These *Jam Problems* are a sampling of the type of problems which could be an the final. These problem are, in no way, meant as a comprehensive review for the cumulative final.

1. **Theorem 1**. For all real numbers x and y, if x is rational, $x \neq 0$ and $y \notin \mathbb{Q}$, then xy is irrational.

1.1. Complete the following definitions.

A real number x is rational provided ____ .

A real number y is irrational provided ____ .

1.2. Symbolically write Theorem 1.

1.3. Prove Theorem 1. (You may use the closure properities of $\mathbb{Q}$.)

2. **Theorem 2**. Let $x, y \in \mathbb{R}$. If y is irrational then $(x + y)$ is irrational or $(x - y)$ is irrational.

2.1. Symbolically write Theorem 2.

2.2. Prove Theorem 2.

3. **Theorem 3**. Let a and b be natural numbers such that

$$a^2 = b^3.$$

Then we have the following.

3a. If a is even then 4 divides a.

3b. If 4 divides a then 4 divides b.

3c. If 4 divides b then 8 divides a.

3d. If a is even then 8 divides a.

Also

3e. there exists $a, b \in \mathbb{N}$ such that $a^2 = b^3$ and a is even but 8 does not divide b.

3.1. Prove Theorem 3 parts 3a–3e. You may use, without proving, the following theorems from class.

Theorem S. An integer z is even if and only if z^2 is even.

Theorem C. An integer z is even if and only if z^3 is even.

4. **Theorem 4**. There does not exist an integer x such that

$$x \equiv 4 \pmod 9 \qquad \text{and} \qquad x \equiv 5 \pmod 6.$$

4.1. Explain why we cannot apply modulo arithmetic to the congruences as they are written in Thm. 4.

4.2. Prove Theorem 4.

5. **Theorem 5**. There is a unique natural number n such that n and $n+1$ are both primes.

5.1. Complete the following definition.

A natural number n is prime provided ______________________________ .

5.2. Symbolically write Theorem 5.

5.3. Prove Theorem 5.

6. **Theorem 6**. Let I be a nonempty arbitrary indexing set and $\{A_i \colon i \in I\}$ be a collection of subsets of some universeral set U. Then

$$\left[\bigcap_{i\in I} A_i\right]^C = \bigcup_{i\in I} (A_i)^C \quad .$$

6.1. Clearly explain why Thm. 6 is true. Use complete sentences. You may (and are encouraged to) use symbolic notation in your explanation. Hint: Write out equivalent statements for $x \in \left[\,\bigcap_{i\in I} A_i\,\right]^C$.

7. A Challenging Problem.

Def. Let $f \colon X \to Y$ be a function from a set X into a set Y. Let $B \subseteq Y$.

The preimage of B under f, denoted by $f^{-1}[B]$, is the set $f^{-1}[B] \stackrel{\text{def}}{=} \{x \in X \, : f(x) \in B\}$.

Note. So $\quad x \in f^{-1}[B] \xLeftrightarrow[\text{of preimage}]{\text{by def.}} f(x) \in B$.

Theorem 7. Let $f \colon X \to Y$ be a function from a set X into a set Y.

Let $B_i \subseteq Y$ for each i in a nonempty index set I. Then

$$f^{-1}\left[\bigcap_{i\in I} B_i\right] \subseteq \bigcap_{i\in I} f^{-1}[B_i] \,.$$

7.1. Clearly explain why Thm. 7 is true. Use complete sentences. You may (and are encouraged to) use symbolic notation in your explanation. Hint: Let ⟨your hypothesis⟩ $x \in f^{-1}\left[\bigcap_{i\in I} B_i\right]$. Write out what implications you get from your hypothesis until you get to your wanted conclusion that $x \in \bigcap_{i\in I} f^{-1}[B_i]$.

8. A Really Challenging Problem.

Theorem 8. For every (strictly) positive real number ϵ there is a (strictly) positive real number δ such that for each real number x, if $2 < x < 3+\delta$ then $4 < x^2 < 9+\epsilon$.

8.1. Fill in the two blanks as so to symbolically write Theorem 8.

$(\forall \epsilon \in \mathbb{R}^{>0})\,(\exists \delta \in \mathbb{R}^{>0})\,(\forall x \in \mathbb{R})$ [(______________) $\Longrightarrow$ (______________)]

8.2. Prove Theorem 8. Hint. Your δ will have a ϵ in it, i.e., δ is a function of ϵ.