ER 1.3.51 Countability. Cantor Diagonalization Argument. §1.3.51 BS4p22

1. Let A be the set of all *finite* sequences of 0's and 1's. To clarify, let

$$A \;=\; \left\{ \; (a_i)_{i=1}^{n} : n \in \mathbb{N} \text{ and } a_i \in \{0,1\} \text{ for each } i \in \mathbb{N}^{\leq n} \; \right\}$$

Prove that A is countably infinite, i.e., is enumerable. Hint. Helpful is Ideas from ER 1.3.12 and 1.3.13.

2. Let B be the set of all *infinite* sequences of 0's and 1's. To clarify,

$$B \;=\; \{ \; (b_i)_{i=1}^{\infty} : b_i \text{ is 0 or 1 for each } i \in \mathbb{N} \; \}$$

Prove that B is uncountable by using a Cantor Diagonalizational Argument.

Recall the picture for the Cantor Diagonalizational Argument we used to show $\mathbb{R}$ is uncountable.

$$\begin{aligned}
x^1 &= 0.d_1^1 \, d_2^1 \, d_3^1 \, d_4^1 \, d_5^1 \cdots \\
x^2 &= 0.d_1^2 \, d_2^2 \, d_3^2 \, d_4^2 \, d_5^2 \cdots \\
x^3 &= 0.d_1^3 \, d_2^3 \, d_3^3 \, d_4^3 \, d_5^3 \cdots \\
x^4 &= 0.d_1^4 \, d_2^4 \, d_3^4 \, d_4^4 \, d_5^4 \cdots \\
&\;\;\vdots
\end{aligned} \tag{CDA}$$

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