Variant of book's ER 2.5.10 Nested Interval Property. §2.5 BS4p53

Let $I_n = [a_n, b_n] \subseteq \mathbb{R}$ with $a_n \leq b_n$. Let the I_n's be nested with $I_n \supseteq I_{n+1}$ for each $n \in \mathbb{N}$.

Let $A = \{a_j \colon j \in \mathbb{N}\}$ and $a_0 = \sup A$.

Let $B = \{b_k \colon k \in \mathbb{N}\}$ and $b_0 = \inf B$.

Prove that

$$\bigcap_{n \in \mathbb{N}} I_n = [a_0, b_0]\,. \tag{1}$$

Hint. In class we showed (thus you can use) the following claims.

Claim 1. The a_j's are increasing.

Claim 2. The b_k's are decreasing.

Claim 3. If $j, k \in \mathbb{N}$ then $a_j \leq b_k$.

Claim 4. $a_0 \in \bigcap_{n \in \mathbb{N}} I_n$.

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