Remark: ER 3.2.51, ER 3.2.52, ER 3.2.53 go together.
In ER 3.2.52 be careful not to make a mistake similar to the mistake in ER 3.2.51.

ER 3.2.51 Find and **explain** the math mistake in the below Flawed Proof of Thm. 1. §3.2 BS4p69-70
In your explanation, refer to the line numbers on the left and the tag number on the right.

Thm. 1. Let $\{s_n\}_n$ and $\{t_n\}_n$ be convergent sequences. Then $\lim_{n\to\infty}(s_n t_n) = \left(\lim_{n\to\infty} s_n\right)\left(\lim_{n\to\infty} t_n\right)$. §3.2 BS4

1 ***Flawed Proof***. Let

$$\lim_{n\to\infty} s_n = S \in \mathbb{R} \qquad \text{and} \qquad \lim_{n\to\infty} t_n = T \in \mathbb{R}. \tag{1.1}$$

2 We will show that $\lim_{n\to\infty} s_n t_n = ST$.

3 Let $\varepsilon > 0$. Since $\lim_{n\to\infty} s_n = S$, there exists $N_S \in \mathbb{N}$ such that if $n \geq N_S$ then

$$|s_n - S| < \left(\frac{1}{|T|+1}\right)\left(\frac{\varepsilon}{2}\right) \tag{1.2}$$

4 Since $\lim_{n\to\infty} t_n = T$, there exists $N_T \in \mathbb{N}$ such that if $n \geq N_T$ then

$$|t_n - T| < \left(\frac{1}{|s_n|+1}\right)\left(\frac{\varepsilon}{2}\right). \tag{1.3}$$

5 Set

$$N = \max\{N_S, N_T\}.$$

6 Let $n \geq N$ with $n \in \mathbb{N}$. Then

$$\begin{aligned} |s_n t_n - ST| &= |s_n t_n - s_n T + s_n T - ST| && (1.4)\\ &= |s_n(t_n - T) + T(s_n - S)| && (1.5) \end{aligned}$$

7 and by the triangle inequality

$$\begin{aligned} &\leq |s_n(t_n - T)| + |T(s_n - S)| && (1.6)\\ &= |s_n|\,|t_n - T| + |T|\,|s_n - S| && (1.7) \end{aligned}$$

8 and by (1.3) and (1.2)

$$\begin{aligned} &< |s_n|\left(\frac{1}{|s_n|+1}\right)\left(\frac{\varepsilon}{2}\right) + |T|\left(\frac{1}{|T|+1}\right)\left(\frac{\varepsilon}{2}\right) && (1.8)\\ &= \left(\frac{|s_n|}{|s_n|+1}\right)\left(\frac{\varepsilon}{2}\right) + \left(\frac{|T|}{|T|+1}\right)\left(\frac{\varepsilon}{2}\right) && (1.9)\\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} && (1.10)\\ &= \varepsilon. && (1.11) \end{aligned}$$

9 Thus, for the fixed $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n \geq N$ then $|s_n t_n - ST| < \varepsilon$.

10 Since $\varepsilon > 0$ was arbitrary, $\lim_{n\to\infty} s_n t_n = ST$. □

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