Practice Problems from Section 3.6. Exercises 3.6.1–3.6.13 except for: 2, 10, 13.
These *Practice Problems* are a sampling of the type of problems which could be on the exam.
These *Practice Problem* are, in no way, meant as a comprehensive review for the exam.

Math is not a spectator sport.
Often we learn more from our failed attempts at a proof rather than immediately looking at the hints or reading a clean proof.
So give these problems a solid attempt before seeking help, e.g.:
looking through: your notes, the texbook, the homework.
Since these problems are not to hand in, you may share:
hints and/or an attempt at a solution for others to comment on.

Some hints are very generous. Do not except such generous hints on the exam.

1. **Theorem 1**. If $n \in \mathbb{N}$ with $n \geq 2$, then

$$\prod_{i=2}^{n} \frac{i^2-1}{i^2} = \frac{n+1}{2n} .$$

1.1. Symbolically write Theorem 1. (Do not forget your quantifiers.)

1.2. Prove Theorem 1 using math induction..

∘. Recall, the sum $\sum_{i=2}^{5} a_i = a_2 + a_3 + a_4 + a_5$ while the product $\prod_{i=2}^{5} a_i = a_2 \cdot a_3 \cdot a_4 \cdot a_5$

hint. The sum $\sum_{i=2}^{n+1} a_i = (a_2 + a_3 + \ldots + a_n) + a_{n+1} = \left(\sum_{i=2}^{n}\right) + a_{n+1}$. This idea is often used in the inductive step. Similarly, for the product $\prod_{i=2}^{n+1} a_i = (a_2 \cdot a_3 \cdot \;\cdots\; \cdot a_n) \cdot a_{n+1} = \left(\prod_{i=2}^{n} a_i\right) \cdot a_{n+1}$.

sw. $\left(\forall n \in \mathbb{N}^{\geq 2}\right) \left[\prod_{i=2}^{n} \frac{i^2-1}{i^2} = \frac{n+1}{2n} \right]$

Proof. We will show that for each natural number n with $n \geq 2$,

$$\prod_{i=2}^{n} \frac{i^2-1}{i^2} = \frac{n+1}{2n} \tag{1.1}$$

by basic induction on the variable n.

For the base step, let $n = 2$. Since

$$\prod_{i=2}^{n} \frac{i^2-1}{i^2} = \prod_{i=2}^{2} \frac{i^2-1}{i^2} = \frac{2^2-1}{2^2} = \frac{3}{4} \qquad \text{and} \qquad \frac{n+1}{2n} = \frac{2+1}{2\,(2)} = \frac{3}{4}.$$

the equation in (1.1) is true when $n = 2$. This completes the base step.

For the inductive step, fix $n \in \mathbb{N}^{\geq 2}$. We assume the inductive hypothesis, which is

$$\prod_{i=2}^{n} \frac{i^2-1}{i^2} = \frac{n+1}{2n}. \tag{IH}$$

We will show the inductive conclusion, which is

$$\prod_{i=2}^{n+1} \frac{i^2-1}{i^2} = \frac{(n+1)+1}{2\,(n+1)}. \tag{IC}$$

We know that ⟨recall $\prod_{i=2}^{n+1} a_i = \left[\prod_{i=2}^{n} a_i\right] \cdot a_{n+1}$⟩

$$\prod_{i=2}^{n+1} \frac{i^2-1}{i^2} = \left[\prod_{i=2}^{n} \frac{i^2-1}{i^2}\right]\left[\frac{(n+1)^2-1}{(n+1)^2}\right].$$

and using the inductive hypothsis (IH) gives

$$= \left[\frac{n+1}{2n}\right]\left[\frac{(n+1)^2-1}{(n+1)^2}\right].$$

and now through algebra ⟨note the algebraic cancellation in next step, which greatly simplies the algebra to come - a good lesson to learn and practice⟩

$$\begin{aligned} &= \frac{(n+1)^2-1}{2n(n+1)} \\ &= \frac{n^2+2n}{2n(n+1)} \\ &= \frac{n}{n}\,\frac{(n^1+2)}{2(n+1)} \\ &= \frac{(n+1)+1}{2(n+1)}. \end{aligned} \tag{1.2}$$

The calulations through (1.2) shows that the inductive conclusion (IC) holds. This completes the inductive step.

Thus the base step and inductive step hold. □

2. **Theorem 2**. If $n \in \mathbb{N}$ with $n \geq 3$, then

$$2n+1 < 2^n .$$

2.1. Symbolically write Theorem 2. (Do not forget your quantifiers.)

2.2. Prove Theorem 2 using math induction.

sw. $\left(\forall n \in \mathbb{N}^{\geq 3}\right)\ [\ 2n+1 < 2^n\ ]$

Proof. We will show that if $n \in \mathbb{N}^{\geq 3}$, then

$$2n+1 < 2^n \tag{2.1}$$

by basic induction on the variable n.

For the base step, let $n = 3$. Since

$$2n+1 = 2\,(3)+1 = 7 \qquad \text{and} \qquad 2^n = 2^3 = 8$$

and $7 < 8$, the equation in (2.1) is true when $n = 3$. This completes the base step.

For the inductive step, fix $n \in \mathbb{N}^{\geq 3}$. We assume the inductive hypothesis, which is

$$2n+1 < 2^n. \tag{IH}$$

We will show the inductive conclusion, which is

$$2\,(n+1)+1 < 2^{n+1}. \tag{IC}$$

From algebra we know

$$\begin{aligned} 2\,(n+1)+1 &= 2n+2+1 \\ &= (2n+1)+2 \end{aligned}$$

and by the (IH)

$$< (2^n)+2$$

and since $3 \le n$ we get $2 < 2^3 \le 2^n$ so

$$\begin{aligned} &< 2^n + 2^n \\ &= 2^1\left(2^n\right) \\ &= 2^{n+1} \end{aligned}$$

and so the inductive conclusion (IC) holds. This completes the inductive step.

Thus the base step and inductive step hold. □

3. **Theorem 3**. If $n \in \mathbb{Z}$ with $n \ge 0$, then

$$3 \mid \left(n^3 + 2n\right).$$

3.1. Symbolically write Theorem 3. (Do not forget your quantifiers.)

3.2. Prove Theorem 3 using math induction.

hint. See book Proposition 4.4 (p178) and (starred) ER 3.4.8a (p181).

sw. $\left(\forall n \in \mathbb{Z}^{\ge 0}\right)\ \left[\ 3 \mid \left(n^3 + 2n\right)\ \right]$

Proof. We will show that if $n \in \mathbb{Z}^{\ge 0}$, then

$$3 \mid \left(n^3 + 2n\right) \tag{3.1}$$

by basic induction on the variable n.

For the base step, let $n = 0$. Then

$$n^3 + 2n = 0^3 + 2(0) = 0.$$

Also

$$3 | 0$$

since $0 = 3j$ where $j = 0$ and $j \in \mathbb{Z}$. Thus (3.1) is true when $n = 0$. This completes the base step.

For the inductive step, fix $n \in \mathbb{Z}^{\ge 0}$. We assume the inductive hypothesis, which is

$$3 \mid \left(n^3 + 2n\right). \tag{IH}$$

We will show the inductive conclusion, which is

$$3 \mid \left(\ (n+1)^3 + 2(n+1)\ \right). \tag{IC}$$

By the (IH), there is a $j \in \mathbb{Z}$ such that

$$n^3 + 2n = 3j. \tag{3.2}$$

From algebra we get

$$\begin{aligned} (n+1)^3 + 2(n+1) &= \left(n^3 + 3n^2 + 3n + 1\right) + (2n + 2) \\ &= \left[\ n^3 + 2n\right] + \left(3n^2 + 3n + 1 + 2\right) \\ &= \left[\ n^3 + 2n\right] + 3\left(n^2 + n + 1\right) \end{aligned}$$

and by (3.2) ⟨which came from the (IH)⟩ and then algebra

$$\begin{aligned} &= [\ 3j\ ] + 3\left(n^2 + n + 1\right) \\ &= 3\left(j + n^2 + n + 1\right) \\ &= 3k \end{aligned} \tag{3.3}$$

where $k = j + n^2 + n + 1$. Since $k \in \mathbb{Z}$, the calculation through (3.3) show that the inductive conclusion (IC) holds. This completes the inductive step.

Thus the base step and inductive step hold. □

4. **Theorem 4**. If $n \in \mathbb{Z}$ with $n \geq 0$, then

$$8 \mid (9^n - 1) .$$

4.1. Symbolically write Theorem 4. (Do not forget your quantifiers.)

4.2. Prove Theorem 4 using math induction.

hint. See book Proposition 4.4 (p178) and (starred) ER 3.4.8a (p181).

sw. $\left(\forall n \in \mathbb{Z}^{\geq 0}\right) \; [\; 8 \mid \; (9^n - 1) \;]$

Proof. We will show that if $n \in \mathbb{Z}^{\geq 0}$, then

$$8 \mid (9^n - 1) \tag{4.1}$$

by basic induction on the variable n.

For the base step, let $n = 0$. Then

$$9^n - 1 = 9^0 - 1 = 1 - 1 = 0.$$

Also

$$8 | 0$$

since $0 = 8j$ where $j = 0$ and $j \in \mathbb{Z}$. Thus (4.1) is true when $n = 0$. This completes the base step.

For the inductive step, fix $n \in \mathbb{Z}^{\geq 0}$. We assume the inductive hypothesis, which is

$$8 \mid (9^n - 1) . \tag{IH}$$

We will show the inductive conclusion, which is

$$8 \mid \left(9^{n+1} - 1\right) . \tag{IC}$$

By the (IH), there is a $j \in \mathbb{Z}$ such that

$$9^n - 1 = 8j$$

and so

$$9^n = 8j + 1. \tag{4.2}$$

From algebra we get

$$9^{n+1} - 1 \;=\; 9\,[\; 9^n \;] - 1$$

and by (4.2) ⟨which came from the (IH)⟩ and then algebra

$$\begin{aligned} &= 9\,[\; 8j + 1 \;] - 1 \\ &= (9)\,(8)\,j + 9 - 1 \\ &= (8)\,(9)\,j + 8 \\ &= 8\,(9j + 1) \\ &= 8k \end{aligned} \tag{4.3}$$

where $k = 9j + 1$. Since $k \in \mathbb{Z}$, the calculation through (4.3) show the inductive conclusion (IC) holds. This completes the inductive step.

Thus the base step and inductive step hold. □

5. **Theorem 5**. Let $\{x_n\}_{n=1}^{\infty}$ be the recursively defined sequence defined by

$$x_1 = 1 \quad , \quad x_2 = 2$$

and

$$\text{when } n \in \mathbb{N}, \qquad x_{n+2} \;=\; \frac{x_{n+1} + x_n}{2} \;. \tag{RD}$$

If $n \in \mathbb{N}$, then

$$1 \leq x_n \leq 2 \;.$$

5.1. Symbolically write Theorem 5. (Do not forget your quantifiers.)

5.2. Prove Theorem 5 using math induction.

hint. Note you need to prove both the lower bound and upper bound.

sw. $\left[\left(x_1 = 1 \;\wedge\; x_2 = 2 \;\wedge\; (\forall n \in \mathbb{N}) \left[\, x_{n+2} \;=\; \frac{x_{n+1} + x_n}{2} \, \right] \right) \Rightarrow (\forall n \in \mathbb{N}) \, [\, 1 \le x_n \le 2 \,] \right]$

Proof. Let the recursively defined sequence $\{x_n\}_{n=0}^{\infty}$ be given by

$$x_1 = 1 \quad , \quad x_2 = 2$$

and

$$\text{when } n \in \mathbb{N}, \qquad x_{n+2} \;=\; \frac{x_{n+1} + x_n}{2} \, . \tag{RD}$$

We will prove that if $n \in \mathbb{N}$ then

$$1 \le x_n \le 2 \tag{5.1}$$

by strong induction on n.

For the base step, consider the cases $n = 1$ and $n = 2$. Then you have

$$1 \le x_1 = 1 \le 2 \quad \text{and} \quad 1 \le x_2 = 2 \le 2.$$

Thus (5.1) holds for $n = 1$ and $n = 2$. This concludes the base step.

For the inductive step, fix $n \in \mathbb{N}^{\ge 2}$. Assume the inductive hypothesis, which is

$$\text{if } j \in \{1, 2, \dots, n\}, \text{ then } 1 \le x_j \le 2. \tag{IH}$$

We will show the inductive conclusion, which is

$$1 \le x_{n+1} \le 2. \tag{IC}$$

Note $n, n-1 \in \mathbb{N}^{\le n}$ since $n \ge 2$ and $n \in \mathbb{N}$. Thus by the inductive hypothesis (IH),

$$1 \le x_n \le 2 \qquad \text{and} \qquad 1 \le x_{n-1} \le 2. \tag{5.2}$$

The inequalities in (5.2) we gives

$$2 \le x_n + x_{n-1} \le 4,$$

which gives

$$1 \le \frac{x_n + x_{n-1}}{2} \le 2. \tag{5.3}$$

Since $\frac{x_n + x_{n-1}}{2} = x_{n+1}$, the inequalities in (5.3) give

$$1 \le x_{n+1} \le 2, \tag{5.4}$$

which is the inductive conclusion (IC).

By strong induction, we have proven that (5.1) holds for all $n \in \mathbb{N}$. □

6. **Theorem 6**. Let $\{x_n\}_{n=1}^{\infty}$ be the recursively defined sequence be

$$x_1 = 1 \quad , \quad x_2 = 1 \quad , \quad x_3 = 1$$

and

$$\text{when } n \in \mathbb{N}^{\ge 4}, \qquad x_n \;=\; x_{n-1} + x_{n-2} + x_{n-3} \, . \tag{RD}$$

If $n \in \mathbb{N}$, then

$$x_n < 2^n \, .$$

6.1. Symbolically write Theorem 6. (Do not forget your quantifiers.)

6.2. Prove Theorem 6 using math induction.

sw. $\left[\left(x_1 = 1 \;\wedge\; x_2 = 1 \;\wedge\; x_3 = 1 \;\wedge\; \left(\forall n \in \mathbb{N}^{\ge 4}\right) [\, x_n = x_{n-1} + x_{n-2} + x_{n-3} \,] \right) \Rightarrow (\forall n \in \mathbb{N}) \, [\, x_n < 2^n \,] \right]$

Proof. Let the recursively defined sequence $\{x_n\}_{n=0}^{\infty}$ be given by

$$x_1 = 1 \quad , \quad x_2 = 1 \quad , \quad x_3 = 1$$

and

$$\text{when } n \in \mathbb{N}^{\ge 4}, \qquad x_n \;=\; x_{n-1} + x_{n-2} + x_{n-3} \, . \tag{RD}$$

We will show that if $n \in \mathbb{N}$, then

$$x_n < 2^n \tag{6.1}$$

by strong induction on n.

For the base step, first let $n = 1$. Then

$$x_n = 1 < 2 = 2^1 = 2^n.$$

When $n = 2$ we have

$$x_n = 1 < 4 = 2^2 = 2^n.$$

When $n = 3$ we have

$$x_n = 1 < 8 = 2^3 = 2^n.$$

Thus (6.1) holds for $n = 1$ and $n = 2$ and $n = 3$. This concludes the base step.

For the inductive step, fix $n \in \mathbb{N}^{\geq 3}$. Assume the inductive hypothesis, which is

$$\text{if } j \in \{1, 2, \ldots, n\}, \text{ then } x_j < 2^j. \tag{IH}$$

We will show the inductive conclusion, which is

$$x_{n+1} < 2^{n+1}. \tag{IC}$$

The recusive definition (RD) is valid for x_k when $k \geq 4$. We have $n \geq 3$, so $n+1 \geq 4$. So by (RD)

$$x_{n+1} = x_n + x_{n-1} + x_{n-2}$$

Since $3 \leq n$ we have $1 \leq n-2$ so $1 \leq n-2 \leq n-1 \leq n$. Thus we can apply the inductive hypothesis (IH) to n, $n-1$ and $n-2$ since

$$n, n-1, n-2 \in \{1, 2, \ldots, n\}.$$

Thus we have by the recursive definition

$$x_{n+1} = x_n + x_{n-1} + x_{n-2}$$

and by (IH)

$$< 2^n + 2^{n-1} + 2^{n-2}$$

and by algebra ⟨look at where you are going: want a 2^{n+1}⟩

$$\begin{aligned} &= 2^{n+1}\left(2^{-1} + 2^{-2} + 2^{-3}\right) \\ &= 2^{n+1}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) \\ &= 2^{n+1}\left(\frac{4+2+1}{8}\right) \\ &= 2^{n+1}\left(\frac{7}{8}\right) \\ &< 2^{n+1}. \end{aligned} \tag{6.2}$$

The calculations through (6.2) show the inductive conclusion (IC) holds. This completes the inductive step.

Thus the base step and the inductive step hold. □

Remark. The calculation in (6.2) could also have been done as below.

$$\begin{aligned} x_{n+1} &= x_n + x_{n-1} + x_{n-2} \\ &< 2^n + 2^{n-1} + 2^{n-2} \\ &= 2^{n-2}\left(2^2 + 2^1 + 2^0\right) \\ &= 2^{n-2}(7) \\ &< 2^{n-2}\left(2^3\right) \\ &= 2^{n+1}. \end{aligned}$$