0$. (Note that this result is an extension of the preceding exercise.)
18. Let f be continuous on $[a, b]$, let $f(x) \ge 0$ for $x \in [a, b]$, and let $M_n := \left(\int_a^b f^n\right)^{1/n}$. Show that $\lim(M_n) = \sup\{f(x) : x \in [a, b]\}$.
19. Suppose that $a > 0$ and that $f \in \mathcal{R}[-a, a]$.
    (a) If f is *even* (that is, if $f(-x) = f(x)$ for all $x \in [0, a]$), show that $\int_{-a}^a f = 2\int_0^a f$.
    (b) If f is *odd* (that is, if $f(-x) = -f(x)$ for all $x \in [0, a]$), show that $\int_{-a}^a f = 0$.
20. Suppose that $f : [a, b] \to \mathbb{R}$ and that $n \in \mathbb{N}$. Let $\mathcal{P}_n$ be the partition of $[a, b]$ into n subintervals having equal lengths, so that $x_i := a + i(b - a)/n$ for $i = 0, 1, \cdots, n$. Let $L_n(f) := S(f; \dot{\mathcal{P}}_{n,l})$ and $R_n(f) := S(f; \dot{\mathcal{P}}_{n,r})$, where $\dot{\mathcal{P}}_{n,l}$ has its tags at the left endpoints, and $\dot{\mathcal{P}}_{n,r}$ has its tags at the right endpoints of the subintervals $[x_{i-1}, x_i]$.
    (a) If f is increasing on $[a, b]$, show that $L_n(f) \le R_n(f)$ and that
    $$0 \le R_n(f) - L_n(f) = \big(f(b) - f(a)\big) \cdot \frac{(b - a)}{n}.$$
    (b) Show that $f(a)(b - a) \le L_n(f) \le \int_a^b f \le R_n(f) \le f(b)(b - a)$.
    (c) If f is decreasing on $[a, b]$, obtain an inequality similar to that in (a).

(d) If $f \in \mathcal{R}[a,b]$ is not monotone, show that $\int_a^b f$ is not necessarily between $L_n(f)$ and $R_n(f)$.

21. If f is continuous on $[-a,a]$, show that $\int_{-a}^{a} f(x^2)\,dx = 2\int_0^a f(x^2)\,dx$.

22. If f is continuous on $[-1,1]$, show that $\int_0^{\pi/2} f(\cos x)\,dx = \int_0^{\pi/2} f(\sin x)\,dx = \frac{1}{2}\int_0^{\pi} f(\sin x)\,dx$. [*Hint:* Examine certain Riemann sums.]

Section 7.3 The Fundamental Theorem

We will now explore the connection between the notions of the derivative and the integral. In fact, there are *two* theorems relating to this problem: one has to do with integrating a derivative, and the other with differentiating an integral. These theorems, taken together, are called the Fundamental Theorem of Calculus. Roughly stated, they imply that the operations of differentiation and integration are inverse to each other. However, there are some subleties that should not be overlooked.

The Fundamental Theorem (First Form)

The First Form of the Fundamental Theorem provides a theoretical basis for the method of calculating an integral that the reader learned in calculus. It asserts that if a function f is the derivative of a function F, *and if f belongs to $\mathcal{R}[a,b]$*, then the integral $\int_a^b f$ can be calculated by means of the evaluation $F\big|_a^b := F(b) - F(a)$. A function F such that $F'(x) = f(x)$ for all $x \in [a,b]$ is called an **antiderivative** or a **primitive of** f on $[a,b]$. Thus, when f has an antiderivative, it is a very simple matter to calculate its integral.

In practice, it is convenient to allow some exceptional points c where $F'(c)$ does not exist in $\mathbb{R}$, or where it does not equal $f(c)$. It turns out that we can permit a *finite* number of such exceptional points.

7.3.1 Fundamental Theorem of Calculus (First Form) *Suppose there is a **finite** set E in $[a,b]$ and functions $f, F : [a,b] \to \mathbb{R}$ such that:*

(a) *F is continuous on $[a,b]$,*

(b) *$F'(x) = f(x)$ for all $x \in [a,b]\backslash E$,*

(c) *f belongs to $\mathcal{R}[a,b]$.*

Then we have

$$\int_a^b f = F(b) - F(a). \tag{1}$$

Proof. We will prove the theorem in the case where $E := \{a,b\}$. The general case can be obtained by breaking the interval into the union of a finite number of intervals (see Exercise 1).

Let $\varepsilon > 0$ be given. Since $f \in \mathcal{R}[a,b]$ by assumption (c), there exists $\delta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ is *any* tagged partition with $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$, then

$$\left|S(f;\dot{\mathcal{P}}) - \int_a^b f\right| < \varepsilon. \tag{2}$$

If the subintervals in $\dot{\mathcal{P}}$ are $[x_{i-1}, x_i]$, then the Mean Value Theorem 6.2.4 applied to F on $[x_{i-1}, x_i]$ implies that there exists $u_i \in (x_{i-1}, x_i)$ such that

$$F(x_i) - F(x_{i-1}) = F'(u_i)\cdot(x_i - x_{i-1}) \qquad \text{for} \quad i = 1, \cdots, n.$$

If we add these terms, note the telescoping of the sum, and use the fact that $F'(u_i) = f(u_i)$, we obtain

$$F(b) - F(a) = \sum_{i=1}^{n} \big(F(x_i) - F(x_{i-1})\big) = \sum_{i=1}^{n} f(u_i)(x_i - x_{i-1}).$$

Now let $\dot{\mathcal{P}}_u := \{([x_{i-1}, x_i], u_i)\}_{i=1}^n$, so the sum on the right equals $S(f;\dot{\mathcal{P}}_u)$. If we substitute $F(b) - F(a) = S(f;\dot{\mathcal{P}}_u)$ into (2), we conclude that

$$\left|F(b) - F(a) - \int_a^b f\right| < \varepsilon.$$

But, since $\varepsilon > 0$ is arbitrary, we infer that equation (1) holds. Q.E.D.

Remark If the function F is differentiable at every point of $[a,b]$, then (by Theorem 6.1.2) hypothesis (a) is automatically satisfied. If f is not defined for some point $c \in E$, we take $f(c) := 0$. Even if F is differentiable at every point of $[a,b]$, condition (c) is *not automatically satisfied*, since there exist functions F such that F' is not Riemann integrable. (See Example 7.3.2(e).)

7.3.2 Examples **(a)** If $F(x) := \frac{1}{2}x^2$ for all $x \in [a,b]$, then $F'(x) = x$ for all $x \in [a,b]$. Further, $f = F'$ is continuous so it is in $\mathcal{R}[a,b]$. Therefore the Fundamental Theorem (with $E = \emptyset$) implies that

$$\int_a^b x\,dx = F(b) - F(a) = \tfrac{1}{2}(b^2 - a^2).$$

(b) If $G(x) := \operatorname{Arctan} x$ for $x \in [a,b]$, then $G'(x) = 1/(x^2+1)$ for all $x \in [a,b]$; also G' is continuous, so it is in $\mathcal{R}[a,b]$. Therefore the Fundamental Theorem (with $E = \emptyset$) implies that

$$\int_a^b \frac{1}{x^2+1}\,dx = \operatorname{Arctan} b - \operatorname{Arctan} a.$$

(c) If $A(x) := |x|$ for $x \in [-10,10]$, then $A'(x) = -1$ if $x \in [-10,0)$ and $A'(x) = +1$ for $x \in (0,10]$. Recalling the definition of the signum function (in 4.1.10(b)), we have $A'(x) = \operatorname{sgn}(x)$ for all $x \in [-10,10]\backslash\{0\}$. Since the signum function is a step function, it belongs to $\mathcal{R}[-10,10]$. Therefore the Fundamental Theorem (with $E = \{0\}$) implies that

$$\int_{-10}^{10} \operatorname{sgn}(x)\,dx = A(10) - A(-10) = 10 - 10 = 0.$$

(d) If $H(x) := 2\sqrt{x}$ for $x \in [0,b]$, then H is continuous on $[0,b]$ and $H'(x) = 1/\sqrt{x}$ for $x \in (0,b]$. Since $h := H'$ is not bounded on $(0,b]$, it does not belong to $\mathcal{R}[0,b]$ no matter how we define $h(0)$. Therefore, the Fundamental Theorem 7.3.1 does not apply. (However, we will see in Example 10.1.10(a) that h is *generalized* Riemann integrable on $[0,b]$.)

(e) Let $K(x) := x^2\cos(1/x^2)$ for $x \in (0,1]$ and let $K(0) := 0$. It follows from the Product Rule 6.1.3(c) and the Chain Rule 6.1.6 that

$$K'(x) = 2x\cos(1/x^2) + (2/x)\sin(1/x^2) \qquad \text{for} \quad x \in (0,1].$$

Further, as in Example 6.1.7(e), it can be shown that $K'(0) = 0$. Thus K is continuous and differentiable at *every point* of $[0, 1]$. Since it can be seen that the function K' is not bounded on $[0, 1]$, it does not belong to $\mathcal{R}[0, 1]$ and the Fundamental Theorem 7.3.1 does not apply to K'. (However, we will see from Theorem 10.1.9 that K' is *generalized* Riemann integrable on $[0, 1]$.) □

The Fundamental Theorem (Second Form)

We now turn to the Fundamental Theorem (Second Form) in which we wish to differentiate an integral involving a variable upper limit.

7.3.3 Definition If $f \in \mathcal{R}[a, b]$, then the function defined by

$$F(z) := \int_a^z f \quad \text{for} \quad z \in [a, b], \tag{3}$$

is called the **indefinite integral** of f with **basepoint** a. (Sometimes a point other than a is used as a basepoint; see Exercise 6.)

We will first show that if $f \in \mathcal{R}[a, b]$, then its indefinite integral F satisfies a Lipschitz condition; hence F is continuous on $[a, b]$.

7.3.4 Theorem *The indefinite integral F defined by (3) is continuous on $[a, b]$. In fact, if $|f(x)| \le M$ for all $x \in [a, b]$, then $|F(z) - F(w)| \le M|z - w|$ for all $z, w \in [a, b]$.*

Proof. The Additivity Theorem 7.2.8 implies that if $z, w \in [a, b]$ and $w \le z$, then

$$F(z) = \int_a^z f = \int_a^w f + \int_w^z f = F(w) + \int_w^z f,$$

whence we have

$$F(z) - F(w) = \int_w^z f.$$

Now if $-M \le f(x) \le M$ for all $x \in [a, b]$, then Theorem 7.1.4(c) implies that

$$-M(z - w) \le \int_w^z f \le M(z - w),$$

whence it follows that

$$|F(z) - F(w)| \le \left|\int_w^z f\right| \le M|z - w|,$$

as asserted. Q.E.D.

We will now show that the indefinite integral F is differentiable at any point where f is continuous.

7.3.5 Fundamental Theorem of Calculus (Second Form) *Let $f \in \mathcal{R}[a, b]$ and let f be continuous at a point $c \in [a, b]$. Then the indefinite integral, defined by (3), is differentiable at c and $F'(c) = f(c)$.*

Proof. We will suppose that $c \in [a, b)$ and consider the right-hand derivative of F at c. Since f is continuous at c, given $\varepsilon > 0$ there exists $\eta_\varepsilon > 0$ such that if $c \le x < c + \eta_\varepsilon$, then

$$f(c) - \varepsilon < f(x) < f(c) + \varepsilon. \tag{4}$$

Let h satisfy $0 < h < \eta_\varepsilon$. The Additivity Theorem 7.2.8 implies that f is integrable on the intervals $[a, c]$, $[a, c + h]$ and $[c, c + h]$ and that

$$F(c + h) - F(c) = \int_c^{c+h} f.$$

Now on the interval $[c, c + h]$ the function f satisfies inequality (4), so that (by Theorem 7.1.4(c)) we have

$$(f(c) - \varepsilon) \cdot h \le F(c + h) - F(c) = \int_c^{c+h} f \le (f(c) + \varepsilon) \cdot h.$$

If we divide by $h > 0$ and subtract $f(c)$, we obtain

$$\left|\frac{F(c + h) - F(c)}{h} - f(c)\right| \le \varepsilon.$$

But, since $\varepsilon > 0$ is arbitrary, we conclude that the right-hand limit is given by

$$\lim_{h \to 0+} \frac{F(c + h) - F(c)}{h} = f(c).$$

It is proved in the same way that the left-hand limit of this difference quotient also equals $f(c)$ when $c \in (a, b]$, whence the assertion follows. Q.E.D.

If f is continuous on all of $[a, b]$, we obtain the following result.

7.3.6 Theorem *If f is continuous on $[a, b]$, then the indefinite integral F, defined by (3), is differentiable on $[a, b]$ and $F'(x) = f(x)$ for all $x \in [a, b]$.*

Theorem 7.3.6 can be summarized: *If f is continuous on $[a, b]$, then its indefinite integral is an antiderivative of f.* We will now see that, in general, the indefinite integral need not be an antiderivative (either because the derivative of the indefinite integral does not exist or does not equal $f(x)$).

7.3.7 Examples (a) If $f(x) := \operatorname{sgn} x$ on $[-1, 1]$, then $f \in \mathcal{R}[-1, 1]$ and has the indefinite integral $F(x) := |x| - 1$ with the basepoint -1. However, since $F'(0)$ does not exist, F is not an antiderivative of f on $[-1, 1]$.

(b) If h denotes Thomae's function, considered in 7.1.6, then its indefinite integral $H(x) := \int_0^x h$ is identically 0 on $[0, 1]$. Here, the derivative of this indefinite integral exists at every point and $H'(x) = 0$. But $H'(x) \ne h(x)$ whenever $x \in \mathbb{Q} \cap [0, 1]$, so that H is not an antiderivative of h on $[0, 1]$. □

Substitution Theorem

The next theorem provides the justification for the "change of variable" method that is often used to evaluate integrals. This theorem is employed (usually implicitly) in the evaluation by means of procedures that involve the manipulation of "differentials", common in elementary courses.

7.3.8 Substitution Theorem *Let $J := [\alpha, \beta]$ and let $\varphi : J \to \mathbb{R}$ have a continuous derivative on J. If $f : I \to \mathbb{R}$ is continuous on an interval I containing $\varphi(J)$, then*

$$\int_\alpha^\beta f(\varphi(t)) \cdot \varphi'(t)\, dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\, dx. \tag{5}$$

The proof of this theorem is based on the Chain Rule 6.1.6, and will be outlined in Exercise 15. The hypotheses that f and φ' are continuous are restrictive, but are used to ensure the existence of the Riemann integral on the left side of (5).

7.3.9 Examples **(a)** Consider the integral $\displaystyle\int_1^4 \frac{\sin\sqrt{t}}{\sqrt{t}}\, dt$.

Here we substitute $\varphi(t) := \sqrt{t}$ for $t \in [1, 4]$ so that $\varphi'(t) = 1/(2\sqrt{t})$ is continuous on $[1, 4]$. If we let $f(x) := 2\sin x$, then the integrand has the form $(f \circ \varphi) \cdot \varphi'$ and the Substitution Theorem 7.3.8 implies that the integral equals $\int_1^2 2\sin x\, dx = -2\cos x|_1^2 = 2(\cos 1 - \cos 2)$.

(b) Consider the integral $\displaystyle\int_0^4 \frac{\sin\sqrt{t}}{\sqrt{t}}\, dt$.

Since $\varphi(t) := \sqrt{t}$ does not have a continuous derivative on $[0, 4]$, the Substitution Theorem 7.3.8 is not applicable, at least with this substitution. (In fact, it is not obvious that this integral exists; however, we can apply Exercise 7.2.11 to obtain this conclusion. We could then apply the Fundamental Theorem 7.3.1 to $F(t) := -2\cos\sqrt{t}$ with $E := \{0\}$ to evaluate this integral.) □

We will give a more powerful Substitution Theorem for the *generalized* Riemann integral in Section 10.1.

Lebesgue's Integrability Criterion

We will now present a statement of the definitive theorem due to Henri Lebesgue (1875–1941) giving a necessary and sufficient condition for a function to be Riemann integrable, and will give some applications of this theorem. In order to state this result, we need to introduce the important notion of a null set.

Warning Some people use the term "null set" as a synonym for the terms "empty set" or "void set" referring to $\emptyset$ (= the set that has no elements). However, we will always use the term "null set" in conformity with our next definition, as is customary in the theory of integration.

7.3.10 Definition **(a)** A set $Z \subset \mathbb{R}$ is said to be a **null set** if for every $\varepsilon > 0$ there exists a countable collection $\{(a_k, b_k)\}_{k=1}^\infty$ of open intervals such that

$$Z \subseteq \bigcup_{k=1}^\infty (a_k, b_k) \quad \text{and} \quad \sum_{k=1}^\infty (b_k - a_k) \le \varepsilon. \tag{6}$$

(b) If $Q(x)$ is a statement about the point $x \in I$, we say that $Q(x)$ holds **almost everywhere** on I (or for **almost every** $x \in I$), if there exists a null set $Z \subset I$ such that $Q(x)$ holds for all $x \in I \backslash Z$. In this case we may write

$$Q(x) \quad \text{for a.e.} \quad x \in I.$$

It is trivial that any subset of a null set is also a null set, and it is easy to see that the union of two null sets is a null set. We will now give an example that may be very surprising.

7.3.11 Example The $\mathbb{Q}_1$ of rational numbers in $[0, 1]$ is a null set.

We enumerate $\mathbb{Q}_1 = \{r_1, r_2, \cdots\}$. Given $\varepsilon > 0$, note that the open interval $J_1 := (r_1 - \varepsilon/4, r_1 + \varepsilon/4)$ contains r_1 and has length $\varepsilon/2$; also the open interval $J_2 := (r_2 - \varepsilon/8, r_2 + \varepsilon/8)$ contains r_2 and has length $\varepsilon/4$. In general, the open interval

$$J_k := \left(r_k - \frac{\varepsilon}{2^{k+1}}, r_k + \frac{\varepsilon}{2^{k+1}}\right)$$

contains the point r_k and has length $\varepsilon/2^k$. Therefore, the union $\bigcup_{k=1}^\infty J_k$ of these open intervals contains every point of $\mathbb{Q}_1$; moreover, the sum of the lengths is $\sum_{k=1}^\infty (\varepsilon/2^k) = \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\mathbb{Q}_1$ is a null set. □

The argument just given can be modified to show that: *Every countable set is a null set.* However, it can be shown that there exist uncountable null sets in $\mathbb{R}$; for example, the Cantor set that will be introduced in Definition 11.1.10.

We now state Lebesgue's Integrability Criterion. It asserts that a bounded function on an interval is Riemann integrable if and only if its points of discontinuity form a null set.

7.3.12 Lebesgue's Integrability Criterion *A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if it is continuous almost everywhere on $[a, b]$.*

A proof of this result will be given in Appendix C. However, we will apply Legesgue's Theorem here to some specific functions, and show that some of our previous results follow immediately from it. We shall also use this theorem to obtain the important Composition and Product Theorems.

7.3.13 Examples **(a)** The step function g in Example 7.1.3(b) is continuous at every point except the point $x = 1$. Therefore it follows from the Lebesgue Integrability Criterion that g is Riemann integrable.

In fact, since every step function has at most a finite set of points of discontinuity, then: *Every step function on $[a, b]$ is Riemann integrable.*

(b) Since it was seen in Theorem 5.5.4 that the set of points of discontinuity of a monotone function is countable, we conclude that: *Every monotone function on $[a, b]$ is Riemann integrable.*

(c) The function G in Example 7.1.3(e) is discontinuous precisely at the points $D := \{1, 1/2, \cdots, 1/n, \cdots\}$. Since this is a countable set, it is a null set and Lebesgue's Criterion implies that G is Riemann integrable.

(d) The Dirichlet function was shown in Example 7.2.2(b) not to be Riemann integrable.

Note that it is discontinuous at *every* point of $[0, 1]$. Since it can be shown that the interval $[0, 1]$ is not a null set, Lebesgue's Criterion yields the same conclusion.

(e) Let $h : [0, 1] \to \mathbb{R}$ be Thomae's function, defined in Examples 5.1.6(h) and 7.1.6.

In Example 5.1.6(h), we saw that h is continuous at every irrational number and is discontinuous at every rational number in $[0, 1]$. By Example 7.3.11, it is discontinuous on a null set, so Lebesgue's Criterion implies that Thomae's function is Riemann integrable on $[0, 1]$, as we saw in Example 7.1.6. □

We now obtain a result that will enable us to take other combinations of Riemann integrable functions.

7.3.14 Composition Theorem *Let $f \in \mathcal{R}[a,b]$ with $f([a,b]) \subseteq [c,d]$ and let $\varphi : [c,d] \to \mathbb{R}$ be continuous. Then the composition $\varphi \circ f$ belongs to $\mathcal{R}[a,b]$.*

Proof. If f is continuous at a point $u \in [a,b]$, then $\varphi \circ f$ is also continuous at u. Since the set D of points discontinuity of f is a null set, it follows that the set $D_1 \subseteq D$ of points of discontinuity of $\varphi \circ f$ is also a null set. Therefore the composition $\varphi \circ f$ also belongs to $\mathcal{R}[a,b]$. Q.E.D.

It will be seen in Exercise 22 that the hypothesis that φ is continuous cannot be dropped. The next result is a corollary of the Composition Theorem.

7.3.15 Corollary *Suppose that $f \in \mathcal{R}[a,b]$. Then its absolute value $|f|$ is in $\mathcal{R}[a,b]$, and*

$$\left|\int_a^b f\right| \le \int_a^b |f| \le M(b-a),$$

where $|f(x)| \le M$ for all $x \in [a,b]$.

Proof. We have seen in Theorem 7.1.5 that if f is integrable, then there exists M such that $|f(x)| \le M$ for all $x \in [a,b]$. Let $\varphi(t) := |t|$ for $t \in [-M, M]$; then the Composition Theorem implies that $|f| = \varphi \circ f \in \mathcal{R}[a,b]$. The first inequality follows from the fact that $-|f| \le f \le |f|$ and 7.1.4(c), and the second from the fact that $|f(x)| \le M$. Q.E.D.

7.3.16 The Product Theorem *If f and g belong to $\mathcal{R}[a,b]$, then the product fg belongs to $\mathcal{R}[a,b]$.*

Proof. If $\varphi(t) := t^2$ for $t \in [-M, M]$, it follows from the Composition Theorem that $f^2 = \varphi \circ f$ belongs to $\mathcal{R}[a,b]$. Similarly, $(f+g)^2$ and g^2 belong to $\mathcal{R}[a,b]$. But since we can write the product as

$$fg = \tfrac{1}{2}\left[(f+g)^2 - f^2 - g^2\right],$$

it follows that $fg \in \mathcal{R}[a,b]$. Q.E.D.

Integration by Parts

We will conclude this section with a rather general form of Integration by Parts for the Riemann integral, and Taylor's Theorem with the Remainder.

7.3.17 Integration by Parts *Let F, G be differentiable on $[a,b]$ and let $f := F'$ and $g := G'$ belong to $\mathcal{R}[a,b]$. Then*

$$(7) \qquad \int_a^b fG = FG\Big|_a^b - \int_a^b Fg.$$

Proof. By Theorem 6.1.3(c), the derivative $(FG)'$ exists on $[a,b]$ and

$$(FG)' = F'G + FG' = fG + Fg.$$

Since F, G are continuous and f, g belong to $\mathcal{R}[a,b]$, the Product Theorem 7.3.16 implies that fG and Fg are integrable. Therefore the Fundamental Theorem 7.3.1 implies that

$$FG\Big|_a^b = \int_a^b (FG)' = \int_a^b fG + \int_a^b Fg,$$

from which (7) follows. Q.E.D.

A special, but useful, case of this theorem is when f and g are continuous on $[a,b]$ and F, G are their indefinite integrals $F(x) := \int_a^x f$ and $G(x) := \int_a^x g$.

We close this section with a version of Taylor's Theorem for the Riemann Integral.

7.3.18 Taylor's Theorem with the Remainder *Suppose that $f', \cdots, f^{(n)}, f^{(n+1)}$ exist on $[a,b]$ and that $f^{(n+1)} \in \mathcal{R}[a,b]$. Then we have*

$$(8) \qquad f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + R_n,$$

where the remainder is given by

$$(9) \qquad R_n = \frac{1}{n!}\int_a^b f^{(n+1)}(t)\cdot(b-t)^n\,dt.$$

Proof. Apply Integration by Parts to equation (9), with $F(t) := f^{(n)}(t)$ and $G(t) := (b-t)^n/n!$, so that $g(t) = -(b-t)^{n-1}/(n-1)!$, to get

$$\begin{aligned} R_n &= \frac{1}{n!}f^{(n)}(t)\cdot(b-t)^n\Big|_{t=a}^{t=b} + \frac{1}{(n-1)!}\int_a^b f^{(n)}(t)\cdot(b-a)^{n-1}\,dt \\ &= -\frac{f^{(n)}(a)}{n!}\cdot(b-a)^n + \frac{1}{(n-1)!}\int_a^b f^{(n)}(t)\cdot(b-t)^{n-1}\,dt. \end{aligned}$$

If we continue to integrate by parts in this way, we obtain (8). Q.E.D.

Exercises for Section 7.3

1. Extend the proof of the Fundamental Theorem 7.3.1 to the case of an arbitrary finite set E.
2. If $n \in \mathbb{N}$ and $H_n(x) := x^{n+1}/(n+1)$ for $x \in [a,b]$, show that the Fundamental Theorem 7.3.1 implies that $\int_a^b x^n\,dx = (b^{n+1} - a^{n+1})/(n+1)$. What is the set E here?
3. If $g(x) := x$ for $|x| \ge 1$ and $g(x) := -x$ for $|x| < 1$ and if $G(x) := \frac{1}{2}|x^2 - 1|$, show that $\int_{-2}^3 g(x)\,dx = G(3) - G(-2) = 5/2$.
4. Let $B(x) := -\frac{1}{2}x^2$ for $x < 0$ and $B(x) := \frac{1}{2}x^2$ for $x \ge 0$. Show that $\int_a^b |x|\,dx = B(b) - B(a)$.
5. Let $f : [a,b] \to \mathbb{R}$ and let $C \in \mathbb{R}$.
   (a) If $\Phi : [a,b] \to \mathbb{R}$ is an antiderivative of f on $[a,b]$, show that $\Phi_C(x) := \Phi(x) + C$ is also an antiderivative of f on $[a,b]$.
   (b) If Φ_1 and Φ_2 are antiderivatives of f on $[a,b]$, show that $\Phi_1 - \Phi_2$ is a constant function on $[a,b]$.
6. If $f \in \mathcal{R}[a,b]$ and if $c \in [a,b]$, the function defined by $F_c(z) := \int_c^z f$ for $z \in [a,b]$ is called the **indefinite integral** of f with **basepoint** c. Find a relation between F_a and F_c.
7. We have seen in Example 7.1.6 that Thomae's function is in $\mathcal{R}[0,1]$ with integral equal to 0. Can the Fundamental Theorem 7.3.1 be used to obtain this conclusion? Explain your answer.
8. Let $F(x)$ be defined for $x \ge 0$ by $F(x) := (n-1)x - (n-1)n/2$ for $x \in [n-1, n)$, $n \in \mathbb{N}$. Show that F is continuous and evaluate $F'(x)$ at points where this derivative exists. Use this result to evaluate $\int_a^b [\![x]\!]\,dx$ for $0 \le a < b$, where $[\![x]\!]$ denotes the greatest integer in x, as defined in Exercise 5.1.4.
9. Let $f \in \mathcal{R}[a,b]$ and define $F(x) := \int_a^x f$ for $x \in [a,b]$.
   (a) Evaluate $G(x) := \int_c^x f$ in terms of F, where $c \in [a,b]$.
   (b) Evaluate $H(x) := \int_x^b f$ in terms of F.
   (c) Evaluate $S(x) := \int_x^{\sin x} f$ in terms of F.

10. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and let $v : [c, d] \to \mathbb{R}$ be differentiable on $[c, d]$ with $v([c, d]) \subseteq [a, b]$. If we define $G(x) := \int_a^{v(x)} f$, show that $G'(x) = f(v(x)) \cdot v'(x)$ for all $x \in [c, d]$.

11. Find $F'(x)$ when F is defined on $[0, 1]$ by:
    (a) $F(x) := \int_0^{x^2} (1 + t^3)^{-1}\, dt$.
    (b) $F(x) := \int_{x^2}^{x} \sqrt{1 + t^2}\, dt$.

12. Let $f : [0, 3] \to \mathbb{R}$ be defined by $f(x) := x$ for $0 \le x < 1$, $f(x) := 1$ for $1 \le x < 2$ and $f(x) := x$ for $2 \le x \le 3$. Obtain formulas for $F(x) := \int_0^x f$ and sketch the graphs of f and F. Where is F differentiable? Evaluate $F'(x)$ at all such points.

13. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $c > 0$, define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) := \int_{x-c}^{x+c} f(t)\, dt$. Show that g is differentiable on $\mathbb{R}$ and find $g'(x)$.

14. If $f : [0, 1] \to \mathbb{R}$ is continuous and $\int_0^x f = \int_x^1 f$ for all $x \in [0, 1]$, show that $f(x) = 0$ for all $x \in [0, 1]$.

15. Use the following argument to prove the Substitution Theorem 7.3.8. Define $F(u) := \int_{\varphi(a)}^{u} f(x)dx$ for $u \in I$, and $H(t) := F(\varphi(t))$ for $t \in J$. Show that $H'(t) = f(\varphi(t))\varphi'(t)$ for $t \in J$ and that

$$\int_{\varphi(a)}^{\varphi(\beta)} f(x)\, dx = F(\varphi(\beta)) = H(\beta) = \int_a^{\beta} f(\varphi(t))\varphi'(t)\, dt.$$

16. Use the Substitution Theorem 7.3.8 to evaluate the following integrals.
    (a) $\displaystyle\int_0^1 t\sqrt{1 + t^2}\, dt$,
    (b) $\displaystyle\int_0^2 t^2(1 + t^3)^{-1/2}\, dt = 4/3$,
    (c) $\displaystyle\int_1^4 \frac{\sqrt{1 + \sqrt{t}}}{\sqrt{t}}\, dt$,
    (d) $\displaystyle\int_1^4 \frac{\cos\sqrt{t}}{\sqrt{t}}\, dt = 2(\sin 2 - \sin 1)$.

17. Sometimes the Substitution Theorem 7.3.8 cannot be applied but the following result, called the "Second Substitution Theorem" is useful. In addition to the hypotheses of 7.3.8, assume that $\varphi'(t) \ne 0$ for all $t \in J$, so the function $\psi : \varphi(J) \to \mathbb{R}$ inverse to φ exists and has derivative $\psi'(\varphi(t)) = 1/\varphi'(t)$. Then

$$\int_a^{\beta} f(\varphi(t))\, dt = \int_{\varphi(a)}^{\varphi(\beta)} f(x)\psi'(x)\, dx.$$

To prove this, let $G(t) := \int_a^t f(\varphi(s))\, ds$ for $t \in J$, so that $G'(t) = f(\varphi(t))$. Note that $K(x) := G(\psi(x))$ is differentiable on the interval $\varphi(J)$ and that $K'(x) = G'(\psi(x))\psi'(x) = f(\varphi \circ \psi(x))\psi'(x) = f(x)\psi'(x)$. Calculate $G(\beta) = K(\varphi(\beta))$ in two ways to obtain the formula.

18. Apply the Second Substition Theorem to evaluate the following integrals.
    (a) $\displaystyle\int_1^9 \frac{dt}{2 + \sqrt{t}}$,
    (b) $\displaystyle\int_1^3 \frac{dt}{t\sqrt{t+1}} = \ln(3 + 2\sqrt{2}) - \ln 3$,
    (c) $\displaystyle\int_1^4 \frac{\sqrt{t}\, dt}{1 + \sqrt{t}}$,
    (d) $\displaystyle\int_1^4 \frac{dt}{\sqrt{t}(t + 4)} = \operatorname{Arctan}(1) - \operatorname{Arctan}(1/2)$.

19. Explain why Theorem 7.3.8 and/or Exercise 7.3.17 cannot be applied to evaluate the following integrals, using the indicated substitution.
    (a) $\displaystyle\int_0^4 \frac{\sqrt{t}\, dt}{1 + \sqrt{t}}$ $\varphi(t) = \sqrt{t}$,
    (b) $\displaystyle\int_0^4 \frac{\cos\sqrt{t}\, dt}{\sqrt{t}}$ $\varphi(t) = \sqrt{t}$,
    (c) $\displaystyle\int_{-1}^1 \sqrt{1 + 2|t|}\, dt$ $\varphi(t) = |t|$,
    (d) $\displaystyle\int_0^1 \frac{dt}{\sqrt{1 - t^2}}$ $\varphi(t) = \operatorname{Arcsin} t$.

20. (a) If Z_1 and Z_2 are null sets, show that $Z_1 \cup Z_2$ is a null set.
    (b) More generally, if Z_n is a null set for each $n \in \mathbb{N}$, show that $\bigcup_{n=1}^{\infty} Z_n$ is a null set. [*Hint:* Given $\varepsilon > 0$ and $n \in \mathbb{N}$, let $\{J_k^n : k \in \mathbb{N}\}$ be a countable collection of open intervals whose union contains Z_n and the sum of whose lengths is $\le \varepsilon/2^n$. Now consider the countable collection $\{J_k^n : n, k \in \mathbb{N}\}$.]

21. Let $f, g \in \mathcal{R}[a, b]$.
    (a) If $t \in \mathbb{R}$, show that $\int_a^b (tf \pm g)^2 \ge 0$.
    (b) Use (a) to show that $2\left|\int_a^b fg\right| \le t\int_a^b f^2 + (1/t)\int_a^b g^2$ for $t > 0$.
    (c) If $\int_a^b f^2 = 0$, show that $\int_a^b fg = 0$.
    (d) Now prove that $\left|\int_a^b fg\right|^2 \le (\int_a^b |fg|)^2 \le (\int_a^b f^2) \cdot (\int_a^b g^2)$. This inequality is called the **Cauchy-Bunyakovsky-Schwarz Inequality** (or simply the **Schwarz Inequality**).

22. Let $h : [0, 1] \to \mathbb{R}$ be Thomae's function and let sgn be the signum function. Show that the composite function $\operatorname{sgn} \circ h$ is not Riemann integrable on $[0, 1]$.

Section 7.4 Approximate Integration

The Fundamental Theorem of Calculus 7.3.1 yields an effective method of evaluating the integral $\int_a^b f$ *provided* we can find an antiderivative F such that $F'(x) = f(x)$ when $x \in [a, b]$. However, when we cannot find such an F, we may not be able to use the Fundamental Theorem. Nevertheless, *when f is continuous*, there are a number of techniques for approximating the Riemann integral $\int_a^b f$ by using sums that resemble the Riemann sums.

One very elementary procedure to obtain quick estimates of $\int_a^b f$, based on Theorem 7.1.4(c), is to note that if $g(x) \le f(x) \le h(x)$ for all $x \in [a, b]$, then

$$\int_a^b g \le \int_a^b f \le \int_a^b h.$$

If the integrals of g and h can be calculated, then we have bounds for $\int_a^b f$. Often these bounds are accurate enough for our needs.

For example, suppose we wish to estimate the value of $\int_0^1 e^{-x^2}\, dx$. It is easy to show that $e^{-x} \le e^{-x^2} \le 1$ for $x \in [0, 1]$, so that

$$\int_0^1 e^{-x}\, dx \le \int_0^1 e^{-x^2}\, dx \le \int_0^1 1\, dx.$$

Consequently, we have $1 - 1/e \le \int_0^1 e^{-x^2}\, dx \le 1$. If we use the mean of the bracketing values, we obtain the estimate $1 - 1/2e \approx 0.816$ for the integral with an error less than $1/2e < 0.184$. This estimate is crude, but it is obtained rapidly and may be quite satisfactory for our needs. If a better approximation is desired, we can attempt to find closer approximating functions g and h.

Taylor's Theorem 6.4.1 can be used to approximate e^{-x^2} by a polynomial. In using Taylor's Theorem, we must get bounds on the remainder term for our calculations to have significance. For example, if we apply Taylor's Theorem to e^{-y} for $0 \le y \le 1$, we get

$$e^{-y} = 1 - y + \tfrac{1}{2}y^2 - \tfrac{1}{6}y^3 + R_3,$$

where $R_3 = y^4 e^{-c}/24$ where c is some number with $0 \le c \le 1$. Since we have no better information as to the location of c, we must be content with the estimate $0 \le R_3 \le y^4/24$. Hence we have

$$e^{-x^2} = 1 - x^2 + \tfrac{1}{2}x^4 - \tfrac{1}{6}x^6 + R_3,$$

where $0 \le R_3 \le x^8/24$, for $x \in [0, 1]$. Therefore, we obtain

$$\int_0^1 e^{-x^2}\,dx = \int_0^1 (1 - x^2 + \tfrac{1}{2}x^4 - \tfrac{1}{6}x^6)\,dx + \int_0^1 R_3\,dx$$
$$= 1 - \tfrac{1}{3} + \tfrac{1}{10} - \tfrac{1}{42} + \int_0^1 R_3\,dx.$$

Since we have $0 \le \int_0^1 R_3\,dx \le \frac{1}{9\cdot 24} = \frac{1}{216} < 0.005$, it follows that

$$\int_0^1 e^{-x^2}\,dx \approx \tfrac{26}{35}\ (\approx 0.7429),$$

with an error less than 0.005.

Equal Partitions

If $f : [a, b] \to \mathbb{R}$ is continuous, we know that its Riemann integral exists. To find an approximate value for this integral with the minimum amount of calculation, it is convenient to consider partitions $\mathcal{P}_n$ of $[a, b]$ into n *equal* subintervals having length $h_n := (b - a)/n$. Hence $\mathcal{P}_n$ is the partition:

$$a < a + h_n < a + 2h_n < \cdots < a + nh_n = b.$$

If we pick our tag points to be the *left endpoints* and the *right endpoints* of the subintervals, we obtain the **nth left approximation** given by

$$L_n(f) := h_n \sum_{k=0}^{n-1} f(a + kh_n),$$

and the **nth right approximation** given by

$$R_n(f) := h_n \sum_{k=1}^{n} f(a + kh_n).$$

It should be noted that it is almost as easy to evaluate *both* of these approximations as only one of them, since they differ only by the terms $f(a)$ and $f(b)$.

Unless we have reason to believe that one of $L_n(f)$ or $R_n(f)$ is closer to the actual value of the integral than the other one, we generally take their mean:

$$\tfrac{1}{2}\left(L_n(f) + R_n(f)\right),$$

which is readily seen to equal

$$T_n(f) := h_n\left(\tfrac{1}{2}f(a) + \sum_{k=1}^{n-1} f(a + kh_n) + \tfrac{1}{2}f(b)\right), \tag{1}$$

as a reasonable approximation to $\int_a^b f$.

However, we note that if f is *increasing* on $[a, b]$, then it is clear from a sketch of the graph of f that

$$L_n(f) \le \int_a^b f \le R_n(f). \tag{2}$$

In this case, we readily see that

$$\left|\int_a^b f - T_n(f)\right| \le \tfrac{1}{2}\left(R_n(f) - L_n(f)\right)$$
$$= \tfrac{1}{2}h_n\,(f(b) - f(a)) = (f(b) - f(a)) \cdot \frac{(b - a)}{2n}.$$

An error estimate such as this is useful, since it gives an upper bound for the error of the approximation in terms of quantities that are known at the outset. In particular, it can be used to determine how large we should choose n in order to have an approximation that will be correct to within a specified error $\varepsilon > 0$.

The above discussion was valid for the case that f is increasing on $[a, b]$. If f is decreasing, then the inequalities in (2) should be reversed. We can summarize both cases in the following statement.

7.4.1 Theorem *If $f : [a, b] \to \mathbb{R}$ is monotone and if $T_n(f)$ is given by (1), then*

$$\left|\int_a^b f - T_n(f)\right| \le |f(b) - f(a)| \cdot \frac{(b - a)}{2n}. \tag{3}$$

7.4.2 Example If $f(x) := e^{-x^2}$ on $[0, 1]$, then f is decreasing. It follows from (3) that if $n = 8$, then $|\int_0^1 e^{-x^2}\,dx - T_8(f)| \le (1 - e^{-1})/16 < 0.04$, and if $n = 16$, then $|\int_0^1 e^{-x^2}\,dx - T_{16}(f)| \le (1 - e^{-1})/32 < 0.02$. Actually, the approximation is considerable better, as we will see in Example 7.4.5. □

The Trapezoidal Rule

The method of numerical integration called the "Trapezoidal Rule" is based on approximating the continuous function $f : [a, b] \to \mathbb{R}$ by a piecewise linear continuous function. Let $n \in \mathbb{N}$ and, as before, let $h_n := (b - a)/n$ and consider the partition $\mathcal{P}_n$. We approximate f by the piecewise linear function g_n that passes through the points $(a + kh_n, f(a + kh_n))$, where $k = 0, 1, \cdots, n$. It seems reasonable that the integral $\int_a^b f$ will be "approximately equal to" the integral $\int_a^b g_n$ when n is sufficiently large (provided that f is reasonably smooth).

Since the area of a trapezoid with horizontal base h and vertical sides l_1 and l_2 is known to be $\frac{1}{2}h(l_1 + l_2)$, we have

$$\int_{a+kh_n}^{a+(k+1)h_n} g_n = \tfrac{1}{2}h_n \cdot \left[f(a + kh_n) + f(a + (k + 1)h_n)\right],$$

for $k = 0, 1, \cdots, n - 1$. Summing these terms and noting that each partition point in $\mathcal{P}_n$ except a and b belongs to two adjacent subintervals, we obtain

$$\int_a^b g_n = h_n\left(\tfrac{1}{2}f(a) + f(a + h_n) + \cdots + f(a + (k - 1)h_n) + \tfrac{1}{2}f(b)\right).$$

But the term on the right is precisely $T_n(f)$, found in (1) as the mean of $L_n(f)$ and $R_n(f)$. We call $T_n(f)$ the **nth Trapezoidal Approximation** of f.

In Theorem 7.4.1 we obtained an error estimate in the case where f is monotone; we now state one without this restriction on f, but in terms of the second derivative f'' of f.

7.4.3 Theorem *Let f, f' and f'' be continuous on $[a, b]$ and let $T_n(f)$ be the nth Trapezoidal Approximation (1). Then there exists $c \in [a, b]$ such that*

$$T_n(f) - \int_a^b f = \frac{(b - a)h_n^2}{12} \cdot f''(c). \tag{4}$$

A proof of this result will be given in Appendix D; it depends on a number of results we have obtained in Chapters 5 and 6.

The equality (4) is interesting in that it can give both an upper bound and a lower bound for the difference $T_n(f) - \int_a^b f$. For example, if $f''(x) \ge A > 0$ for all $x \in [a, b]$, then (4) implies that this difference always exceeds $\frac{1}{12}A(b-a)h_n^2$. If we only have $f''(x) \ge 0$ for $x \in [a, b]$, which is the case when f is **convex** (= **concave upward**), then the Trapezoidal Approximation is always *too large*. The reader should draw a figure to visualize this.

However, it is usually the upper bound that is of greater interest.

7.4.4 Corollary *Let f, f' and f'' be continuous, and let $|f''(x)| \le B_2$ for all $x \in [a, b]$. Then*

$$(5)\qquad \left|T_n(f) - \int_a^b f\right| \le \frac{(b-a)h_n^2}{12}\cdot B_2 = \frac{(b-a)^3}{12n^2}\cdot B_2.$$

When an upper bound B_2 can be found, (5) can be used to determine how large n must be chosen in order to be certain of a desired accuracy.

7.4.5 Example If $f(x) := e^{-x^2}$ on $[0, 1]$, then a calculation shows that $f''(x) = 2e^{-x^2}(2x^2 - 1)$, so that we can take $B_2 = 2$. Thus, if $n = 8$, then

$$\left|T_8(f) - \int_0^1 f\right| \le \frac{2}{12\cdot 64} = \frac{1}{384} < 0.003.$$

On the other hand, if $n = 16$, then we have

$$\left|T_{16}(f) - \int_0^1 f\right| \le \frac{2}{12\cdot 256} = \frac{1}{1536} < 0.00066.$$

Thus, the accuracy in this case is considerably better than predicted in Example 7.4.2 □

The Midpoint Rule

One obvious method of approximating the integral of f is to take the Riemann sums evaluated at the *midpoints* of the subintervals. Thus, if $\mathcal{P}_n$ is the equally spaced partition given before, the **Midpoint Approximation** of f is given by

$$\begin{aligned} M_n(f) &:= h_n\left(f(a + \tfrac{1}{2}h_n) + f\left(a + \tfrac{3}{2}h_n\right) + \cdots + f\left(a\left(n - \tfrac{1}{2}\right)h_n\right)\right) \\ (6)\qquad &= h_n \sum_{k=1}^{n} f\left(a + \left(k - \tfrac{1}{2}\right)h_n\right). \end{aligned}$$

Another method might be to use piecewise linear functions that are *tangent* to the graph of f at the midpoints of these subintervals. At first glance, it seems as if we would need to know the slope of the tangent line to the graph of f at each of the midpoints $a + (k - \frac{1}{2}h_n)$ $(k = 1, 2, \cdots, n)$. However, it is an exercise in geometry to show that the *area* of the trapezoid whose top is this tangent line at the midpoint $a + (k - \frac{1}{2})h_n$ is equal to the *area* of the rectangle whose height is $f(a + (k - \frac{1}{2})h_n)$. (See Figure 7.4.1.) Thus, this area is given by (6), and the "Tangent Trapezoid Rule" turns out to be the same as the "Midpoint Rule". We now state a theorem showing that the Midpoint Rule gives better accuracy than the Trapezoidal Rule by a factor of 2.

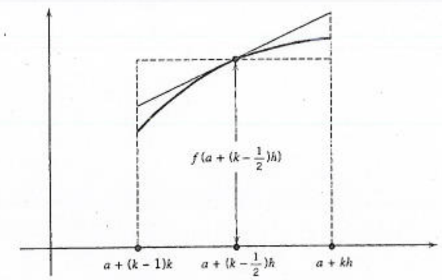


Figure 7.4.1 The tangent trapezoid.

7.4.6 Theorem *Let f, f', and f'' be continuous on $[a, b]$ and let $M_n(f)$ be the nth Midpoint Approximation (6). Then there exists $\gamma \in [a, b]$ such that*

$$(7)\qquad \int_a^b f - M_n(f) = \frac{(b-a)h_n^2}{24}\cdot f''(\gamma).$$

The proof of this result is in Appendix D.

As in the case with Theorem 7.4.3, formula (7) can be used to give both an upper bound and a lower bound for the difference $\int_a^b f - M_n(f)$, although it is an upper bound that is usually of greater interest. In contrast with the Trapezoidal Rule, if the function is **convex**, then the Midpoint Approximation is always too small.

The next result is parallel to Corollary 7.4.4.

7.4.7 Corollary *Let f, f', and f'' be continuous, and let $|f''(x)| \le B_2$ for all $x \in [a, b]$. Then*

$$(8)\qquad \left|M_n(f) - \int_a^b f\right| \le \frac{(b-a)h_n^2}{24}\cdot B_2 = \frac{(b-a)^3}{24n^2}\cdot B_2.$$

Simpson's Rule

The final approximation procedure that we will consider usually gives a better approximation than either the Trapezoidal or the Midpoint Rule and requires essentially no extra calculation. However, the convexity (or the concavity) of f does not give any information about the error for this method.

Whereas the Trapezoidal and Midpoint Rules were based on the approximation of f by piecewise linear functions, Simpson's Rule approximates the graph of f by parabolic arcs. To help motivate the formula, the reader may show that if three points

$$(-h, y_0), \quad (0, y_1), \quad \text{and} \quad (h, y_2)$$

are given, then the quadratic function $q(x) := Ax^2 + Bx + C$ that passes through these points has the property that

$$\int_{-h}^{h} q = \tfrac{1}{3}h(y_0 + 4y_1 + y_2).$$

Now let f be a continuous function on $[a, b]$ and let $n \in \mathbb{N}$ be *even*, and let $h_n := (b-a)/n$. On each "double subinterval"

$$[a, a+2h_n], \quad [a+2h_n, a+4h_n], \quad \cdots, \quad [b-2h_n, b],$$

we approximate f by $n/2$ quadratic functions that agree with f at the points

$$y_0 := f(a), \quad y_1 := f(a+h_n), \quad y_2 := f(a+2h_n), \quad \cdots, \quad y_n := f(b).$$

These considerations lead to the **nth Simpson Approximation**, defined by

$$\begin{aligned} S_n(f) := \tfrac{1}{3}h_n\big(&f(a) + 4f(a+h_n) + 2f(a+2h_n) + 4f(a+3h_n) \\ &+ 2f(a+4h_n) + \cdots + 2f(b-2h_n) + 4f(b-h_n) + f(b)\big). \end{aligned} \tag{9}$$

Note that the coefficients of the values of f at the $n+1$ partition points follow the pattern $1, 4, 2, 4, 2, \cdots, 4, 2, 4, 1$.

We now state a theorem that gives an estimate about the accuracy of the Simpson approximation; it involves the *fourth derivative* of f.

7.4.8 Theorem *Let $f, f', f'', f^{(3)}$ and $f^{(4)}$ be continuous on $[a, b]$ and let $n \in \mathbb{N}$ be even. If $S_n(f)$ is the nth Simpson Approximation (9), then there exists $c \in [a, b]$ such that*

$$S_n(f) - \int_a^b f = \frac{(b-a)h_n^4}{180} \cdot f^{(4)}(c). \tag{10}$$

A proof of this result is given in Appendix D.

The next result is parallel to Corollaries 7.4.4 and 7.4.7.

7.4.9 Corollary *Let $f, f', f'', f^{(3)}$ and $f^{(4)}$ be continuous on $[a, b]$ and let $|f^{(4)}(x)| \le B_4$ for all $x \in [a, b]$. Then*

$$\left|S_n(f) - \int_a^b f\right| \le \frac{(b-a)h_n^4}{180} \cdot B_4 = \frac{(b-a)^5}{180n^4} \cdot B_4. \tag{11}$$

Successful use of the estimate (11) depends on being able to find an upper bound for the fourth derivative.

7.4.10 Example If $f(x) := 4e^{-x^2}$ on $[0, 1]$ then a calculation shows that

$$f^{(4)}(x) = 4e^{-x^2}(4x^4 - 12x^2 + 3),$$

whence it follows that $|f^{(4)}(x)| \le 20$ for $x \in [0, 1]$, so we can take $B_4 = 20$. It follows from (11) that if $n = 8$ then

$$\left|S_8(f) - \int_0^1 f\right| \le \frac{1}{180 \cdot 8^4} \cdot 20 = \frac{1}{36{,}864} < 0.000\,03$$

and that if $n = 16$ then

$$\left|S_{16}(f) - \int_0^1 f\right| \le \frac{1}{589{,}824} < 0.000\,001\,7. \qquad \square$$

Remark The nth Midpoint Approximation $M_n(f)$ can be used to "step up" to the $(2n)$th Trapezoidal and Simpson Approximations by using the formulas

$$T_{2n}(f) = \tfrac{1}{2}M_n(f) + \tfrac{1}{2}T_n(f) \qquad \text{and} \qquad S_{2n}(f) = \tfrac{2}{3}M_n(f) + \tfrac{1}{3}T_n(f),$$

that are given in the Exercises. Thus once the initial Trapezoidal Approximation $T_1 = T_1(f)$ has been calculated, only the Midpoint Approximations $M_n = M_n(f)$ need be found. That is, we employ the following sequence of calculations:

$$T_1 = \tfrac{1}{2}(b-a)\big(f(a) + f(b)\big);$$

$$\begin{array}{lll} M_1 = (b-a)f\left(\tfrac{1}{2}(a+b)\right), & T_2 = \tfrac{1}{2}M_1 + \tfrac{1}{2}T_1, & S_2 = \tfrac{2}{3}M_1 + \tfrac{1}{3}T_1; \\ M_2, & T_4 = \tfrac{1}{2}M_2 + \tfrac{1}{2}T_2, & S_4 = \tfrac{2}{3}M_2 + \tfrac{1}{3}T_2; \\ M_4, & T_8 = \tfrac{1}{2}M_4 + \tfrac{1}{2}T_4, & S_8 = \tfrac{2}{3}M_4 + \tfrac{1}{3}T_4; \\ \cdots & \cdots \quad \cdots & \cdots \end{array}$$

Exercises for Section 7.4

1. Use the Trapezoidal Approximation with $n = 4$ to evaluate $\ln 2 = \int_1^2 (1/x)\,dx$. Show that $0.6866 \le \ln 2 \le 0.6958$ and that
$$0.0013 < \frac{1}{768} \le T_4 - \ln 2 \le \frac{1}{96} < 0.0105.$$
2. Use the Simpson Approximation with $n = 4$ to evaluate $\ln 2 = \int_1^2 (1/x)\,dx$. Show that $0.6927 \le \ln 2 \le 0.6933$ and that
$$0.000\,016 < \frac{1}{2^5} \cdot \frac{1}{1920} \le S_4 - \ln 2 \le \frac{1}{1920} < 0.000\,521.$$
3. Let $f(x) := (1+x^2)^{-1}$ for $x \in [0, 1]$. Show that $f''(x) = 2(3x^2 - 1)(1+x^2)^{-3}$ and that $|f''(x)| \le 2$ for $x \in [0, 1]$. Use the Trapezoidal Approximation with $n = 4$ to evaluate $\pi/4 = \int_0^1 f(x)\,dx$. Show that $|T_4(f) - (\pi/4)| \le 1/96 < 0.0105$.
4. If the Trapezoidal Approximation $T_n(f)$ is used to approximate $\pi/4$ as in Exercise 3, show that we must take $n \ge 409$ in order to be sure that the error is less than 10^{-6}.
5. Let f be as in Exercise 3. Show that $f^{(4)}(x) = 24(5x^4 - 10x^2 + 1)(1+x^2)^{-5}$ and that $|f^{(4)}(x)| \le 96$ for $x \in [0, 1]$. Use Simpson's Approximation with $n = 4$ to evaluate $\pi/4$. Show that $|S_4(f) - (\pi/4)| \le 1/480 < 0.0021$.
6. If the Simpson Approximation $S_n(f)$ is used to approximate $\pi/4$ as in Exercise 5, show that we must take $n \ge 28$ in order to be sure that the error is less than 10^{-6}.
7. If p is a polynomial of degree at most 3, show that the Simpson Approximations are exact.
8. Show that if $f''(x) \ge 0$ on $[a, b]$ (that is, if f is convex on $[a, b]$), then for any natural numbers m, n we have $M_n(f) \le \int_a^b f(x)\,dx \le T_m(f)$. If $f''(x) \le 0$ on $[a, b]$, this inequality is reversed.
9. Show that $T_{2n}(f) = \frac{1}{2}[M_n(f) + T_n(f)]$.
10. Show that $S_{2n}(f) = \frac{2}{3}M_n(f) + \frac{1}{3}T_n(f)$.
11. Show that one has the estimate $\left|S_n(f) - \int_a^b f(x)\,dx\right| \le \left[(b-a)^2/18n^2\right]B_2$, where $B_2 \ge |f''(x)|$ for all $x \in [a, b]$.

12. Note that $\int_0^1 (1-x^2)^{1/2}\,dx = \pi/4$. Explain why the error estimates given by formulas (4), (7), and (10) cannot be used. Show that if $h(x) = (1-x^2)^{1/2}$ for x in $[0,1]$, then $T_n(h) \le \pi/4 \le M_n(h)$. Calculate $M_8(h)$ and $T_8(h)$.

13. If h is as in Exercise 12, explain why $K := \int_0^{1/\sqrt{2}} h(x)\,dx = \pi/8 + 1/4$. Show that $|h''(x)| \le 2^{3/2}$ and that $|h^{(4)}(x)| \le 9 \cdot 2^{7/2}$ for $x \in [0, 1/\sqrt{2}]$. Show that $|K - T_n(h)| \le 1/12n^2$ and that $|K - S_n(h)| \le 1/10n^4$. Use these results to calculate π.

In Exercises 14–20, approximate the indicated integrals, giving estimates for the error. Use a calculator to obtain a high degree of precision.

14. $\int_0^2 (1+x^4)^{1/2}\,dx$. 15. $\int_0^2 (4+x^3)^{1/2}\,dx$. 16. $\int_0^1 \frac{dx}{1+x^3}$.

17. $\int_0^\pi \frac{\sin x}{x}\,dx$. 18. $\int_0^{\pi/2} \frac{dx}{1+\sin x}$. 19. $\int_0^{\pi/2} \sqrt{\sin x}\,dx$.

20. $\int_0^1 \cos(x^2)\,dx$.

Section 7.1

1. (a) $\|\mathcal{P}_1\| = 2$ (b) $\|\mathcal{P}_2\| = 2$ (c) $\|\mathcal{P}_3\| = 1.4$ (d) $\|\mathcal{P}_4\| = 2$.

2. (a) $0^2 \cdot 1 + 1^2 \cdot 1 + 2^2 \cdot 2 = 0 + 1 + 8 = 9$
(b) 37 (c) 13 (d) 33.

5. (a) If $u \in [x_{i-1}, x_i]$, then $x_{i-1} \le u$ so that $c_1 \le t_i \le x_i \le x_{i-1} + \|\dot{\mathcal{P}}\|$ whence $c_1 - \|\dot{\mathcal{P}}\| \le x_{i-1} \le u$. Also $u \le x_i$ so that $x_i - \|\dot{\mathcal{P}}\| \le x_{i-1} \le t_i \le c_2$, whence $u \le x_i \le c_2 + \|\dot{\mathcal{P}}\|$.

10. g is not bounded. Take rational tags.

12. Let $\mathcal{P}_n$ be the partition of $[0,1]$ into n equal parts. If $\dot{\mathcal{P}}_n$ is this partition with rational tags, then $S(f; \dot{\mathcal{P}}_n) = 1$, while if $\dot{\mathcal{Q}}_n$ is this partition with irrational tags, then $S(f; \dot{\mathcal{Q}}_n) = 0$.

13. Argue as in Example 7.1.3(d).

15. If $\|\dot{\mathcal{P}}\| < \delta_\varepsilon := \varepsilon/4\alpha$, then the union of the subintervals in $\dot{\mathcal{P}}$ with tags in $[c,d]$ contains the interval $[c+\delta_\varepsilon, d-\delta_\varepsilon]$ and is contained in $[c-\delta_\varepsilon, d+\delta_\varepsilon]$. Therefore $\alpha(d-c-2\delta_\varepsilon) \le S(\varphi; \dot{\mathcal{P}}) \le \alpha(d-c+2\delta_\varepsilon)$, whence $|S(\varphi;\dot{\mathcal{P}}) - \alpha(d-c)| \le 2\alpha\delta_\varepsilon < \varepsilon$.

16. (b) In fact, $(x_i^2 + x_i x_{i-1} + x_{i-1}^2) \cdot (x_i - x_{i-1}) = x_i^3 - x_{i-1}^3$.
(c) The terms in $S(Q; \dot{\mathcal{Q}})$ telescope.

18. Let $\dot{\mathcal{P}} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ be a tagged partition of $[a,b]$ and let $\dot{\mathcal{Q}} := \{([x_{i-1}+c, x_i+c], t_i+c)\}_{i=1}^n$ so that $\dot{\mathcal{Q}}$ is a tagged partition of $[a+c, b+c]$ and $\|\dot{\mathcal{Q}}\| = \|\dot{\mathcal{P}}\|$. Moreover, $S(g;\dot{\mathcal{Q}}) = S(f;\dot{\mathcal{P}})$ so that $|S(g;\dot{\mathcal{Q}}) - \int_a^b f| = |S(f;\dot{\mathcal{P}}) - \int_a^b f| < \varepsilon$ when $\|\dot{\mathcal{Q}}\| < \delta_\varepsilon$.

Section 7.4

1. Use (4) with $n = 4$, $a = 1$, $b = 2$, $h = 1/4$. Here $1/4 \le f''(c) \le 2$, so $T_4 \approx 0.697\,02$.

3. $T_4 \approx 0.782\,79$.

4. The index n must satisfy $2/12n^2 < 10^{-6}$; hence $n > 1000/\sqrt{6} \approx 408.25$.

5. $S_4 \approx 0.785\,39$.

6. The index n must satisfy $96/180n^4 < 10^{-6}$; hence $n \ge 28$.

12. The integral is equal to the area of one quarter of the unit circle. The derivatives of h are unbounded on $[0,1]$. Since $h''(x) \le 0$, the inequality is $T_n(h) < \pi/4 < M_n(h)$. See Exercise 8.

13. Interpret K as an area. Show that $h''(x) = -(1-x^2)^{3/2}$ and that $h^{(4)}(x) = -3(1+4x^2)(1-x^2)^{-7/2}$. To eight decimal places, $\pi = 3.141\,592\,65$.

14. Approximately 3.653 484 49. 15. Approximately 4.821 159 32.

16. Approximately 0.835 648 85. 17. Approximately 1.851 937 05.

18. 1. 19. Approximately 1.198 140 23.

20. Approximately 0.904 524 24.

Section 7.2

2. If the tags are all rational, then $S(h;\dot{\mathcal{P}}) \ge 1$, while if the tags are all irrational, then $S(h;\dot{\mathcal{P}}) = 0$.

3. Let $\dot{\mathcal{P}}_n$ be the partition of $[0,1]$ into n equal subintervals with $t_1 = 1/n$ and $\dot{\mathcal{Q}}_n$ be the same subintervals tagged by irrational points.

5. If $c_1, \cdots, c_n$ are the distinct values taken by φ, then $\varphi^{-1}(c_j)$ is the union of a finite collection $\{J_{j1}, \cdots, J_{jr_j}\}$ of disjoint subintervals of $[a,b]$. We can write $\varphi = \sum_{j=1}^n \sum_{k=1}^{r_j} c_j \varphi_{J_{jk}}$.

6. Not necessarily.

8. If $f(c) > 0$ for some $c \in (a,b)$, there exists $\delta > 0$ such that $f(x) > \frac{1}{2}f(c)$ for $|x - c| \le \delta$. Then $\int_a^b f \ge \int_{c-\delta}^{c+\delta} f \ge (2\delta)\frac{1}{2}f(c) > 0$. If c is an endpoint, a similar argument applies.

10. Use Bolzano's Theorem 5.3.7.

12. Indeed, $|g(x)| \le 1$ and is continuous on every interval $[c, 1]$ where $0 < c < 1$. The preceding exercise applies.

13. Let $f(x) := 1/x$ for $x \in (0,1]$ and $f(0) := 0$.

16. Let $m := \inf f(x)$ and $M := \sup f$. By Theorem 7.1.4(c), we have $m(b-a) \le \int_a^b f \le M(b-a)$. By Bolzano's Theorem 5.3.7, there exists $c \in [a,b]$ such that $f(c) = (\int_a^b f)/(b-a)$.

19. (a) Let $\dot{\mathcal{P}}_n$ be a sequence of tagged partitions of $[0,a]$ with $\|\dot{\mathcal{P}}_n\| \to 0$ and let $\dot{\mathcal{P}}_n^*$ be the corresponding "symmetric" partition of $[-a,a]$. Show that $S(f;\dot{\mathcal{P}}_n^*) = 2S(f;\dot{\mathcal{P}}_n) \to 2\int_0^a f$.

21. Note that $x \mapsto f(x^2)$ is an even continuous function.

22. Let $x_i := i(\pi/2)$ for $i = 0, 1, \cdots, n$. Then we have that $(\pi/2n)\sum_{i=0}^{n-1} f(\cos x_i) = (\pi/2n)\sum_{k=1}^{n} f(\sin x_k)$.

Section 7.3

1. Suppose that $E := \{a = c_0 < c_1 < \cdots < c_m = b\}$ contains the points in $[a,b]$ where the derivative $F'(x)$ either does not exist, or does not equal $f(x)$. Then $f \in \mathcal{R}[c_{i-1}, c_i]$ and $\int_{c_{i-1}}^{c_i} f = F(c_i) - F(c_{i-1})$. Exercise 7.2.14 and Corollary 7.2.10 imply that $f \in \mathcal{R}[a,b]$ and that $\int_a^b f = \sum_{i=1}^m (F(c_i) - F(c_{i-1})) = F(b) - F(a)$.

2. $E = \emptyset$. 3. Let $E := \{-1, 1\}$. If $x \notin E$, $G'(x) = g(x)$.

4. Indeed, $B'(x) = |x|$ for all x. 6. $F_c = F_a - \int_a^c f$.

7. Let h be Thomae's function. There is no function $H : [0,1] \to \mathbb{R}$ such that $H'(x) = h(x)$ for x in some nondegenerate open interval; otherwise Darboux's Theorem 6.2.12 would be contradicted on this interval.

9. (a) $G(x) = F(x) - F(c)$, (b) $H(x) = F(b) - F(x)$, (c) $S(x) = F(\sin x) - F(x)$.

10. Use Theorem 7.3.6 and the Chain Rule 6.1.6.

11. (a) $F'(x) = 2x(1+x^6)^{-1}$ (b) $F'(x) = (1+x^2)^{1/2} - 2x(1+x^4)^{1/2}$.

13. $g'(x) = f(x+c) - f(x-c)$.

16. (a) Take $\varphi(t) = 1 + t^2$ to get $\frac{1}{3}(2^{3/2} - 1)$.
(b) Take $\varphi(t) = 1 + t^3$ to get $\frac{4}{3}$.
(c) Take $\varphi(t) = 1 + \sqrt{t}$ to get $\frac{4}{3}(3^{3/2} - 2^{3/2})$.
(d) Take $\varphi(t) = t^{1/2}$ to get $2(\sin 2 - \sin 1)$.

18. (a) Take $x = \varphi(t) = t^{1/2}$, so $t = \psi(x) = x^2$ to get $4(1 - \ln(5/3))$.
(b) Take $x = \varphi(t) = (t+1)^{1/2}$, so $t = \psi(x) = x^2 - 1$ to get $\ln(3 + 2\sqrt{2}) - \ln 3$.
(c) Take $x = \varphi(t) = t^{1/2}$ to get $2(3/2 + \ln 3/2)$.
(d) Take $x = \varphi(t) = t^{1/2}$ to get $\operatorname{Arctan} 1 - \operatorname{Arctan}(1/2)$.

19. In (a) – (c) $\varphi'(0)$ does not exist. For (a), integrate over $[c, 4]$ and let $c \to 0+$. For (c), the integrand is even so the integral equals $2\int_0^1 (1+t)^{1/2}\,dt$.

20. (b) $\bigcup_n Z_n$ is contained in $\bigcup_{n,k} J_k^n$ and the sum of the lengths of these intervals is $\le \sum_n \varepsilon/2^n = \varepsilon$.

21. (a) The Product Theorem 7.3.16 applies.
(b) We have $\mp 2t\int_a^b fg \le t^2 \int_a^b f^2 + \int_a^b g^2$.
(c) Let $t \to \infty$ in (b).
(d) If $\int_a^b f^2 \ne 0$, let $t = \left(\int_a^b g^2 / \int_a^b f^2\right)^{1/2}$ in (b).

22. Note that $\operatorname{sgn} \circ h$ is Dirichlet's function, which is not Riemann integrable.