Exam 1 includes Direct Proofs (with quantifiers). Read the **whole** first page and then do (A)–(I).

☺ ☺ A good source of exam problems. ☺ ☺

Provide a proof or a counterexample for each of these statements.

(A) For each positive integer x, the quantity $x^2 + x + 41$ is a prime.

(B) $(\forall x \in \mathbb{R})\ (\exists y \in \mathbb{R})\ [\ x + y = 0\ ]$.

(C) $(\forall x \in \mathbb{R})\ (\forall y \in \mathbb{R})\ [\ (x > 1 \wedge y > 0) \implies y^x > x]$

(D) For integers a, b, and c, if a divides bc, then a divides b or a divides c.

(E) For integers a, b, c, and d, if a divides $b - c$ and a divides $c - d$, then a divides $b - d$.

(F) For each positive real number x, the inequality $x^2 - x \geq 0$ holds.

(G) For all positive real numbers x, we have $2^x > x + 1$.

(H) For every positive real number x, there is a positive real number y less than x with the property that for all positive real numbers z, we have $yz \geq z$.

(I) For every positive real number x, there is a positive real number y with the property that if $y < x$, then for all positive real numbers z, it holds that $yz \geq z$.

From: *A Transition To Advanced Mathematics* by Smith, Eggen, St. Andre. (6^{th}Ed. 1.6.5/p54 = 7^{th}Ed. 1.6.4/p57).

Warnings

You can think of these statements as conjecture. It should be understood from the instructions that first we need to decide of the statement/conjecture is true or false. After this,

- if the statement is true, then you should provide a proof (following the Writing Guidelines)
- if the stament is false, then you should provide a counterexample clearly explaining, in complete sentences, why the counterexample shows the statement is false.

Avoid common mistakes seen this semester on homework:

- "x is a positive number" provided $x > 0$ while "x is a nonnegative number" provided $x \geq 0$
- "x is less than y" provided $x < y$ while "x is less than or equal to y" provided $x \leq y$
- don't forget needed parentheses, e.g., $a|b - 1$ does not make sense and should be written as $a|\,(b-1)$.

Observe

Each of these exercises are of the form

$$(\forall x \in U)\ [P(x)] \tag{$\forall$}$$

T. One way to prove that such a ($\forall$) statement is true is to fix an arbitrary element $x_0 \in U$ and show/argue that (for some reason) $P(x_0)$ must be true.

F. One way to show that such a ($\forall$) statement is false is to find a counterexample, which is a (specific element) $x_0 \in U$ such that $P(x_0)$ is false (be sure to say why $P(x_0)$ is false).

Give these Practice Exercises a good honest shot and work together on the Piazza site before looking at hints on the next page.

True/False solutions for the original given statements

(A) false
(B) true
(C) false

(D) false
(E) true
(F) false

(G) false
(H) false
(I) true

More on (A)

A. **Conjecture A**: For each positive integer x, the quantity $x^2 + x + 41$ is a prime.

A1. Symbolically write Conjecture A, as it is stated. In the open sentence, you may use English words.

$$\left(\forall x \in \mathbb{Z}^{>0}\right) \left[x^2 + x + 41 \text{ is prime}\right]$$

Remarks:

- You should know: the open sentence in a statement of the form $(\forall x \in U)\,[P(x)]$ is $P(x)$.
- Although $\mathbb{Z}^{>0} = \mathbb{N}$, since the instructions say **as it is stated**, we should use $\mathbb{Z}^{>0}$ rather than $\mathbb{N}$.

A2. Symbolically write a negation of Conjecture A. You may not use a math symbol for not (e.g., $\sim$). In the open sentence, you may use English words.

$$\left(\exists x \in \mathbb{Z}^{>0}\right) \left[x^2 + x + 41 \text{ is not prime}\right]$$

A3. Is Conjecture A true or false? (circle one) true $\boxed{\text{false}}$

A4. Is the negation of Conjecture A true or false? (circle one) $\boxed{\text{true}}$ false

A5. Provide a proof or a counterexample to Conjecture A. *Counterexample.* The statement "for all positive integers x, the quantity $x^2 + x + 41$ is a prime" is false, as shown by the counterexample of $x_0 = 41$. For if $x_0 = 41$, then

$$\begin{aligned} x_0^2 + x_0 + 41 &= (41)^2 + 41 + 41 \\ &= 41\,(41 + 1 + 1) \\ &= 41\,(43)\,. \end{aligned}$$

Note the quantity $(41)\,(43)$ is not prime.

Thus, when $x_0 = 41$, the quantity $x_0^2 + x_0 + 41 = (41)^2 + 41 + 41$ is not prime. $\square$

More with (H) and (I

►. Symbolically write (with universe $\mathbb{R}^{>0}$), the statements as given. You may not use English words.

(I). For every positive real number x, there is a positive real number y with the property that if $y < x$, then for all positive real numbers z, it holds that $yz \geq z$.

$$\left(\forall x \in \mathbb{R}^{>0}\right) \left(\exists y \in \mathbb{R}^{>0}\right) \left[\, y < x \;\Rightarrow\; \left(\forall z \in \mathbb{R}^{>0}\right) [yz \geq z] \,\right]$$

(H). For every positive real number x, there is a positive real number y less than x with the property that for all positive real numbers z, we have $yz \geq z$.

$$\left(\forall x \in \mathbb{R}^{>0}\right) \left(\exists y \in \mathbb{R}^{>0}\right) \left[\, y < x \;\wedge\; \left(\forall z \in \mathbb{R}^{>0}\right) [yz \geq z] \,\right]$$

$\sim$(H). A negation of the Conjecture H that uses neither $\wedge$ nor $\vee$. $[\sim(P \wedge Q)\,] \equiv [\,P \Rightarrow (\sim Q)\,]$.

$$\left(\exists x \in \mathbb{R}^{>0}\right) \left(\forall y \in \mathbb{R}^{>0}\right) \left[\, y < x \;\Rightarrow\; \left(\exists z \in \mathbb{R}^{>0}\right) [yz < z] \,\right]$$

When negating quantified statements, we often use the logical equivalencies (from our §2.2 Handout):

$$\begin{aligned} [\sim(P \Rightarrow Q)\,] &\equiv [\,P \wedge (\sim Q)\,] && \text{(how do you break a promise?)} \\ [\sim(P \wedge Q)\,] &\equiv [\,P \Rightarrow (\sim Q)\,] && \text{(not in book)} \\ [\sim(P \wedge Q)\,] &\equiv [\,(\sim P) \vee (\sim Q)\,] && \text{(De Morgans Law)} \\ [\sim(P \vee Q)\,] &\equiv [\,(\sim P) \wedge (\sim Q)\,] && \text{(De Morgans Law)}\,. \end{aligned}$$

You should have a working knowledge of the logical equivalencies from this handout.