Definition. Let $a \in \mathbb{R}$ and $\varepsilon > 0$. Then the ε-neighborhood about a is the set

$$N_\varepsilon(a) \;:=\; \{x \in \mathbb{R} \colon\ |x-a| < \varepsilon\}\,.$$

Recall for two real numbers x and a, the distance $d(x,a)$ between x and a is

$$d(x,a) \;:=\; |x-a|$$

and for an $\varepsilon > 0$

$$|x-a| < \varepsilon \quad \Longleftrightarrow \quad -\varepsilon < x-a < \varepsilon \quad \Longleftrightarrow \quad a-\varepsilon < x < a+\varepsilon\ .$$

Thus we can think of an ε-neighborhood of a in several ways (see book's Figure 2.2.4 on page 35): Def2.2.7 BS4p35

$$\begin{aligned} N_\varepsilon(a) &= \{x \in \mathbb{R} \colon d(x,a) < \varepsilon\} \\ &= \{x \in \mathbb{R} \colon\ |x-a| < \varepsilon\} \\ &= \{x \in \mathbb{R} \colon\ -\varepsilon < x-a < \varepsilon\} \\ &= \{x \in \mathbb{R} \colon a-\varepsilon < x < a+\varepsilon\} \\ &= (a-\varepsilon, a+\varepsilon) \quad \rightsquigarrow \text{an open interval} \end{aligned}$$

An ε-neighborhood about a is also called an ε-neighborhood of a.

BTW: The notation $N_\varepsilon(a)$ for a Neighborhood is more common than the book's notation of $V_\varepsilon(a)$.

Variant of book's ER 2.2.17 §2.2 BS4p36

Let $a_1, a_2 \in \mathbb{R}$ such that $a_1 \neq a_2$.

1. Find an $\varepsilon > 0$ such that $N_\varepsilon(a_1)$ and $N_\varepsilon(a_2)$ are disjoint.

Explain geometrically (not algebraically) why your ε works.

2. Finish the statement of Lemma 1 below by giving an upper bound for ε (natually, ε is strictly positive).

Your solution should be of the form $0 < \varepsilon <$(somthing) or $0 < \varepsilon \leq$ (somthing). ⟨Justification not needed.⟩

Lemma 1. Let $a_1, a_2 \in \mathbb{R}$ such that $a_1 \neq a_2$. Then $N_\varepsilon(a_1) \cap N_\varepsilon(a_2) = \emptyset$ if and only if $0 < \varepsilon \ldots$

. . . .