These *Practice Problems* are a sampling of the type of problems which could be an the final. These practice problem are, in no way, meant as a comprehensive review for the cumulative final.

1. **Theorem 1**. For all real numbers x and y, if x is rational, $x \neq 0$ and $y \notin \mathbb{Q}$, then xy is irrational.

1.1. Complete the following definitions.

A real number x is rational provided ______________________________ .

A real number y is irrational provided ______________________________ .

1.2. Symbolically write Theorem 1.

1.3. Prove Theorem 1. (You may use the closure properities of $\mathbb{Q}$.)

2. **Theorem 2**. There does not exist an integer x such that

$$x \equiv 4 \pmod 9 \qquad \text{and} \qquad x \equiv 5 \pmod 6.$$

2.1. Explain why we cannot apply modulo arithmetic to the congruences as they are written in Thm. 2.

2.2. Prove Theorem 2.

3. **Theorem 3**. There is a unique natural number n such that n and $n + 1$ are both primes.

3.1. Complete the following definition.

A natural number n is prime provided ______________________________ .

3.2. Symbolically write Theorem 3.

3.3. Prove Theorem 3.

4. **Theorem 4**. Let I be a nonempty arbitrary indexing set and $\{A_i : i \in I\}$ be a collection of subsets of some universeral set U. Then

$$\left[\bigcap_{i \in I} A_i \right]^C = \bigcup_{i \in I} (A_i)^C \; .$$

4.1. Clearly explain why Thm. 4 is true. Use complete sentences. You may (and are encouraged to) use symbolic notation in your explanation. Hint: Write out equivalent statements for $x \in \left[\bigcap_{i \in I} A_i \right]^C$.

5. **Theorem 5**. For all integers x and y,

$$(x + y)^7 \equiv \left(x^7 + y^7\right) \pmod 7.$$

5.1. Symbolically write Theorem 5. (Hint. Do not forget your quantifiers.)

5.2. Pascal's Triangle ⟨and the Binomial Theorem⟩ are helpful in expanding $(x+y)^n$, where $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$. We used Pascal's Triangle in ER 3.1.6c. If you need a review, here is a link: Algebra 2.

5.3. Prove Theorem 5.

6. **Theorem 6**. For every (strictly) positive real number ε there is a (strictly) positive real number δ such that for each real number x, if $0 \leq x < 3 + \delta$ then $x^2 < 9 + \varepsilon$.

6.1. Fill in the two blanks as so to symbolically write Theorem 6.

$(\forall \varepsilon \in \mathbb{R}^{>0})\, (\exists \delta \in \mathbb{R}^{>0})\, (\forall x \in \mathbb{R})\, [\ ($ ______________ $) \implies ($ ______________ $)\ ]$

6.2. Prove Theorem 6. Hint. Your δ will have a ε in it, i.e., δ is a function of ε.

7. **Theorem 7**. Each point on or inside the circle whose equation is

$$(x-1)^2+(y-2)^2=4$$

is also inside the circle whose equation is

$$(x-4)^2+y^2=42\ .$$

hint. **Definition**. The point $(x_0, y_0) \in \mathbb{R}^2$ is:

- inside the circle whose equation is $(x-h)^2+(y-k)^2=r^2$ provided $(x_0-h)^2+(y_0-k)^2<r^2$
- on the circle whose equation is $(x-h)^2+(y-k)^2=r^2$ provided $(x_0-h)^2+(y_0-k)^2=r^2$
- outside the circle $(x-h)^2+(y-k)^2=r^2$ provided $(x_0-h)^2+(y_0-k)^2>r^2$.

▷. Compare to ER 3.6.1.

hint. Symbolically looks: $\left(\forall (x,y)\in\mathbb{R}^2\right)\ [\ P(x,y) \implies Q(x,y)\ ]$. The open sentences $P(x,y)$ and $Q(x,y)$ will be inequalities.

7.1. Symbolically write Theorem 7.

7.2. Prove Theorem 7.

8. **Lemma 8**. The product of two consecutive integers is even.

Theorem 8. If u is an odd integer, then the equation $x^3-x-u=0$ (in the variable x) does not have a solution that is an integer.

hint. A solution in $\mathbb{R}$ to the equation $x^3-x-u=0$ (in the variable x) is any $n\in\mathbb{R}$ satisfying $n^3-n-u=0$.

hint. The product of 2 consecutive integers can be expressed as $n(n+1)$ for some $n\in\mathbb{Z}$.

⊛. On Problem 8, you may and are encouraged to use the Previously Shown Results which we refer to as Lemma POO and friends (rather than the definition of even/odd).

8.1. Prove Lemma 8.

8.2. Symbolically write Theorem 8.

8.3. Prove Theorem 8

9. **Theorem 9**. Let $x, y\in\mathbb{R}$. If y is irrational then $(x+y)$ is irrational or $(x-y)$ is irrational.

9.1. Symbolically write Theorem 9.

9.2. Prove Theorem 9.

10. **Theorem 10**. For each $n\in\mathbb{N}$ with $n\geq 2$,

$$1+2^n<3^n\ . \tag{10.1}$$

Symbolically write Theorem 10. Prove Theorem 10.

11. **Theorem 11**. Consider a sequence $\{a_n\}_{n=1}^{\infty}$ recusively defined by $a_1=a_2=a_3=1$ while for $n\in\mathbb{N}$

$$a_{n+3}=a_{n+2}+a_{n+1}+a_n\ . \tag{11.1}$$

For each $n\in\mathbb{N}$ with $n>1$,

$$a_n\leq 2^{n-2}\ . \tag{11.2}$$

Symbolically write Theorem 11. Prove Theorem 11.