▶. **Goal**. To define preciously: $\lim_{x \to p} f(x) = L$ where $f\colon D_f \to \mathbb{R}$ and $p, L \in \widehat{\mathbb{R}}$ and D_f is: an interval of $\mathbb{R}$, or $\mathbb{R}$, or $\mathbb{N}$.

☺. We will find one Limit Recipe (LR) that gives the definition for the 81 cases of choices for: p, L, D_f.

Set-Up for this handout

▷. Let $f\colon D_f \to \mathbb{R}$ and D_f, p, and L are as in above goal.

▷. Let $T = \mathbb{R}$. ⟨think of T as a subset of the function f's domain (so x-axis) or codomain (so y-axis)⟩

▷. Let $t_0 \in \widehat{\mathbb{R}}$, and if $t_0 \in \mathbb{R}$ we can also have t_0^+ and t_0^-. ⟨t_0 will be p (on x-axis) or L (on y-axis)⟩

Neighborhood (NH)

▷. $\mathrm{NH}_T(t_0)$ denotes an arbitrary **neighborhood about** t_0 (in T).

▷. The definition/form of $\mathrm{NH}_T(t_0)$ depends of form of t_0. ⟨think of $\mathrm{NH}_T(t_0)$ as all points *close* to t_0⟩

▷. When $t_0 \in \mathbb{R}$, an arbitrary neighborhood is, for some $\varepsilon > 0$,

1. $N_\varepsilon(t_0) \overset{\text{def}}{=} \{t \in T\colon d(t, t_0) < \varepsilon\} \overset{\text{i.e.}}{=} \{t \in T\colon |t - t_0| < \varepsilon\} \overset{\text{i.e.}}{=} (t_0 - \varepsilon, t_0 + \varepsilon)$
2. $N_\varepsilon(t_0^+) \overset{\text{def}}{=} (t_0, t_0 + \varepsilon)$
3. $N_\varepsilon(t_0^-) \overset{\text{def}}{=} (t_0 - \varepsilon, t_0)$.

▷. When $t_0 \in \{\pm\infty\}$, an arbitrary neighborhood is, for some $M \in \mathbb{R}$,

4. $N_M(^+\infty) \overset{\text{def}}{=} \{t \in T\colon M < t\} \overset{\text{i.e.}}{=} (M, \infty)$
5. $N_M(^-\infty) \overset{\text{def}}{=} \{t \in T\colon t < M\} \overset{\text{i.e.}}{=} (-\infty, M)$.

▷. E.g., the collection of all neighborhoods $\mathrm{NH}_T(t_0)$ will be,
∘ when $t_0 \in \mathbb{R}$, $\{N_\varepsilon(t_0) : \varepsilon > 0\} \overset{\text{i.e.}}{=} \{\{t \in T\colon d(t, t_0) < \varepsilon\} : \varepsilon > 0\} \overset{\text{i.e.}}{=} \{(t_0 - \varepsilon, t_0 + \varepsilon) : \varepsilon > 0\}$
∘ when $t_0 \in \infty$, $\{N_M(\infty) : M \in \mathbb{R}\} \overset{\text{i.e.}}{=} \{\{t \in T\colon M < t\} : M \in \mathbb{R}\} \overset{\text{i.e.}}{=} \{(M, \infty) : M \in \mathbb{R}\}$.

Deleted Neighborhood (NH′)

Def. The deleted neighborhood of $\mathrm{NH}_T(t_0)$ is ⟨delete-out t_0 from $\mathrm{NH}_T(t_0)$⟩

$$\mathrm{NH}'_T(t_0) \overset{\text{def}}{=} \mathrm{NH}_T(t_0) \setminus \{t_0\}.$$

Rmk. Note $\mathrm{NH}'_T(t_0) = \mathrm{NH}_T(t_0) \Leftrightarrow t_0 \notin \mathrm{NH}_T(t_0)$
∘ So $\mathrm{NH}'_T(p) = \mathrm{NH}_T(p)$ when $p = \pm\infty$ or, for some $t_0 \in \mathbb{R}$, the p is t_0^+ or t_0^-
∘ and $\mathrm{NH}'_T(p) \neq \mathrm{NH}_T(p)$ when $p \in \mathbb{R}$.

Limit Point of the set D_f

Def. We say $p \in \widehat{\mathbb{R}}$ is a limit point of the set D_f $\Leftrightarrow$
each deleted neighborhood $\mathrm{NH}'_{\mathbb{R}}(p)$ contains a point in D_f.
Thus $p \in \widehat{\mathbb{R}}$ is not a limit point of the set D_f $\Leftrightarrow$
there is a deleted neighborhood $\mathrm{NH}'_{\mathbb{R}}(p)$ that does not contain a point from D_f.

Ex. The set of limit points of $[a, b]$ is $[a, b]$. The set of limit points of (a, b) is $[a, b]$.
The set of limit points of $\mathbb{N}$ is $\{\infty\}$.

Warning

⚠. The notation $\mathrm{NH}_T(t_0)$ is for *Thinking Land* only and never should be used in a proof.
In proofs, use notation that specify how the NH look, e.g.: $N_\varepsilon(p)$, $(p - \varepsilon, p + \varepsilon)$, $N_M(\infty)$, (M, ∞).

Recall. Have $f\colon D_f \to \mathbb{R}$ where D_f is: an interval of $\mathbb{R}$, or $\mathbb{R}$, or $\mathbb{N}$.
Think of as $f\colon X \cap D_f \to Y$ where $X = \mathbb{R} = Y$.
Also have $p, L \in \widehat{\mathbb{R}}$ (or, for some $x_0 \in \mathbb{R}$, the p can be x_0^+ or x_0^-).
We require p to be a limit point of D_f. $\langle$guaranties the uniqueness of the limit $L\rangle$

LR. The Limit Recipe (LR) for

$$\lim_{x\to p} f(x) = L \tag{L}$$

is

$$\underbrace{(\forall\ \mathrm{NH}_Y(L))}_{1^{\text{st}}\text{ step}}\ \underbrace{(\exists\ \mathrm{NH}_X(p))}_{2^{\text{nd}}\text{ step}}\ \underbrace{(\forall x \in D_f)\ [\,x \in \mathrm{NH}'_X(p)}_{3^{\text{rd}}\text{ step}}\ \underbrace{\Rightarrow\ f(x) \in \mathrm{NH}_Y(L)}_{4^{\text{th}}\text{ step}}\,]. \tag{LR}$$

The steps of a proof are indicated.

Rmk. The LR provides (the $\boxed{\forall\exists}$) definitions of the limit for the various setting.

Ex. Let $f\colon \mathbb{N} \to \mathbb{R}$ and $L \in \mathbb{R}$. Give the $\boxed{\forall\exists}$ definition for $\lim_{n\to\infty} f(n) = L$.