Ideas/Defs

Let $\{s_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Define two corresponding sequences $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$, which examine the behavior of the *tails of* $\{s_n\}_{n=1}^{\infty}$, by

$$\begin{aligned} a_k &= \inf_{n\geq k} s_n = \text{glb}\{s_n : n \geq k\} \in \mathbb{R}\cup\{-\infty\} \quad \text{an nondecreasing} \nearrow \text{sequence} \\ b_k &= \sup_{n\geq k} s_n = \text{lub}\{s_n : n \geq k\} \in \mathbb{R}\cup\{\infty\} \quad \text{an nonincreasing} \searrow \text{sequence.} \end{aligned}$$

Since $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ are monotone sequences, their limits exist in the extended ($\widehat{\mathbb{R}}$) sense.
Now define (and examine) the *limit superior* and *limit inferior* of the sequence $\{s_n\}_{n=1}^{\infty}$.

$$\begin{aligned} \liminf_{n\to\infty} s_n &\overset{\text{notation}}{=} \varliminf_{n\to\infty} s_n \overset{\text{def}}{=} \lim_{k\to\infty} \inf_{n\geq k} s_n = \lim_{k\to\infty} a_k \overset{\text{Thm 4.1.6}}{=} \sup_{k\in\mathbb{N}} a_k \in \widehat{\mathbb{R}} \\ \limsup_{n\to\infty} s_n &\overset{\text{notation}}{=} \varlimsup_{n\to\infty} s_n \overset{\text{def}}{=} \lim_{k\to\infty} \sup_{n\geq k} s_n = \lim_{k\to\infty} b_k \overset{\text{Thm 4.1.6}}{=} \inf_{k\in\mathbb{N}} b_k \in \widehat{\mathbb{R}} \end{aligned}$$

Claim 1. $\varliminf_{n\to\infty} s_n \leq \varlimsup_{n\to\infty} s_n$

Claim 2. $\{s_n\}_n$ is bounded below $\Leftrightarrow \varliminf_{n\to\infty} s_n \in \widehat{\mathbb{R}} \setminus \{-\infty\}$.
$\{s_n\}_n$ is bounded above $\Leftrightarrow \varlimsup_{n\to\infty} s_n \in \widehat{\mathbb{R}} \setminus \{+\infty\}$.

Claim 3. Let $\varlimsup_{n\to\infty} s_n \in \mathbb{R}$.

$$\varlimsup_{n\to\infty} s_n \leq \beta \iff \forall\varepsilon > 0 \;\; \exists N \in \mathbb{N} \;\; \forall n \geq N \;\; , \;\; s_n < \beta + \varepsilon$$

Claim 4. Let $\varlimsup_{n\to\infty} s_n \in \mathbb{R}$.

$$\varlimsup_{n\to\infty} s_n \geq \beta \iff \forall\varepsilon > 0 \;\; \forall N \in \mathbb{N} \;\; \exists n \geq N \;\; , \;\; \beta - \varepsilon < s_n$$

Claim 5. Let $\varliminf_{n\to\infty} s_n \in \mathbb{R}$.

$$\varliminf_{n\to\infty} s_n \geq \alpha \iff \forall\varepsilon > 0 \;\; \exists N \in \mathbb{N} \;\; \forall n \geq N \;\; , \;\; \alpha - \varepsilon < s_n$$

Claim 6. Let $\varliminf_{n\to\infty} s_n \in \mathbb{R}$.

$$\varliminf_{n\to\infty} s_n \leq \alpha \iff \forall\varepsilon > 0 \;\; \forall N \in \mathbb{N} \;\; \exists n \geq N \;\; , \;\; s_n < \alpha + \varepsilon$$

Claim 7. $\varliminf_{n\to\infty} s_n = -\varlimsup_{n\to\infty} (-s_n)$

Claim 8. $\lim_{n\to\infty} s_n$ exists in the extended sense (i.e. in $\widehat{\mathbb{R}}$) $\iff \varliminf_{n\to\infty} s_n = \varlimsup_{n\to\infty} s_n$.
In this case, $\varliminf_{n\to\infty} s_n = \lim_{n\to\infty} s_n = \varlimsup_{n\to\infty} s_n$.

Corollary. Let $\overline{\lim}_{n\to\infty} s_n \in \mathbb{R}$. Then $\overline{\lim}_{n\to\infty} s_n = \beta$ if and only if

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \quad , \quad s_n < \beta + \varepsilon \tag{3}$$

and

$$\forall \varepsilon > 0 \quad \forall N \in \mathbb{N} \quad \exists n \geq N \quad , \quad \beta - \varepsilon < s_n \tag{4}$$

Corollary. Let $\underline{\lim}_{n\to\infty} s_n \in \mathbb{R}$. Then $\underline{\lim}_{n\to\infty} s_n = \alpha$ if and only if

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \quad , \quad \alpha - \varepsilon < s_n \tag{5}$$

and

$$\forall \varepsilon > 0 \quad \forall N \in \mathbb{N} \quad \exists n \geq N \quad , \quad s_n < \alpha + \varepsilon \tag{6}$$

Claim 9.

(1) $\overline{\lim}_{n\to\infty} s_n = \infty \Leftrightarrow \forall M \in \mathbb{R}\ \forall N \in \mathbb{N}\ \exists n \geq N$ s.t. $s_n \geq M$

(2) $\underline{\lim}_{n\to\infty} s_n = -\infty \Leftrightarrow \forall M \in \mathbb{R}\ \forall N \in \mathbb{N}\ \exists n \geq N$ s.t. $s_n \leq M$

(3) $\overline{\lim}_{n\to\infty} s_n = -\infty \Leftrightarrow \lim_{n\to\infty} s_n = -\infty$

(4) $\underline{\lim}_{n\to\infty} s_n = \infty \Leftrightarrow \lim_{n\to\infty} s_n = \infty$

Claim 10. There exists a subsequence $\{s_{n_k}\}_{k=1}^{\infty}$ of $\{s_n\}_n$ s.t. $\lim_{k\to\infty} s_{n_k} = \overline{\lim}_{n\to\infty} s_n \in \widehat{\mathbb{R}}$.
Also there exists a subsequence $\left\{s_{n_j}\right\}_{j=1}^{\infty}$ of $\{s_n\}_n$ s.t. $\lim_{j\to\infty} s_{n_j} = \underline{\lim}_{n\to\infty} s_n \in \widehat{\mathbb{R}}$.
Corollaries: • A sequence that is bounded has a subsequence that converges in $\mathbb{R}$.

- A sequence that is not bounded above has a subsequence that converges to ∞.
- A sequence that is not bounded below has a subsequence that converges to $-\infty$.

Claim 11. Let $\{s_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{s_n\}_n$ such that $\lim_{k\to\infty} s_{n_k} \in \widehat{\mathbb{R}}$.
Then $\underline{\lim}_{n\to\infty} s_n \ \leq\ \lim_{k\to\infty} s_{n_k} \leq \overline{\lim}_{n\to\infty} s_n$.

Claim 12.

$$\overline{\lim_{n\to\infty}}\ s_n \ = \ \sup\left\{ \lim_{k\to\infty} s_{n_k} \ : \ \{s_{n_k}\}_{k=1}^{\infty} \text{ is a subsequence of } \{s_n\}_n \text{ s.t. } \lim_{k\to\infty} s_{n_k} \in \widehat{\mathbb{R}} \right\}$$

$$\underline{\lim_{n\to\infty}}\ s_n \ = \ \inf\left\{ \lim_{k\to\infty} s_{n_k} \ : \ \{s_{n_k}\}_{k=1}^{\infty} \text{ is a subsequence of } \{s_n\}_n \text{ s.t. } \lim_{k\to\infty} s_{n_k} \in \widehat{\mathbb{R}} \right\}$$

Claim 13. provided it makes sense:

$$\underline{\lim_{n\to\infty}}\ s_n + \underline{\lim_{n\to\infty}}\ t_n \overset{(2)}{\leq} \underline{\lim_{n\to\infty}}\ (s_n + t_n) \overset{\text{(Claim 1)}}{\leq} \overline{\lim_{n\to\infty}}\ (s_n + t_n) \overset{(1)}{\leq} \overline{\lim_{n\to\infty}}\ s_n + \overline{\lim_{n\to\infty}}\ t_n$$

even more ....

$$\underline{\lim_{n\to\infty}}\ s_n + \underline{\lim_{n\to\infty}}\ t_n \overset{(2)}{\leq} \underline{\lim_{n\to\infty}}\ (s_n + t_n) \overset{(4)}{\leq} \underline{\lim_{n\to\infty}}\ s_n + \overline{\lim_{n\to\infty}}\ t_n \overset{(3)}{\leq} \overline{\lim_{n\to\infty}}\ (s_n + t_n) \overset{(1)}{\leq} \overline{\lim_{n\to\infty}}\ s_n + \overline{\lim_{n\to\infty}}\ t_n$$

(1) takes some work. Then (2) follows from (1) and Claim 7.

(3) follows from (1) and a clever $\triangle$-inequality-like argument. Then (4) follows from (3) and Claim 7.