TS§3.4 p146-147

▶. **Proposition 3.27**. If n is an integer, then 3 divides $n^3 - n$.
Symbolically write: $(\forall n \in \mathbb{Z})\ [\ 3 \mid (n^3 - n)\ ]$.

▷. Do any proof methods that we have learned so far seem as if they would work?
Well ... not really. So let's try a proof by cases. (If need be, review (the linked) Pascal's triangle.)

Proof. Let $n \in \mathbb{Z}$. We will show that 3 divides $n^3 - n$ by examining the cases for the remainder when n is divided by 3. By the Division Algorithm, there exist unique integers q and r such that

$$n = 3q + r \tag{1}$$

and

$$r \in \{0, 1, 2\}. \tag{2}$$

We will examine ⟨each of⟩ the 3 possible cases in (2) for the remainder: (1) r=0, (2) r=1, (3) r=2.

For case 1, let $r = 0$. By (1) we have $n = 3q$ for some $q \in \mathbb{Z}$. <Let our WTS guide our algebra. WTF$_{\text{ind}}$ $j \in \mathbb{Z}$ s.t. $3j = n^3 - n$, i.e., $n^3 - n = 3j$.> Thus

$$\begin{aligned} n^3 - n &= (3q)^3 - (3q) \\ &= 3^3 q^3 - 3q \\ &= 3\left(3^2 q^3 - q\right) \\ &= 3j_1 \end{aligned} \tag{3}$$

where $j_1 = 3^2 q^3 - q$. Note $j \in \mathbb{Z}$ by the closure properties of $\mathbb{Z}$ since $q \in \mathbb{Z}$. Thus the calculation in (3) show that $3 \mid (n^3 - n)$. This complete case 1.

For case 2, let $r = 1$. By (1) we have $n = 3q + 1$ for some $q \in \mathbb{Z}$. Thus ⟨use Pascal's triangle⟩

$$\begin{aligned} n^3 - n &= (3q+1)^3 - (3q+1) \\ &= \left(1 \cdot 3^3 q^3 + 3 \cdot 3^2 q^2 + 3 \cdot 3^1 q + 1 \cdot 1\right) - (3q+1) \\ &= 3^3 q^3 + 3^3 q^2 + 3\left(3^1 - 1\right) q + 0 \\ &= 3\left(3^2 q^3 + 3^2 q^2 + 2q\right) \\ &= 3j_2 \end{aligned} \tag{4}$$

where $j_2 = 3^2 q^3 + 3^2 q^2 + 2q$. Note $j_2 \in \mathbb{Z}$ by the closure properties of $\mathbb{Z}$ since $q \in \mathbb{Z}$. Thus the calculation in (4) show that $3 \mid (n^3 - n)$. This complete case 2.

For case 3, let $r = 2$. By (1) we have $n = 3q + 2$ for some $q \in \mathbb{Z}$. Thus ⟨use Pascal's triangle⟩

$$\begin{aligned} n^3 - n &= (3q+2)^3 - (3q+2) \\ &= \left[\ 1 \cdot \left(3^3 q^3\right) + 3 \cdot \left(3^2 q^2\right)\ \left(2^1\right) + 3 \cdot \left(3^1 q^1\right)\ \left(2^2\right) + 1 \cdot \left(2^3\right)\ \right] - (3q+2) \\ &= 3^3 q^3 + 3^3 \cdot 2 \cdot q^2 + 3\left(3^1 \cdot 2^2 - 1\right) q^1 + 6 \\ &= 3\left(3^2 q^3 + 3^2 \cdot 2 \cdot q^2 + \left(3 \cdot 2^2 - 1\right) q^1 + 2\right). \\ &= 3j_3 \end{aligned} \tag{5}$$

where $j_3 = 3^2 q^3 + 3^2 \cdot 2q^2 + \left(3 \cdot 2^2 - 1\right) q + 2$. Note $j_3 \in \mathbb{Z}$ by the closure properties of $\mathbb{Z}$ since $q \in \mathbb{Z}$. Thus the calculation in (5) show that $3 \mid (n^3 - n)$. This complete case 3.

We have just shown that for each possible case $3 \mid (n^3 - n)$. Thus have show that if $n \in \mathbb{Z}$ then $3 \mid (n^3 - n)$. □

▶. Question 1. Can we simplify the algebra some by using the Division Algorithm$^+$?

▶. Question 2. Can we prove this proposition by using Modulo Arithmetic instead of the Division Algorithm (or Division Algorithm$^+$)?