Setup. Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the natural numbers and $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ be the integers.
Let $P(n)$ be an *open sentence* (also called *predicate*) in the variable n, where n is in a subset S of $\mathbb{Z}$.
So when a specified value for $n \in S$ is assigned, $P(n)$ is a statement. Sometimes we denote $P(n)$ by P_n.
For a $n_0 \in \mathbb{N}$, let $\mathbb{N}^{\geq n_0} := \{n \in \mathbb{N} \colon n \geq n_0\} \overset{\text{i.e.}}{=} \{n_0, n_0+1, n_0+2, n_0+3, \dots\} \overset{\text{so}}{\subset} \mathbb{N}$.
For a $n_0 \in \mathbb{Z}$, let $\mathbb{Z}^{\geq n_0} := \{n \in \mathbb{Z} \colon n \geq n_0\} \overset{\text{i.e.}}{=} \{n_0, n_0+1, n_0+2, n_0+3, \dots\} \overset{\text{so}}{\subset} \mathbb{Z}$.

INDUCTION

1. Induction (basic form)

If

BASE STEP: $P(1)$ is true

INDUCTIVE STEP: for each $n \in \mathbb{N}$: $\underbrace{[\ P(n) \text{ is true}\ ]}_{\text{inductive hypothesis}} \implies \underbrace{[\ P(n+1) \text{ is true}\ ]}_{\text{inductive conclusion}}$

then $P(n)$ is true for each $n \in \mathbb{N}$.

2. Induction (doesn't matter where you start form) ⟨instead of starting at 1, let's start at $n_0 \in \mathbb{Z}$⟩

Fix $n_0 \in \mathbb{Z}$.
If

BASE STEP: $P(n_0)$ is true

INDUCTIVE STEP: for each $n \in \mathbb{Z}^{\geq n_0}$: $\underbrace{[\ P(n) \text{ is true}\ ]}_{\text{inductive hypothesis}} \implies \underbrace{[\ P(n+1) \text{ is true}\ ]}_{\text{inductive conclusion}}$

then $P(n)$ is true for each $n \in \mathbb{Z}^{\geq n_0}$.

3. Strong Induction (also called complete induction)

Fix $n_0 \in \mathbb{Z}$.
If

BASE STEP: $P(n_0)$ is true

INDUCTIVE STEP: for each $n \in \mathbb{Z}^{\geq n_0}$: $\underbrace{[\ P(j) \text{ is true for } j \in \{n_0, 1+n_0, \dots, n\}\ ]}_{\text{inductive hypothesis}} \Rightarrow \underbrace{[\ P(n+1) \text{ is true}\ ]}_{\text{inductive conclusion}}$

then $P(n)$ is true for each $n \in \mathbb{Z}^{\geq n_0}$.

GUIDELINES FOR WRITING AN INDUCTION PROOF

When writing an induction proof, remember to *keep your reader informed*; thus, you should:

(1) say what you are trying to show inductively
(2) say what your induction variable is (e.g., if you are trying to show $(\forall j \in \mathbb{N})\, [P(j)]$ then say: We will show that *blub* holds for each $j \in \mathbb{N}$ by induction on j.)
(3) indicate where your base step begins
(4) indicate where your base step ends
(5) indicate where your inductive step begins
(6) clearly state your inductive hypothesis (IH)
(7) clearly state your inductive conclusion (IC)
(8) indicate where your inductive step ends.

1. Show that $\prod_{i=1}^{n+1}\left(1-\frac{1}{i+1}\right)=\frac{1}{n+2}$ for each $n \in \mathbb{N}$.

Proof. We shall show that for each $n \in \mathbb{N}$

$$\prod_{i=1}^{n+1}\left(1-\frac{1}{i+1}\right)=\frac{1}{n+2} \tag{1}$$

by induction on n.

For the base step, let $n=1$. Then

$$\prod_{i=1}^{n+1}\left(1-\frac{1}{i+1}\right)=\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)=\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)=\frac{2}{6}=\frac{1}{3}=\frac{1}{1+2}=\frac{1}{n+2}.$$

Thus equation (1) holds when $n=1$. This concludes the base step.

For the inductive step, let $n \in \mathbb{N}$ and assume that

$$\prod_{i=1}^{n+1}\left(1-\frac{1}{i+1}\right)=\frac{1}{n+2}. \tag{IH}$$

We will show that

$$\prod_{i=1}^{(n+1)+1}\left(1-\frac{1}{i+1}\right)=\frac{1}{(n+1)+2}. \tag{IC}$$

Using the definition of a product we can see that

$$\prod_{i=1}^{n+2}\left(1-\frac{1}{i+1}\right)=\left(\prod_{i=1}^{n+1}\left(1-\frac{1}{i+1}\right)\right)\left(1-\frac{1}{(n+2)+1}\right)$$

and by the inductive hypothesis (IH)

$$=\left(\frac{1}{n+2}\right)\left(1-\frac{1}{n+3}\right)$$

and now by some algebra

$$\begin{aligned}
&=\left(\frac{1}{n+2}\right)\left(\frac{(n+3)-1}{n+3}\right)\\
&=\left(\frac{1}{n+2}\right)\left(\frac{n+2}{n+3}\right)\\
&=\frac{1}{n+3}\\
&=\frac{1}{(n+1)+2}\ .
\end{aligned}$$

Therefore the inductive conclusion (IC) holds. This completes the inductive step.

Thus, by induction, equation (1) holds for each $n \in \mathbb{N}$. □

2. Show that $2^n > n^2$ each natural number n strictly larger than 4.

Proof. We shall show that if $n \in \mathbb{N}^{\geq 5}$ then

$$2^n > n^2 \tag{1}$$

by induction on n.

For the base step, let $n = 5$. Then

$$2^n = 2^5 = 32 > 25 = 5^2 = n^2.$$

Thus, inequality (1) holds when $n = 5$.

For the inductive step, fix a natural number $n \in \mathbb{N}^{\geq 5}$. Assume that

$$2^n > n^2. \tag{IH}$$

We need to show that

$$2^{(n+1)} > (n+1)^2. \tag{IC}$$

Using the inductive hypothesis (IH) and algebra, we have

$$\begin{aligned} 2^{(n+1)} &= 2(2^n) \\ &> 2(n^2) \\ &= n^2 + n^2 \\ &= n^2 + (n)(n) \end{aligned}$$

and since $n > 4$

$$\begin{aligned} &> n^2 + (4)(n) \\ &= n^2 + 2n + 2n. \end{aligned}$$

and since $n > 4$ it is clear that $2n > 1$ so

$$\begin{aligned} &> n^2 + 2n + 1 \\ &= (n+1)^2. \end{aligned}$$

Therefore, the inequality (IC) holds. This completes the inductive step.

Thus, by induction, inequality (1) holds for each natural number $n \in \mathbb{N}^{\geq 5}$. □

3. Each natural number n has a factorization as

$$n = 2^k m$$

for some k is some nonnegative integer and some odd natural number m.
Written symbolically: $(\forall n \in \mathbb{N})\ (\exists k \in \mathbb{Z})\ (\exists m \in \mathbb{N})\ \big[k \geq 0\ \wedge\ m \text{ is odd }\ \wedge\ n = 2^k m\big]$.

Proof. We shall show that if $n \in \mathbb{N}$ then n can be written as

$$n = 2^k m \qquad \text{for some } k \in \mathbb{N} \cup \{0\} \text{ and odd natural number } m \tag{1}$$

by strong induction on n.

For the base step, let $n = 1$. Then

$$n = 1 = 2^0 \cdot 1 = 2^k m$$

where $k = 0 \in \mathbb{N} \cup \{0\}$ and $m = 1 \in \mathbb{N}$ is odd. Thus (1) holds when $n = 1$. This completes the base step.

For the inductive step, fix $n \in \mathbb{N}$ and assume the inductive hypotheses, which is that for each $j \in \{1, 2, \ldots, n\}$

$$j = 2^a b \qquad \text{for some } a \in \mathbb{N} \cup \{0\} \text{ and odd natural number } b. \tag{IH}$$

To show the inductive conclusion, which is

$$n + 1 = 2^k m \qquad \text{for some } k \in \mathbb{N} \cup \{0\} \text{ and odd natural number } m, \tag{IC}$$

we consider two cases: n is even and n is odd.

For the first cases, let n be an even integer. Then $n+1$ is odd and so

$$n + 1 = 2^0\,(n+1) = 2^k m$$

where $k = 0 \in \mathbb{N} \cup \{0\}$ and $m = n + 1$ is an odd integer. Thus (IC) holds for the first case.

For the second case, let n be an odd integer. Then $n+1$ is even; thus, there is $i \in \mathbb{N}$ such that

$$n + 1 = 2i. \tag{2}$$

Note that $i \in \{1, 2, \ldots, n\}$ since $i \in \mathbb{N}$ and

$$1 \leq i < 2i = n + 1.$$

Thus by the inductive hypotheses (IH), there exists $a \in \mathbb{N} \cup \{0\}$ and an odd natural number b such that

$$i = 2^a b. \tag{3}$$

Equations (2) and (3) give,

$$n + 1 = 2i = 2\,(2^a b) = 2^{a+1} b = 2^k m$$

where $m{:=}b$ is an odd natural number and $k{:=}a + 1 \in \mathbb{N} \cup \{0\}$ (since $a \in \mathbb{N} \cup \{0\}$). Thus (IC) holds for the second case.

This completes the inductive step. So, by strong induction, equation (1) holds for all $n \in \mathbb{N}$. ☐