These *Jam Problems* are a sampling of the type of problems which could be an the final.
These problem are, in no way, meant as a comprehensive review for the cumulative final.

Math is not a spectator sport.
Often we learn more from our failed attempts at a proof rather than reading a clean proof.
So give these problems a solid attempt before seeking help, e.g.:
looking through your notes and/or book, looking at the below hints, or looking at Piazza.
Some hints are very generous. Do not except such generous hints on the exam.
Since these problems are not to hand in, on Piazza you may share:
hints and/or an attempt at a solution for others to comment on.

1. **Theorem 1**. For all real numbers x and y, if x is rational, $x \neq 0$ and $y \notin \mathbb{Q}$, then xy is irrational.

1.1. Complete the following definitions.

A real number x is rational provided there exists $n \in \mathbb{Z}$ and $d \in \mathbb{N}$ such that $x = \frac{n}{d}$.

A real number y is irrational provided y is not rational.

1.2. Symbolically write Theorem 1.

1.3. Prove Theorem 1. (You may use the closure properities of $\mathbb{Q}$.)

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HINTS.

1. Symbolically written, Thm. 1 is

$$\left(\forall\,(x,y) \in \mathbb{R}^2\right)\left[(x \in \mathbb{Q} \wedge x \neq 0 \wedge y \notin Q) \implies xy \notin \mathbb{Q}\right]. \tag{1.1}$$

Knowing a real number is irrational does not give us much of a *bird in the hand*. On the other hand, knowing a real number is rational give us some *bird in the hand* (namely a numerator and a denominator). As stated, Thm. 1 has two irrational numbers ⟨y and xy⟩ but only one rational number ⟨x⟩. So let's look at some negations of Thm. 1 in attempt to get more rational numbers ⟨as so to get more birds in the hand⟩. Below are some negations of Thm 1 ⟨some more useful than others⟩.

$$\sim\ \left[\ \left(\forall\,(x,y) \in \mathbb{R}^2\right)\left[(x \in \mathbb{Q} \wedge x \neq 0 \wedge y \notin Q) \implies xy \notin \mathbb{Q}\right]\ \right]$$

$$\left(\exists\,(x,y) \in \mathbb{R}^2\right)\ \sim\ \left[(x \in \mathbb{Q} \wedge x \neq 0 \wedge y \notin Q) \implies xy \notin \mathbb{Q}\right]$$

⟨Recall think of $P \Rightarrow Q$ as a *promise*. So the negation of $P \Rightarrow Q$ is to *break our promise*. Thus $\sim [P \Rightarrow Q] \equiv [P \wedge (\sim Q)]$.⟩

$$\left(\exists\,(x,y) \in \mathbb{R}^2\right)\ \left[(x \in \mathbb{Q} \wedge x \neq 0 \wedge y \notin Q) \wedge xy \in \mathbb{Q}\right]$$

$$\left(\exists\,(x,y) \in \mathbb{R}^2\right)\ \left[x \neq 0 \wedge x \in \mathbb{Q} \wedge xy \in \mathbb{Q} \wedge y \notin \mathbb{Q}\right] \tag{1.2}$$

The statement in (1.2) gives us more to work with ⟨more of a bird in the hand⟩ than (1.1). So let's try a proof by contradiction and assume (1.2) holds ⟨i.e., the negation of what we want to show holds⟩ and go looking for a contradition. Our birds in the hand are: $x \in \mathbb{Q}$ and $xy \in \mathbb{Q}$. Can we somehow use that $x \in \mathbb{Q}$ and $xy \in \mathbb{Q}$ to find a contradition to $y \notin \mathbb{Q}$? The key observation is that, since $x \neq 0$, we can write $y = \left(\frac{1}{x}\right)(xy)$. Actually, Thm. 1 is the book's Proposition 3.19 (p. 123) so you can read the proof in the book. Please note that in Version 2.1 there is a small typo (book forgot to assume $x \neq 0$) but the proof in Version 3 is correct.

2. **Theorem 2**. Let $x, y \in \mathbb{R}$. If y is irrational then $(x+y)$ is irrational or $(x-y)$ is irrational.

2.1. Symbolically write Theorem 2.

2.2. Prove Theorem 2.

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HINTS.

2. Symbolically written, Thm. 2 is

$$\left(\forall\,(x,y)\in\mathbb{R}^2\right)[\ y\notin\mathbb{Q}\implies\{\ (x+y)\notin\mathbb{Q}\vee(x-y)\notin\mathbb{Q}\ \}\ ]. \tag{2.1}$$

Knowing a real number is irrational does not give us much of a *bird in the hand*. On the other hand, knowing a real number is rational give us some *bird in the hand* (namely a numerator and a denominator). As stated, all the real numbers in Thm. 2 are irrational. So let's look at some negations of Thm. 2 in attempt to get some rational numbers ⟨as so to get some birds in the hand⟩. Below are some negations of Thm 2 ⟨some more useful than others⟩.

$$\sim\ \left[\ \left(\forall\,(x,y)\in\mathbb{R}^2\right)[\ y\notin\mathbb{Q}\implies\{\ (x+y)\notin\mathbb{Q}\vee(x-y)\notin\mathbb{Q}\ \}\ ]\ \right]$$
$$\left(\exists\,(x,y)\in\mathbb{R}^2\right)\ \sim\ [\ y\notin\mathbb{Q}\implies\{\ (x+y)\notin\mathbb{Q}\vee(x-y)\notin\mathbb{Q}\ \}\ ]$$

⟨Recall think of $P\Rightarrow Q$ as a *promise*. So the negation of $P\Rightarrow Q$ is to *break our promise*. Thus $\sim[P\Rightarrow Q]\equiv[P\wedge(\sim Q)]$.⟩

$$\left(\exists\,(x,y)\in\mathbb{R}^2\right)\ [\ y\notin\mathbb{Q}\wedge\ \sim\{\ (x+y)\notin\mathbb{Q}\vee(x-y)\notin\mathbb{Q}\ \}\ ]$$
$$\left(\exists\,(x,y)\in\mathbb{R}^2\right)\ [\ y\notin\mathbb{Q}\ \wedge\ (x+y)\in\mathbb{Q}\ \wedge\ (x-y)\in\mathbb{Q}\ ]$$
$$\left(\exists\,(x,y)\in\mathbb{R}^2\right)\ [\ (x+y)\in\mathbb{Q}\ \wedge\ (x-y)\in\mathbb{Q}\ \wedge\ y\notin\mathbb{Q}\ ] \tag{2.2}$$

The statement in (2.2) gives us more to work with ⟨more of a bird in the hand⟩ than (2.1). So let's try a proof by contradiction and assume (2.2) holds ⟨i.e., the negation of what we want to show holds⟩ and go looking for a contradiction. Our birds in the hand are: $x+y\in\mathbb{Q}$ and $x-y\in\mathbb{Q}$. Can we somehow use that $x+y\in\mathbb{Q}$ and $x-y\in\mathbb{Q}$ to find a contradition to $y\notin\mathbb{Q}$? The key observations are: $(x+y)-(x-y)=2y$ and $y=\frac{1}{2}(2y)$.

Proof. We shall show Theorem 2 by contradiction. Thus let $x, y\in\mathbb{R}$ satisfy that

$$x+y\in\mathbb{Q}\qquad\text{and}\qquad x-y\in\mathbb{Q}\qquad\text{and}\qquad y\notin\mathbb{Q}. \tag{2.3}$$

We shall find a contradiction ⟨to our above assumption in (2.3)⟩.

Since both $x+y$ and $x-y$ are rational, $\mathbb{Q}$ is closed under subtraction, and

$$720\,(x+y)-(x-y)=2y$$

we have that $2y\in\mathbb{Q}$. Since $\frac{1}{2}$ is also rational, $\mathbb{Q}$ is closed under multiplication, and

$$y=\frac{1}{2}\,(2y)$$

we get that $y\in\mathbb{Q}$. However we assumed that $y\notin\mathbb{Q}$. So we have a contradiction.

This completes the proof by contradiction of Theorem 2. □

3. **Theorem 3**. Let a and b be natural numbers such that

$$a^2 = b^3.$$

Then we have the following.

3a. If a is even then 4 divides a.
3b. If 4 divides a then 4 divides b.
3c. If 4 divides b then 8 divides a.
3d. If a is even then 8 divides a.

Also

3e. there exists $a, b \in \mathbb{N}$ such that $a^2 = b^3$ and a is even but 8 does not divide b.

3.1. Prove Theorem 3 parts 3a–3e. You may use, without proving, the following theorems from class.
Theorem S. An integer z is even if and only if z^2 is even.
Theorem C. An integer z is even if and only if z^3 is even.

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HINTS. Below are *Thinking Lands*. Let $a, b \in \mathbb{N}$ satisfy

$$a^2 = b^3. \tag{3.1}$$

3a. Let a be even. Then

$$a \text{ even} \xRightarrow{\text{Thm. S}} a^2 \text{ even} \xRightarrow{(3.1)} b^3 \text{ even} \xRightarrow{\text{Thm. C}} b \text{ even}.$$

So $\exists j, k \in \mathbb{Z}$ s.t. $a = 2k$ and $b = 2j$. So

$$a^2 = b^3 \xRightarrow{\text{Alg}} 2^2k^2 = 2^3j^3 \xRightarrow{\text{Alg}} k^2 = 2j^3 \xRightarrow[\text{even}]{\text{def.}} k^2 \text{ even} \xRightarrow{\text{Thm. S}} k \text{ even} \xRightarrow[\text{even}]{\text{def.}} \exists n \in \mathbb{Z} \text{ s.t. } k = 2n.$$

Know $a = 2k$ so get

$$a = 2k = 2(2n) = 4n \xRightarrow[\text{divides}]{\text{def.}} 4 \mid a.$$

3b. Let 4 divide a. Then

$$4 \mid a \xRightarrow[\text{divides}]{\text{def.}} \exists n \in \mathbb{Z} \text{ s.t. } a = 4n \xRightarrow{(3.1)} b^3 = (4n)^2 = 2\left(8n^2\right) \xRightarrow[\text{even}]{\text{def.}} b^3 \text{ even} \xRightarrow{\text{Thm. C}} b \text{ even}$$

$$\xRightarrow[\text{even}]{\text{def.}} \exists k \in \mathbb{Z} \text{ s.t. } b = 2k \xRightarrow{(3.1)} (2k)^3 = (4n)^2 \xRightarrow[4=2^2]{\text{Alg}} 2^3k^3 = 2^4n^2 \xRightarrow{\text{Alg}} k^3 = 2n^2$$

$$\xRightarrow[\text{even}]{\text{def.}} k^3 \text{ even} \xRightarrow{\text{Thm. C}} k \text{ even} \xRightarrow[\text{even}]{\text{def.}} \exists j \in \mathbb{Z} \text{ s.t. } k = 2j \xRightarrow[\text{together}]{\text{put}} b = 2k = 2(2j) = 4j \xRightarrow[\text{divides}]{\text{def.}} 4 \mid b.$$

3c. Let 4 divide b. Then

$$4 \mid b \xRightarrow[\text{divides}]{\text{def.}} \exists k \in \mathbb{Z} \text{ s.t. } b = 4k = 2(2k) \xRightarrow[\text{even}]{\text{def.}} b \text{ even} \xRightarrow{\text{Thm. C}} b^3 \text{ even} \xRightarrow{(3.1)} a^2 \text{ even}$$

$$\xRightarrow{\text{Thm. S}} a \text{ even} \xRightarrow[\text{3a.}]{\text{part}} 4 \mid a \xRightarrow[\text{divides}]{\text{def.}} \exists j \in \mathbb{Z} \text{ s.t. } a = 4j.$$

So now have $j, k \in \mathbb{Z}$ s.t. $a = 4j$ and $b = 4k$. So get

$$a^2 = b^3 \implies (4j)^2 = (4k)^3 \xRightarrow{\text{Alg}} 4^2j^2 = 4^3k^3 \xRightarrow{\text{Alg}} j^2 = 4k^3 = 2(2k^3) \xRightarrow[\text{even}]{\text{def.}} j^2 \text{ even}$$

$$\xRightarrow{\text{Thm. S}} j \text{ even} \xRightarrow[\text{even}]{\text{def.}} \exists n \in \mathbb{Z} \text{ s.t. } j = 2n \xRightarrow{\text{Alg}} a = 4j = 4(2n) = 8n \xRightarrow[\text{divides}]{\text{def.}} 8 \mid a.$$

3d. Follows from parts: 3a, 3b, 3c.

3e. $a = 8$ and $b = 4$. Think prime fractorizations: $a = \prod_{i=1}^{n} (p_i)^{k_i}$ and $b = \prod_{i=1}^{m} (q_i)^{j_i}$. Since a is even $\boxed{p_1 = 2}$.

$$\prod_{i=1}^{n} (p_i)^{2k_i} \underset{(3.1)}{\overset{\text{by}}{=}} \prod_{i=1}^{m} (q_i)^{3j_i} \xRightarrow[\text{prime factoriz.}]{\text{uniqueness of}} n = m \text{ and } \forall i \in \{1, 2, \ldots, n\} : p_i = q_i \text{ and } 2k_i = 3j_i.$$

So can take $n = m = 1$ and $p_1 = q_1 = 2$ and $\langle$for $2k_i = 3j_i\rangle$ $k_1 = 3$, and $j_1 = 2$.

4. **Theorem 4**. There does not exist an integer x such that

$$x \equiv 4 \pmod 9 \qquad \text{and} \qquad x \equiv 5 \pmod 6.$$

4.1. Explain why we cannot apply modulo arithmetic to the congruences as they are written in Thm. 4.

4.2. Prove Theorem 4.

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HINTS.

4.1. To apply modulo arithmetic to two congruences, the two congruences must be of the same modulo. As the two congruences in Thm. 4 are stated, one congruence is modulo 9 while the other congruence is modulo 6.

4.2. *Thinking Land.* The fact that such an x does not exist does not give us much of a *bird in the hand*. On the other hand, if such an x were to exist, we would have some *birds in the hand*. So let's try a proof by contradiction and assume the negation of Thm. 4 holds ⟨i.e.,such an x does exist⟩ and go looking for a contradition.

The negation of Thm. 4 is that there exists $x \in \mathbb{Z}$ such that $x \equiv 4 \pmod 9$ and $x \equiv 5 \pmod 6$. Given our answer to part 4.1, we cannot apply modulo arithmetic (MA) to the given congruences as stated. But since 3 divides both 9 and 6, each of the given congruences should imply some congruence modulo 3. Then we can apply MA to these congruences modulo 3.

Proof. We shall show Theorem 4 by contradiction. Thus let $x \in \mathbb{Z}$ satisfy

$$x \equiv 4 \pmod 9 \qquad \text{and} \qquad x \equiv 5 \pmod 6. \tag{4.1}$$

We shall find a contradiction.

Since $x \equiv 4 \pmod 9$, there exists $k \in \mathbb{Z}$ such that $x = 9k + 4$. Thus

$$x = 9k + 4 = 3(3k) + 3 + 1 = 3(3k + 1) + 1$$

and $3k + 1 \in \mathbb{Z}$ by closure properties of $\mathbb{Z}$; thus,

$$x \equiv 1 \pmod 3. \tag{4.2}$$

Similarly, since $x \equiv 5 \pmod 6$, there exists $j \in \mathbb{Z}$ such that $x = 6j + 5$. Thus

$$x = 6j + 5 = 3(2j) + 3 + 2 = 3(2j + 1) + 2$$

and $2j + 1 \in \mathbb{Z}$ by closure properties of $\mathbb{Z}$; thus,

$$x \equiv 2 \pmod 3. \tag{4.3}$$

Since there exists a unique $r \in \{0, 1, 2\}$ such that $x \equiv r \pmod 3$, equations (4.2) and (4.3) provide a contradiction to our assumption in (4.1).

We have just shown that assuming Theorem 4 is false leads to a contradiction. Thus Theorem 4 is true. □

5. **Theorem 5**. There is a unique natural number n such that n and $n+1$ are both primes.

5.1. Complete the following definition.

A natural number n is prime provided $n \neq 1$ and only natural numbers that are factors of n are 1 and n .

5.2. Symbolically write Theorem 5.

5.3. Prove Theorem 5.

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HINTS.

5.1.
$$(\exists! n \in \mathbb{N}) \, [\ (n \text{ is prime}) \ \wedge \ (n+1 \text{ is prime})\ ]$$

5.2. First agrue there exists an $n \in \mathbb{N}$ such that both n and $n+1$ are primes by constructing/finding such an n. Will get $n = 2$ since both 2 are 3 are primes. Next are that $n = 2$ is the only possible (i.e., unique) n such that both n and $n+1$ are primes. Since we will use the below lemma twice, we will begin by stating and proving the lemma.

Lemma P. The only even prime number is 2.

Proof of Lemma P. We know that 2 is an even prime number. Let p be an arbitrary even prime number. We shall show that $p = 2$.

Since p is prime, if a natural number k divides p then $k \in \{1, p\}$. Since p is even, we know $2 \mid p$. So $2 \in \{1, p\}$. Thus $p = 2$.

We have just shown that the only even prime is 2. □

Proof of Thm. 5. We shall show Theorem 5, which says that there exists unique natural number n such that n and $n+1$ are both primes, by first showing that there existence such an n and then showing that this n is the only possible n.

Towards the *existence* part, let $n = 2$. Then $n + 1 = 3$. Note 2 and 3 are both primes. Thus $n = 2$ satisfies that both n and $n+1$ are prime. This complete the existence part.

Towards the *uniqueness* part, let $n \in \mathbb{N}$ be such that both n and $n+1$ are primes. We will show that $n = 2$ by cases: n is even and n is odd.

For case 1, let n be even. Then by Lemma P, $n = 2$.

For case 2, let n be odd. Then by Lemma SOO, $n+1$ is even. We are assuming that $n+1$ is prime so by Lemma P, $n + 1 = 2$. Thus $n = 1$. But n is prime and 1 is not prime so n cannot be 1. Thus case 2 cannot occur.

We have just shown that if $n \in \mathbb{N}$ such that both n and $n+1$ are primes, then $n = 2$.

This completes the proof. □

6. **Theorem 6**. Let I be a nonempty arbitrary indexing set and $\{A_i : i \in I\}$ be a collection of subsets of some universeral set U. Then

$$\left[\bigcap_{i\in I} A_i\right]^C = \bigcup_{i\in I} (A_i)^C \ .$$

6.1. Clearly explain why Thm. 6 is true. Use complete sentences. You may (and are encouraged to) use symbolic notation in your explanation. Hint: Write out equivalent statements for $x \in \left[\, \bigcap_{i\in I} A_i \,\right]^C$.

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HINTS. For sets S and T, to show that $S = T$ we often show that $S \subseteq T$ and $T \subseteq S$, i.e., we show *set containment holds in both directions*. Note the below proof is an example of where both directions can easily be *done at the same time*.

Proof. Let I be a nonempty arbitrary indexing set and $\{A_i : i \in I\}$ be a collection of subsets of some universeral set U. We shall show that

$$\left[\bigcap_{i\in I} A_i\right]^C = \bigcup_{i\in I} (A_i)^C \ . \tag{6.1}$$

Let $x \in U$. Note the following statements are equivalent, using symbolic notation when helpful.

$$x \in \left[\bigcap_{i\in I} A_i\right]^C \tag{6.2}$$

and by definition of complement

$$x \notin \bigcap_{i\in I} A_i$$

and by definition of intersection

$$\sim \ \{\ (\forall i \in I)\, [x \in A_i]\ \}$$

and by rules of negation

$$(\exists i \in I)\, [x \notin A_i]$$

and by definition of complement

$$(\exists i \in I)\, [x \in (A_i)^c]$$

and by definition of union

$$x \in \bigcup_{i\in I} (A_i)^c \tag{6.3}$$

Thus (6.2) and (6.3) are equivalent, showing that

$$x \in \left[\bigcap_{i\in I} A_i\right]^C \qquad \text{if and only if} \qquad x \in \bigcup_{i\in I} (A_i)^c$$

Thus (6.1) holds. □

7. A Challenging Problem.

Def. Let $f\colon X \to Y$ be a function from a set X into a set Y. Let $B \subseteq Y$.

The preimage of B under f, denoted by $f^{-1}[B]$, is the set $f^{-1}[B] \stackrel{\text{def}}{=} \{x \in X : f(x) \in B\}$.

Note. So $\quad x \in f^{-1}[B] \xLeftrightarrow[\text{of preimage}]{\text{by def.}} f(x) \in B.$

Theorem 7. Let $f\colon X \to Y$ be a function from a set X into a set Y.

Let $B_i \subseteq Y$ for each i in a nonempty index set I. Then

$$f^{-1}\left[\bigcap_{i\in I} B_i\right] \subseteq \bigcap_{i\in I} f^{-1}[B_i].$$

7.1. Clearly explain why Thm. 7 is true. Use complete sentences. You may (and are encouraged to) use symbolic notation in your explanation. Hint: Let $\langle$your hypothesis$\rangle$ $x \in f^{-1}\left[\bigcap_{i\in I} B_i\right]$. Write out what implications you get from your hypothesis until you get to your wanted conclusion that $x \in \bigcap_{i\in I} f^{-1}[B_i]$.

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HINTS. Compare the below proof to the proof of DeMorgan's law for sets. Recall

$$\boxed{x \in f^{-1}[B] \xLeftrightarrow[\text{of preimage}]{\text{by def.}} f(x) \in B}.$$

Proof. Consider the function $f\colon X \to Y$. Let $B_i \subseteq Y$ for each i in a nonempty index set I. Let $\langle$since $f^{-1}\left[\bigcap_{i\in I} B_i\right] \subseteq X$, let's call an element in $f^{-1}\left[\bigcap_{i\in I} B_i\right]$ by $x\rangle$

$$x \in f^{-1}\left[\bigcap_{i\in I} B_i\right].$$

We shall show that

$$x \in \bigcap_{i\in I} f^{-1}[B_i].$$

Since $x \in f^{-1}\left[\bigcap_{i\in I} B_i\right]$, by definition of preimage, we get

$$f(x) \in \bigcap_{i\in I} B_i.$$

By definition of intersection, we get

$$(\forall i \in I)\ [f(x) \in B_i].$$

By definition of preimage, we get

$$(\forall i \in I)\ \left[x \in f^{-1}[B_i]\right].$$

By definition of intersection, we get

$$x \in \bigcap_{i\in I} f^{-1}[B_i].$$

We have just shown that if $x \in f^{-1}\left[\bigcap_{i\in I} B_i\right]$ then $x \in \bigcap_{i\in I} f^{-1}[B_i]$. This completes the proof. $\square$

8. A Really Challenging Problem.

Theorem 8. For every (strictly) positive real number ϵ there is a (strictly) positive real number δ such that for each real number x, if $2 < x < 3 + \delta$ then $4 < x^2 < 9 + \epsilon$.

8.1. Fill in the two blanks as so to symbolically write Theorem 8.

$$(\forall \epsilon \in \mathbb{R}^{>0})\,(\exists \delta \in \mathbb{R}^{>0})\,(\forall x \in \mathbb{R})\,[\,(\ \underline{\quad 2 < x < 3 + \delta \quad}\ ) \implies (\ \underline{\quad 4 < x^2 < 9 + \epsilon \quad}\ )\,]$$

8.2. Prove Theorem 8. Hint. Your δ will have a ϵ in it, i.e., δ is a function of ϵ.

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HINTS. *Thinking Land.* Fix $\epsilon \in \mathbb{R}^{>0}$ (i.e., $\epsilon > 0$). We want to find the $\delta > 0$ but to do this we first have to do some calculations. So let's look at what such a $\delta > 0$ would have to look like.

Let's say we know that $2 < x < 3 + \delta$. Then algebra gives us that $4 < x^2 < 9 + 6\delta + \delta^2$. So if $9 + 6\delta + \delta^2 = 9 + \epsilon$, then we would be done. But note $9 + 6\delta + \delta^2 = 9 + \epsilon \Leftrightarrow \delta^2 + 6\delta - \epsilon = 0 \Leftrightarrow \delta = \frac{-6 \pm \sqrt{36+4\epsilon}}{2}$. Algebra gives $\frac{-6 \pm \sqrt{36+4\epsilon}}{2} = \frac{-6 \pm \sqrt{4(9+\epsilon)}}{2} = -3 \pm \sqrt{9+\epsilon}$. So take $\delta := \sqrt{9+\epsilon} - 3 \overset{\text{so}}{>} 0$.

For guidance with the proof's first paragraph, look at the symbolic writting of Theorem 8.

Proof. Let $\epsilon > 0$. Set $\delta := \sqrt{9+\epsilon} - 3$. Note $\delta > 0$ since $\delta = \sqrt{9+\epsilon} - 3 > \sqrt{9} - 3 = 0$. Let $x \in \mathbb{R}$ satisfy $2 < x < 3 + \delta$. We shall show that $4 < x^2 < 9 + \epsilon$.
Since we know that $2 < x$ we have $4 < x^2$. Towards the upper bound on x^2, since $x < 3 + \delta$

$$x^2 < [3 + \delta]^2$$

and since $\delta = \sqrt{9+\epsilon} - 3$

$$< \left[\ 3 + \left(\sqrt{9+\epsilon} - 3\right)\ \right]^2$$

and now by algebra

$$= \left[\ \left(\sqrt{9+\epsilon}\right)\ \right]^2$$
$$= 9 + \epsilon.$$

We have just shown that $4 < x^2 < 9 + \epsilon$.

This completes the proof. $\square$