To help us write our proofs more efficiently (as so to save time on an exam) and clearer, below are **two** color coded proofs from class.
Compare the color coding of the (WTS) to the color coding in the proof.
Do you see that the symbolic writing of the (WTS) provides a format of the proof.

1. Prove that

$$\lim_{n\to\infty} \frac{n^5}{n^2+7n-17} = \infty$$

by using the definition of diverges to infinity. Recall that, by definition, $\lim_{n\to\infty} x_n = \infty$ provided

$$(\forall M \in \mathbb{R})\ (\exists N \in \mathbb{N})\ (\forall n \in \mathbb{N})\ [\ n \geq N \implies x_n > M\ ],$$

or equivalently,

$$\left(\forall M \in \mathbb{R}^{>0}\right)\ (\exists N \in \mathbb{N})\ (\forall n \in \mathbb{N})\ [\ n \geq N \implies x_n > M\ ]. \qquad \text{(WTS)}$$

Proof. Fix $M > 0$. Using the Archimedean property, pick $N \in \mathbb{N}$ such that

$$N > \sqrt[3]{8M}. \qquad (1)$$

Note that (1) gives

$$N^3 > 8M. \qquad (2)$$

Fix $n \in \mathbb{N}$ such that $n \geq N$. Then ⟨note, in (WTS), here have $x_n = \frac{n^5}{n^2+7n-17}$⟩

$$\begin{aligned}\frac{n^5}{n^2+7n-17} &> \frac{n^5}{n^2+7n}\\ &\geq \frac{n^5}{n^2+7n^2}\\ &= \frac{n^5}{8n^2}\\ &= \frac{n^3}{8}\end{aligned}$$

and since $n \geq N$

$$\geq \frac{N^3}{8}$$

and by our choice of N (see (2))

$$\begin{aligned}&\geq \frac{8M}{8}\\ &= M.\end{aligned}$$

We have just shown that if $n \geq N$ then

$$\frac{n^5}{n^2+7n-17} > M.$$

Thus, by definition of diverges to infinity,

$$\lim_{n\to\infty} \frac{n^5}{n^2+7n-17} = \infty,$$

as needed. ☐

2. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences of strictly positive numbers such that

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \infty$$

and

$$\{x_n\}_{n=1}^{\infty} \text{ is bounded above by } U \in \mathbb{R}.$$

Using the definition of convergence, prove that

$$\lim_{n\to\infty} y_n = 0.$$

HINT. Start by fixing $\epsilon > 0$. Since $\frac{x_n}{y_n} \to \infty$, you can make $\frac{x_n}{y_n}$ as big as you need by taking n *big enough*.

. . . .

LTGBG. Want to show

$$(\forall \epsilon > 0)\ (\exists N \in \mathbb{N})\ (\forall n \in \mathbb{N})\ [\ n \geq N \implies |y_n - 0| < \epsilon\ ]. \tag{WTS}$$

. . . .

Proof. LTGBG. We shall show $\lim_{n\to\infty} y_n = 0$.

Fix $\varepsilon > 0$. Since $\lim_{n\to\infty} \frac{x_n}{y_n} = \infty$, using the definition of divergent to ∞, pick $N \in \mathbb{N}$ such that

$$\text{if} \quad n \geq N \qquad \text{then} \quad \frac{U}{\varepsilon} < \frac{x_n}{y_n}. \tag{1}$$

Note that (1) gives, since $U \geq x_{17} > 0$ and each of ε, x_n, and y_n is strictly positive,

$$\text{if} \quad n \geq N \qquad \text{then} \quad 0 < \frac{y_n}{x_n} < \frac{\varepsilon}{U}. \tag{2}$$

Fix $n \in \mathbb{N}$ such that $n \geq N$. Then, since $y_n > 0$

$$|y_n - 0| = y_n$$

and since $x_n \neq 0$

$$= \frac{y_n}{x_n}\, x_n$$

and since U is an upper bound of $\{x_n\}_n$

$$\leq U\, \frac{y_n}{x_n}$$

and by (2)

$$< U\, \frac{\varepsilon}{U}$$

$$= \varepsilon.$$

We have just shown that if $n \geq N$ then

$$|y_n - 0| < \varepsilon.$$

Thus, by definition of convergence, we have that $\lim_{n\to\infty} y_n = 0$, as needed. ☐