These *Practice Problems* are a sampling of the type of problems which could be an the final. These practice problem are, in no way, meant as a comprehensive review for the cumulative final.

Math is not a spectator sport.
Often we learn more from our failed attempts at a proof rather than reading a clean proof.
So give these problems a solid attempt before seeking help, e.g.:
your notes and/or book, hints below, or our discussion thread.
Some hints are very generous. Do not except such generous hints on the exam.
Since these problems are not to hand in, on our discussion thread you may share:
hints and/or an attempt at a solution for others to comment on.

1. **Theorem 1**. For all real numbers x and y, if x is rational, $x \neq 0$ and $y \notin \mathbb{Q}$, then xy is irrational.

1.1. Complete the following definitions.

A real number x is rational provided <u>there exists $n \in \mathbb{Z}$ and $d \in \mathbb{N}$ such that $x = \frac{n}{d}$</u> .

A real number y is irrational provided <u>y is not rational</u> .

1.2. Symbolically write Theorem 1.

1.3. Prove Theorem 1. (You may use the closure properities of $\mathbb{Q}$.)

. . . .

1. Symbolically written, Thm. 1 is

$$\left(\forall\,(x,y) \in \mathbb{R}^2\right)\left[(x \in \mathbb{Q} \wedge x \neq 0 \wedge y \notin \mathbb{Q}) \implies xy \notin \mathbb{Q}\right]. \tag{1.1}$$

Thinking Land.

Knowing a real number is irrational does not give us much of a *bird in the hand*. On the other hand, knowing a real number is rational give us some *bird in the hand* (namely a numerator and a denominator). As stated, Thm. 1 has two irrational numbers ⟨y and xy⟩ but only one rational number ⟨x⟩. So let's look at some negations of Thm. 1 in attempt to get more rational numbers ⟨as so to get more birds in the hand⟩. Below are some negations of Thm 1 ⟨some more useful than others⟩.

$$\sim \left[\ \left(\forall\,(x,y) \in \mathbb{R}^2\right)\left[(x \in \mathbb{Q} \wedge x \neq 0 \wedge y \notin Q) \implies xy \notin \mathbb{Q}\right]\ \right]$$

$$\left(\exists\,(x,y) \in \mathbb{R}^2\right) \sim \left[(x \in \mathbb{Q} \wedge x \neq 0 \wedge y \notin Q) \implies xy \notin \mathbb{Q}\right]$$

⟨Recall think of $P \Rightarrow Q$ as a *promise*. So the negation of $P \Rightarrow Q$ is to *break our promise*. Thus $\sim [P \Rightarrow Q] \equiv [P \wedge (\sim Q)]$.⟩

$$\left(\exists\,(x,y) \in \mathbb{R}^2\right)\ \left[(x \in \mathbb{Q} \wedge x \neq 0 \wedge y \notin Q) \wedge xy \in \mathbb{Q}\right]$$

$$\left(\exists\,(x,y) \in \mathbb{R}^2\right)\ \left[x \neq 0 \wedge x \in \mathbb{Q} \wedge xy \in \mathbb{Q} \wedge y \notin \mathbb{Q}\right] \tag{1.2}$$

The statement in (1.2) gives us more to work with ⟨more of a bird in the hand⟩ than (1.1). So let's try a proof by contradiction and assume (1.2) holds ⟨i.e., the negation of what we want to show holds⟩ and go looking for a contradition. Our birds in the hand are: $x \in \mathbb{Q}$ and $xy \in \mathbb{Q}$. Can we somehow use that $x \in \mathbb{Q}$ and $xy \in \mathbb{Q}$ to find a contradition to $y \notin \mathbb{Q}$? The key observation is that, since $x \neq 0$, we can write $y = \left(\frac{1}{x}\right)(xy)$. Thanks to our *Thinking Land*, we can now do the proof.

▷. Theorem 1 is the book's Proposition 3.19 (p. 123) so please see the book for the proof.

2. **Theorem 2**. There does not exist an integer x such that

$$x \equiv 4 \pmod 9 \qquad \text{and} \qquad x \equiv 5 \pmod 6.$$

2.1. Explain why we cannot apply modulo arithmetic to the congruences as they are written in Thm. 2.

2.2. Prove Theorem 2.

. . . .

2.1. To apply modulo arithmetic to two congruences, the two congruences must be of the same modulo. As the two congruences in Thm. 2 are stated, one congruence is modulo 9 while the other congruence is modulo 6.

2.2. *Thinking Land.* The fact that such an x does not exist does not give us much of a *bird in the hand.* On the other hand, if such an x were to exist, we would have some *birds in the hand.* So let's try a proof by contradiction and assume the negation of Thm. 2 holds ⟨i.e.,such an x does exist⟩ and go looking for a contradition.

The negation of Thm. 2 is that there exists $x \in \mathbb{Z}$ such that $x \equiv 4 \pmod 9$ and $x \equiv 5 \pmod 6$. Given our answer to part 2.1, we cannot apply modulo arithmetic (MA) to the given congruences as stated. But since 3 divides both 9 and 6, each of the given congruences should imply some congruence modulo 3. Then we can apply MA to these congruences modulo 3.

Proof. We shall show Theorem 2 by contradiction. Thus let $x \in \mathbb{Z}$ satisfy

$$x \equiv 4 \pmod 9 \qquad \text{and} \qquad x \equiv 5 \pmod 6. \tag{2.1}$$

We shall find a contradiction.

Since $x \equiv 4 \pmod 9$, there exists $k \in \mathbb{Z}$ such that $x = 9k + 4$. Thus

$$x = 9k + 4 = 3(3k) + 3 + 1 = 3(3k + 1) + 1$$

and $3k + 1 \in \mathbb{Z}$ by closure properties of $\mathbb{Z}$; thus,

$$x \equiv 1 \pmod 3. \tag{2.2}$$

Similarly, since $x \equiv 5 \pmod 6$, there exists $j \in \mathbb{Z}$ such that $x = 6j + 5$. Thus

$$x = 6j + 5 = 3(2j) + 3 + 2 = 3(2j + 1) + 2$$

and $2j + 1 \in \mathbb{Z}$ by closure properties of $\mathbb{Z}$; thus,

$$x \equiv 2 \pmod 3. \tag{2.3}$$

Since there exists a unique $r \in \{0, 1, 2\}$ such that $x \equiv r \pmod 3$, equations (2.2) and (2.3) provide a contradiction to our assumption in (2.1).

We have just shown that assuming Theorem 2 is false leads to a contradiction. Thus Theorem 2 is true. □

3. **Theorem 3**. There is a unique natural number n such that n and $n+1$ are both primes.

3.1. Complete the following definition.

A natural number n is prime provided $n \neq 1$ and only natural numbers that are factors of n are 1 and n .

3.2. Symbolically write Theorem 3.

3.3. Prove Theorem 3.

. . . .

▷. $$(\exists! n \in \mathbb{N}) \, [\, (n \text{ is prime}) \;\wedge\; (n+1 \text{ is prime}) \,]$$

TL. First agrue there exists an $n \in \mathbb{N}$ such that both n and $n+1$ are primes by constructing/finding such an n. Will get $n = 2$ since both 2 are 3 are primes. Next are that $n = 2$ is the only possible (i.e., unique) n such that both n and $n+1$ are primes. Since we will use the below lemma twice, we will begin by stating and proving the lemma.

Lemma P. The only even prime number is 2.

Proof of Lemma P. We know that 2 is an even prime number. Let p be an arbitrary even prime number. We shall show that $p = 2$.

Since p is prime, the only natural numbers that divide p are 1 and p. But 2 divides p since p is even. So 2 must be 1 or p. Thus $p = 2$.

We have just shown that the only even prime is 2. □

Proof of Thm. 3. We shall show Theorem 3, which says that there exists unique natural number n such that n and $n+1$ are both primes, by first showing that there existence such an n and then showing that this n is the only possible n.

Towards the *existence* part, let $n = 2$. Then $n + 1 = 3$. Note 2 and 3 are both primes. Thus $n = 2$ satisfies that both n and $n+1$ are prime. This complete the existence part.

Towards the *uniqueness* part, let $n \in \mathbb{N}$ be such that both n and $n+1$ are primes. We will show that $n = 2$ by cases: n is even and n is odd.

For case 1, let n be even. Then by Lemma P, $n = 2$.

For case 2, let n be odd. Then by Lemma SOO, $n+1$ is even. We are assuming that $n+1$ is prime so by Lemma P, $n + 1 = 2$. Thus $n = 1$. But n is prime and 1 is not prime so n cannot be 1. Thus case 2 cannot occur.

We have just shown that if $n \in \mathbb{N}$ such that both n and $n+1$ are primes, then $n = 2$.

This completes the proof. □

4. **Theorem 4**. Let I be a nonempty arbitrary indexing set and $\{A_i \colon i \in I\}$ be a collection of subsets of some universeral set U. Then

$$\left[\bigcap_{i \in I} A_i \right]^C = \bigcup_{i \in I} (A_i)^C \ .$$

4.1. Clearly explain why Thm. 4 is true. Use complete sentences. You may (and are encouraged to) use symbolic notation in your explanation. Hint: Write out equivalent statements for $x \in \left[\bigcap_{i \in I} A_i \right]^C$.

. . . .

For sets S and T, to show that $S = T$ we often show that $S \subseteq T$ and $T \subseteq S$, i.e., we show *set containment holds in both directions*. Note the below proof is an example of where both directions can easily be *done at the same time*.

Proof. Let I be a nonempty arbitrary indexing set and $\{A_i \colon i \in I\}$ be a collection of subsets of some universeral set U. We shall show that

$$\left[\bigcap_{i \in I} A_i \right]^C = \bigcup_{i \in I} (A_i)^C \ . \tag{4.1}$$

Let $x \in U$. Note the following statements are equivalent, using symbolic notation when helpful.

$$x \in \left[\bigcap_{i \in I} A_i \right]^C \tag{4.2}$$

and by definition of complement

$$x \notin \bigcap_{i \in I} A_i$$

and by definition of intersection

$$\sim \ \{ \ (\forall i \in I) \, [x \in A_i] \ \}$$

and by rules of negation

$$(\exists i \in I) \, [x \notin A_i]$$

and by definition of complement

$$(\exists i \in I) \, [x \in (A_i)^c]$$

and by definition of union

$$x \in \bigcup_{i \in I} (A_i)^c \tag{4.3}$$

Thus (4.2) and (4.3) are equivalent, showing that

$$x \in \left[\bigcap_{i \in I} A_i \right]^C \quad \text{if and only if} \quad x \in \bigcup_{i \in I} (A_i)^c$$

Thus (4.1) holds. □

5. Theorem 5. For all integers x and y,

$$(x+y)^7 \equiv \left(x^7+y^7\right) \pmod 7.$$

5.1. Symbolically write Theorem 5. (Hint. Do not forget your quantifiers.)

$$(\forall\,(x,y)\in\mathbb{Z}^2)\ \left[\ (x+y)^7 \equiv (x^7+y^7)\ \pmod 7\ \right]$$

5.2. Pascal's Triangle ⟨and the Binomial Theorem⟩ are helpful in expanding $(x+y)^n$, where $n\in\mathbb{N}$ and $x,y\in\mathbb{R}$. We used Pascal's Triangle in ER 3.1.6c. If you need a review, here is a link: Algebra 2.

5.3. Prove Theorem 5.

. . . .

Proof. Let $x,y\in\mathbb{Z}$. We shall show that

$$(x+y)^7 \equiv \left(x^7+y^7\right) \pmod 7.$$

by showing that 7 divides $(x+y)^7-(x^7+y^7)$.

With the help of Pascal's triangle we get

$$(x+y)^7 = 1x^7y^0+7x^6y^1+21x^5y^2+35x^4y^3+35x^3y^4+21x^2y^5+7x^1y^6+1x^0y^7. \tag{5.1}$$

Using (5.1) and algebra we see

$$\begin{aligned}(x+y)^7-\left(x^7+y^7\right) &= 7x^6y^1+21x^5y^2+35x^4y^3+35x^3y^4+21x^2y^5+7x^1y^6\\ &= 7\left(x^6y^1+3x^5y^2+5x^4y^3+5x^3y^4+3x^2y^5+x^1y^6\right)\\ &= 7j\end{aligned}$$

where $j=x^6y^1+3x^5y^2+5x^4y^3+5x^3y^4+3x^2y^5+x^1y^6$. Note $j\in\mathbb{Z}$ by the closure properties of the integers since $x,y\in\mathbb{Z}$. Thus we have shown $(x+y)^7-(x^7+y^7)=7j$ for some $j\in\mathbb{Z}$, which gives that 7 divides $(x+y)^7-(x^7+y^7)$. Since 7 divides $(x+y)^7-(x^7+y^7)$, by definition of modulo congruence,

$$(x+y)^7 \equiv \left(x^7+y^7\right) \pmod 7.$$

This completes the proof. □

6. **Theorem 6.** For every (strictly) positive real number ε there is a (strictly) positive real number δ such that for each real number x, if $0 \le x < 3+\delta$ then $x^2 < 9+\varepsilon$.

6.1. Fill in the two blanks as so to symbolically write Theorem 6.

$$(\forall \varepsilon \in \mathbb{R}^{>0})\,(\exists \delta \in \mathbb{R}^{>0})\,(\forall x \in \mathbb{R})\left[\,\left(\underline{\quad 0 \le x < 3+\delta \quad}\right) \implies \left(\underline{\quad x^2 < 9+\varepsilon \quad}\right)\right]$$

6.2. Prove Theorem 6. Hint. Your δ will have a ε in it, i.e., δ is a function of ε.

. . . .

Thinking Land. Fix $\varepsilon \in \mathbb{R}^{>0}$ (i.e., $\varepsilon > 0$). We want to find the $\delta > 0$ but to do this we first have to do some calculations. So let's look at what such a $\delta > 0$ would have to look like.

Let's say we know that $0 \le x < 3+\delta$. Then algebra gives us that $|x| \le 3+\delta$ so $x^2 < 9+6\delta+\delta^2$. So if $9+6\delta+\delta^2 = 9+\varepsilon$, then we would be done. But note $9+6\delta+\delta^2 = 9+\varepsilon \Leftrightarrow \delta^2+6\delta-\varepsilon = 0 \Leftrightarrow \delta = \frac{-6\pm\sqrt{36+4\varepsilon}}{2}$. Algebra gives $\frac{-6\pm\sqrt{36+4\varepsilon}}{2} = \frac{-6\pm\sqrt{4(9+\varepsilon)}}{2} = -3\pm\sqrt{9+\varepsilon}$. So take $\delta := \sqrt{9+\varepsilon}-3 \overset{\text{so}}{>} 0$.

⚠. For guidance with the proof's first paragraph, look at the symbolic writing of Theorem 6.

Proof. Let $\varepsilon > 0$. Take

$$\delta := \sqrt{9+\varepsilon} - 3. \tag{6.1}$$

Note $\delta > 0$ since $\delta = \sqrt{9+\varepsilon} - 3 > \sqrt{9} - 3 = 0$. Let $x \in \mathbb{R}$ satisfy ⟨the hypothesis⟩

$$0 \le x < 3+\delta. \tag{6H}$$

We shall show ⟨the conclusion⟩

$$x^2 < 9+\varepsilon. \tag{6C}$$

Towards (6C), our hypothesis in (6H) gives $|x| < 3+\delta$. Thus

$$x^2 < [3+\delta]^2$$

and by our choice of $\delta = \sqrt{9+\varepsilon} - 3$ in (6.1)

$$< \left[\,3 + \left(\sqrt{9+\varepsilon} - 3\right)\,\right]^2$$
$$= \left[\,\left(\sqrt{9+\varepsilon}\right)\,\right]^2$$
$$= 9+\varepsilon.$$

We have just shown that $x^2 < 9+\varepsilon$. Thus (6C) holds.

We have just shown that given any $\varepsilon > 0$, by taking $\delta = \sqrt{9+\varepsilon} - 3$, which is strictly positive, if $x \in \mathbb{R}$ satisfies $0 \le x < 3+\delta$ then $x^2 < 9+\varepsilon$. This completes the proof. □

7. **Theorem 7**. Each point on or inside the circle whose equation is

$$(x-1)^2+(y-2)^2=4$$

is also inside the circle whose equation is

$$(x-4)^2+y^2=42\ .$$

hint. **Definition**. The point $(x_0, y_0) \in \mathbb{R}^2$ is:

- inside the circle whose equation is $(x-h)^2+(y-k)^2=r^2$ provided $(x_0-h)^2+(y_0-k)^2<r^2$
- on the circle whose equation is $(x-h)^2+(y-k)^2=r^2$ provided $(x_0-h)^2+(y_0-k)^2=r^2$
- outside the circle $(x-h)^2+(y-k)^2=r^2$ provided $(x_0-h)^2+(y_0-k)^2>r^2$.

▷. Compare to ER 3.6.1.

hint. Symbolically looks: $(\forall (x,y) \in \mathbb{R}^2)\ [\ P(x,y) \implies Q(x,y)\ ]$. The open sentences $P(x,y)$ and $Q(x,y)$ will be inequalities.

7.1. Symbolically write Theorem 7.

7.2. Prove Theorem 7.

. . . .

▷. $(\forall (x,y) \in \mathbb{R}^2)\ \left[\ (x-1)^2+(y-2)^2 \le 4 \implies (x-4)^2+y^2<42\ \right]$

Proof. Let $(x,y) \in \mathbb{R}^2$ satisfy

$$(x-1)^2+(y-2)^2 \le 4. \tag{7.1}$$

We shall show that $(x-4)^2+y^2<42$.

Since (7.1) implies

$$(x-1)^2 \le (x-1)^2+(y-2)^2 \le 4,$$

we have $|x-1|^2 \le 2^2$. Similarly $|y-2|^2 \le 2^2$. Thus we have the following inequalities.

$$\begin{aligned}
|x-1| &\le 2 &&\text{and}&& |y-2| \le 2.\\
-2 \le x-1 &\le 2 &&\text{and}&& -2 \le y-2 \le 2.\\
-1 \le x &\le 3 &&\text{and}&& 0 \le y \le 4.\\
-5 \le x-4 &\le -1 &&\text{and}&& |y| \le 4.\\
|x-4| &\le 5 &&\text{and}&& |y| \le 4.\\
(x-4)^2 &\le 25 &&\text{and}&& y^2 \le 16.
\end{aligned} \tag{7.2}$$

By (7.2) we now see that

$$(x-4)^2+y^2 \le 25+16=41<42.$$

We have just shown that if $(x-1)^2+(y-2)^2 \le 4$ then $(x-4)^2+y^2<42$.

This completes the proof that each point on or inside the circle with equation $(x-1)^2+(y-2)^2=4$ is also inside the circle whose equation is $x^2+y^2=26$. □

▷. A proof of Theorem 7 will a long string of inequalities for x and y.
In the above proof, an inequality for both x and y are shown on the same line.
It is also fine for first do a string for just the x's and then to do another string for just the y's.
The choose is yours.

8. **Lemma 8**. The product of two consecutive integers is even.

Theorem 8. If u is an odd integer, then the equation $x^3 - x - u = 0$ (in the variable x) does not have a solution that is an integer.

hint. A solution in $\mathbb{R}$ to the equation $x^3 - x - u = 0$ (in the variable x) is any $n \in \mathbb{R}$ satisfying $n^3 - n - u = 0$.

hint. The product of 2 consecutive integers can be expressed as $n\,(n+1)$ for some $n \in \mathbb{Z}$.

⊛. On Problem 8, you may and are encouraged to use the Previously Shown Results which we refer to as Lemma POO and friends (rather than the definition of even/odd).

8.1. Prove Lemma 8.

8.2. Symbolically write Theorem 8.

8.3. Prove Theorem 8

. . . .

Proof of Lemma 8. Let $n \in \mathbb{Z}$. We shall show $n\,(n+1)$ is even by cases: n is odd or n is even.

For case 1, let n be odd. By Lemma SOO, $n+1$ is even. By Lemma PEA, $n\,(n+1)$ is even.

For case 2, let n be even. Then $n\,(n+1)$ is even by Lemma PEA. □

sw. SW for Thm 8. $$(\forall u \in \mathbb{Z}) \left[\underbrace{u \text{ is odd}}_{\text{hypothesis}} \implies \underbrace{(\forall n \in \mathbb{Z}) \left[n^3 - n - u \neq 0\right]}_{\text{conclusion}} \right]$$

Thinking Land for Thm 8. Thm 8 in the original form just above contains a conditional statement $P \Rightarrow Q$. The $\neq$ in the conclusion not so nice. Recall a conditional statement is logically equivalent to its contrapositive, i.e., $[P \Rightarrow Q] \equiv [(\sim Q) \Rightarrow (\sim P)]$. So we can use the contrapositive to get rid of the $\neq$. The contrapositive tell us that Thm 8 is equivalent to

$$(\forall u \in \mathbb{Z}) \left[\underbrace{(\exists n \in \mathbb{Z}) \left[n^3 - n - u = 0\right]}_{\text{hypothesis}} \implies \underbrace{u \text{ is even}}_{\text{conclusion}} \right]$$

The contrapositive's hypothesis gives us an $n \in \mathbb{Z}$ such that $n^3 - n - u = 0$; indeed, this given n is a *bird in the hand*. For the contrapositive's conclusion we need to show this given n is even.

Proof of Thm 8. Let u be an integer. We shall show Theorem 8 by contrapositive. Thus let $n \in \mathbb{Z}$ satisfy

$$n^3 - n - u = 0. \tag{8.1}$$

We need to show that n is even.

Since

$$n^3 - n \;=\; n\left(n^2 - 1\right) \;=\; n\,(n-1)\,(n+1)$$

equation (8.1) gives

$$u \;=\; (n-1)\,(n)\,(n+1)\,. \tag{8.2}$$

Being the product of 2 consecutive integers, $(n-1)\,(n)$ is even by Lemma 8. Since the product of the even integer and any integer is even (see Lemma PEA), we have $(n-1)\,(n)\,(n+1)$ is even. Thus (8.2) gives that u is even.

We have just shown that if u is odd then there does not exist $n \in \mathbb{Z}$ such that $n^3 - n - u = 0$. □

9. **Theorem 9**. Let $x, y \in \mathbb{R}$. If y is irrational then $(x+y)$ is irrational or $(x-y)$ is irrational.

9.1. Symbolically write Theorem 9.

9.2. Prove Theorem 9.

. . . .

9. Symbolically written, Thm. 9 is

$$\left(\forall\,(x,y)\in\mathbb{R}^2\right)\left[\; y\notin\mathbb{Q} \implies \{\; (x+y)\notin\mathbb{Q} \vee (x-y)\notin\mathbb{Q} \;\}\;\right]. \tag{9.1}$$

Thinking Land.

Knowing a real number is irrational does not give us much of a *bird in the hand*. On the other hand, knowing a real number is rational give us some *bird in the hand* (namely a numerator and a denominator). As stated, all the real numbers in Thm. 9 are irrational. So let's look at some negations of Thm. 9 in attempt to get some rational numbers ⟨as so to get some birds in the hand⟩. Below are some negations of Thm 9 ⟨some more useful than others⟩.

$$\sim\;\left[\;\left(\forall\,(x,y)\in\mathbb{R}^2\right)\left[\; y\notin\mathbb{Q} \implies \{\; (x+y)\notin\mathbb{Q} \vee (x-y)\notin\mathbb{Q} \;\}\;\right]\;\right]$$

$$\left(\exists\,(x,y)\in\mathbb{R}^2\right)\;\sim\;\left[\; y\notin\mathbb{Q} \implies \{\; (x+y)\notin\mathbb{Q} \vee (x-y)\notin\mathbb{Q} \;\}\;\right]$$

⟨Recall think of $P \Rightarrow Q$ as a *promise*. So the negation of $P \Rightarrow Q$ is to *break our promise*. Thus $\sim[P \Rightarrow Q] \equiv [P \wedge (\sim Q)]$.⟩

$$\left(\exists\,(x,y)\in\mathbb{R}^2\right)\;\left[\; y\notin\mathbb{Q} \wedge \;\sim\{\; (x+y)\notin\mathbb{Q} \vee (x-y)\notin\mathbb{Q} \;\}\;\right]$$

$$\left(\exists\,(x,y)\in\mathbb{R}^2\right)\;\left[\; y\notin\mathbb{Q} \;\wedge\; (x+y)\in\mathbb{Q} \;\wedge\; (x-y)\in\mathbb{Q} \;\right]$$

$$\left(\exists\,(x,y)\in\mathbb{R}^2\right)\;\left[\; (x+y)\in\mathbb{Q} \;\wedge\; (x-y)\in\mathbb{Q} \;\wedge\; y\notin\mathbb{Q} \;\right] \tag{9.2}$$

The statement in (9.2) gives us more to work with ⟨more of a bird in the hand⟩ than (9.1). So let's try a proof by contradiction and assume (9.2) holds ⟨i.e., the negation of what we want to show holds⟩ and go looking for a contradition. Our birds in the hand are: $x+y\in\mathbb{Q}$ and $x-y\in\mathbb{Q}$. Can we somehow use that $x+y\in\mathbb{Q}$ and $x-y\in\mathbb{Q}$ to find a contradition to $y\notin\mathbb{Q}$? The key observations are: $(x+y)-(x-y)=2y$ and $y=\frac{1}{2}\,(2y)$.

Proof. We shall show Theorem 9 by contradiction. Thus let $x, y \in \mathbb{R}$ satisfy that

$$x+y\in\mathbb{Q} \qquad \text{and} \qquad x-y\in\mathbb{Q} \qquad \text{and} \qquad y\notin\mathbb{Q}. \tag{9.3}$$

We shall find a contradiction ⟨to our above assumption in (9.3)⟩.

Since both $x+y$ and $x-y$ are rational, $\mathbb{Q}$ is closed under subtraction, and

$$(x+y)-(x-y)=2y$$

we have that $2y\in\mathbb{Q}$. Since $\frac{1}{2}$ is also rational, $\mathbb{Q}$ is closed under multiplication, and

$$y=\frac{1}{2}\,(2y)$$

we get that $y\in\mathbb{Q}$. However we assumed that $y\notin\mathbb{Q}$. So we have a contradiction.

This completes the proof by contradiction of Theorem 9. □

10. **Theorem 10**. For each $n \in \mathbb{N}$ with $n \geq 2$,

$$1 + 2^n < 3^n \ . \tag{10.1}$$

Symbolically write Theorem 10. Prove Theorem 10.

. . . .

$\left(\forall n \in \mathbb{N}^{\geq 2}\right) \ [1 + 2^n < 3^n]$.

Generalized Induction.

Base Step: show (10.1) holds when $n = 2$.

Inductive Step: Fix $n \in \mathbb{N}^{\geq 2}$.

Main calculation is:

$1 + 2^{n+1} = 1 + (2)\,(2^n) \overset{\text{(IH)}}{<} 1 + (2)\,(3^n - 1) = (2)\,(3^n) - 1 < (3)\,3^n - 1 < 3^{n+1} - 0 = 3^{n+1}$.

11. **Theorem 11**. Consider a sequence $\{a_n\}_{n=1}^{\infty}$ recusively defined by $a_1 = a_2 = a_3 = 1$ while for $n \in \mathbb{N}$

$$a_{n+3} = a_{n+2} + a_{n+1} + a_n \ . \tag{11.1}$$

For each $n \in \mathbb{N}$ with $n > 1$,

$$a_n \leq 2^{n-2} \ . \tag{11.2}$$

Symbolically write Theorem 11. Prove Theorem 11.

. . . .

$\left(\forall n \in \mathbb{N}^{\geq 2}\right) \ [\ (a_1 = 1 \wedge a_2 = 1 \wedge a_3 = 1 \wedge (\forall j \in \mathbb{N})\,[a_{j+3} = a_{j+2} + a_{j+1} + a_j]) \implies a_n \leq 2^{n-2}\ ]$.

Induction (of some kind). Keeping in mind that the recursive definition (RD) *kicks in* when $n = 4$, there are several (equivalent) ways to express the recursive definition (RD), e.g.:

$$\text{For each } n \in \mathbb{Z}^{\geq 1}\text{ , define} \qquad a_{n+3} = a_{n+2} + a_{n+1} + a_n \tag{RD$_1$}$$

$$\text{For each } n \in \mathbb{Z}^{\geq 2}\text{ , define} \qquad a_{n+2} = a_{n+1} + a_n + a_{n-1} \tag{RD$_2$}$$

$$\text{For each } n \in \mathbb{Z}^{\geq 3}\text{ , define} \qquad a_{n+1} = a_n + a_{n-1} + a_{n-2} \tag{RD$_3$}$$

$$\text{For each } n \in \mathbb{Z}^{\geq 4}\text{ , define} \qquad a_n = a_{n-1} + a_{n-2} + a_{n-3}. \tag{RD$_4$}$$

Note (RD$_1$)–(RD$_4$) are equivalent. So use which ever one(s) you find most useful.

For the time being, let's just blindly sketch out how the inductive step might look, without verifing we can actully do each step. We will worry about verifying the step later (by properly setting up our induction). In the inductive step, we would want to assume the inductive hypothesis (IH) and show the inductive conclusion (IC), which would look like

$$a_{n+1} \leq 2^{(n+1)-2} \ , \tag{IC$_1$}$$

or equivalently ⟨by just algebra⟩

$$a_{n+1} \leq 2^{n-1} \ . \tag{IC$_2$}$$

So, **when n is big enough** ⟨with big enough to be determined⟩ would get from the recusive definititon

$$a_{n+1} = a_n + a_{n-1} + a_{n-2} \tag{used RD}$$

and by the (IH) applied to a_n, a_{n-1}, and a_{n-1} ... going to need strong induction!

$$\leq 2^{(n)-2} + 2^{(n-1)-2} + 2^{(n-2)-2} \qquad \text{(used IH)}$$

let's clean off the mud by some simple algebra

$$= 2^{n-2} + 2^{n-3} + 2^{n-4} \qquad (11.3)$$

... now it looks like we could do some algebra to get

$\leq 2^{n-1}$ and we would be done !

So we will use strong induction. In order to do the inductive step, how big will n have to be? Well

- To be able use the recusive definition in (used RD), we would need ⟨applying (RD) to a_{n+1}, so would need $n+1 \geq 4$ so⟩ $n \geq 3$.
- To be able to apply the (IH) in (used IH), would need n and $n-1$ and $n-2$ in the set $\{2, 3, \ldots n\}$. Thus need $n-2 \geq 2$, i.e., need $n \geq 4$.

So for the inductive step, we need both $n \geq 3$ and $n \geq 4$, thus we need $n \geq 4$.

Thus, use strong induction. In the Base Step, show that (11.2) holds for $n = 2$, and $n = 3$ and $n = 4$. For the Inductive Step, fix $n \geq 4$. Assume the inductive hypothesis, which is

$$\text{if } \; j \in \{2, \ldots, n\} \quad \text{then} \quad a_j \leq 2^{j-2}. \qquad \text{(IH)}$$

We want to show the inductive conclusion, which is

$$a_{n+1} \leq 2^{n-1} \; . \qquad (\text{IC}_2)$$

Well, since $n \geq 4$, know $n+1 \geq 3$ and so the recursive definition gives

$$a_{n+1} \overset{\text{by (RD)}}{=} a_n + a_{n-1} + a_{n-2}$$

and since $n \geq 4$, we know $n, n-1, n-2 \in \{2, \ldots, n\}$ and so can apply the (IH) to j being n, $n-1$ and $n-2$ to get

$$\overset{\text{by (IH)}}{\leq} 2^{(n)-2} + 2^{(n-1)-2} + 2^{(n-2)-2}$$

Now look at where you are heading (i.e., want to show the inductive conclusion (IC_2) and figure out how to get there.