MARK BOX		
PROBLEM	POINTS	
0A	11	
0B	8	
0C	8	
0D	3	
Total for 0	30	
1	10	
2	10	
3	10	
4-13	40=10x4	
%	100	

NAME: Solutions

PIN: 17

INSTRUCTIONS

- Instructions for the different types of problems on this exam.
  1. Problem 0.
     - Fill-in the blank/box or circle the correct answer (can circle at most one choice).
     - No need to show work.
  2. Multiple Choice.
     - Circle the correct answer. Circle at most one choice.
     - No need to show work (but can do scratch work if needed). No partial credit.
  3. Work Out.
     - Partial credit possible.
     - Justify your answer **below** the provided box/line.
       * Show all your work. Work in a logical fashion. Use proper notation.
       * Explain your thoughts (using English words).
       * You will be graded on the quality and correctness of your justification.
     - Then put your answer **in/on** the provided box/line.
     - No points for a correct answer that does not have a justification.
- The MARK BOX above indicates the problems (check you have them all) along with their points.
- Write your PIN on the top center on **each page**.
- **Write on only the front side of the paper**. Only the front side will graded.
- If you run out of space for a solution at the place the solution should go, then leave me a note at the where you solution should be telling me where to go look for the rest of your solution.
- During the exam, the use of unauthorized materials is prohibited. Unauthorized materials include: books, notes, eletronic devices (e.g., cell phone, smart watch, earbuds). Unauthorized materials (including cell phones) must be in a secured (e.g. zipped up, snapped closed) bag placed completely under your desk or, if you did not bring such a bag, then place on the front desk during the exam (and pick it after the exam). **So no electronic devices allowed in your pockets**.
- At a student's request, a clock will be projected to the screen.
- During this exam, do not leave your seat without permission.
- If you have a question, then raise your hand.
- When you finish: turn your exam over, put your pencil down and raise your hand.
- No portion, nor the statements of the problems, may leave the classroom.
- Not following instructions can result in a lose of points. Cheating is grounds for a F in the course.
- This exam covers (from *Calculus* by Thomas, 13^{th} ed., ET): §10.7–10.10, 11.1–11.4.

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the instructions on the exam.

Signature : ____________________

The rest of this page is left intentionally blank.
You may use the space as scratch paper.
Nothing written below will be looked at nor graded.

0A. Power Series Condsider a (formal) power series

$$h(x) \;=\; \sum_{n=0}^{\infty} a_n\,(x-x_0)^n\;, \tag{0.1}$$

with radius of convergence $R \in [0,\infty]$.
(Here $x_0 \in \mathbb{R}$ is fixed and $\{a_n\}_{n=0}^{\infty}$ is a fixed sequence of real numbers.)

- For this part, answer the 4 questions by circling one and only one choice. Abbreviations: AC for absolutely convergent, CC for conditionally convergent, and DV for divergent.

  (1) For $x = x_0$, the power series $h(x)$ in (0.1)
  a. is always AC b. is always CC c. is always DV
  d. can do anything, i.e., there are examples showing that it can be AC, CC, or DV.

  (2) For $x \in \mathbb{R}$ such that $|x - x_0| < R$, the power series $h(x)$ in (0.1)
  a. is always AC b. is always CC c. is always DV
  d. can do anything, i.e., there are examples showing that it can be AC, CC, or DV.

  (3) For $x \in \mathbb{R}$ such that $|x - x_0| > R$ the power series $h(x)$ in (0.1)
  a. is always AC b. is always CC c. is always DV
  d. can do anything, i.e., there are examples showing that it can be AC, CC, or DV.

  (4) If $0 < R < \infty$, then for the endpoints $x = x_0 \pm R$, the power series $h(x)$ in (0.1)
  a. is always AC b. is always CC c. is always DV
  d. can do anything, i.e., there are examples showing that it can be AC, CC, or DV.

- For this part, fill in the 7 boxes.
  Let $R > 0$ and donsider the function $y = h(x)$ defined by the power series in (0.1).

  (1) The function $y = h(x)$ is <u>always differentiable</u> on the interval $\boxed{(x_0 - R, x_0 + R)}$ (make this interval as large as it can be, but still keeping the statement true). Furthermore, on this interval

  $$h'(x) \;=\; \sum_{n=\boxed{1}}^{\infty} \boxed{n\,a_n\,(x-x_0)^{n-1}}\;. \tag{0.2}$$

  What can you say about the radius of convergence of the power series in (0.2)?

  $\boxed{\text{The power series in (0.2) has the same raduis of convergence as the power series in (0.1).}}$

  (2) The function $y = h(x)$ <u>always has an antiderivative</u> on the interval $\boxed{(x_0 - R, x_0 + R)}$ (make this interval as large as it can be, but still keeping the statement true). Futhermore, if α and β are in this interval, then

  $$\int_{x=\alpha}^{x=\beta} h(x)\,dx \;=\; \sum_{n=\boxed{0}}^{\infty} \boxed{\frac{a_n}{n+1}\,(x-x_0)^{n+1}}\Bigg|_{\mathbf{x}=\alpha}^{\mathbf{x}=\beta}\;.$$

0B. Taylor/Maclaurin Polynomials and Series.

For this part, fill-in the 9 boxes.

Let $y = f(x)$ be a function with derivatives of all orders in an interval I containing x_0.

Let $y = P_N(x)$ be the N^{th}-order Taylor polynomial of $y = f(x)$ about x_0.

Let $y = R_N(x)$ be the N^{th}-order Taylor remainder of $y = f(x)$ about x_0.

Let $y = P_\infty(x)$ be the Taylor series of $y = f(x)$ about x_0.

Let c_n be the n^{th} Taylor coefficient of $y = f(x)$ about x_0.

a. The formula for c_n is

$$c_n = \boxed{\frac{f^{(n)}(x_0)}{n!}}$$

b. In open form (i.e., with ... and without a $\sum$-sign)

$$P_N(x) = \boxed{f(x_0) + f'(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x-x_0)^N}$$

c. In closed form (i.e., with a $\sum$-sign and without ...)

$$P_N(x) = \boxed{\sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}\,(x-x_0)^n}$$

d. In open form (i.e., with ... and without a $\sum$-sign)

$$P_\infty(x) = \boxed{f(x_0) + f'(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \ldots}$$

e. In closed form (i.e., with a $\sum$-sign and without ...)

$$P_\infty(x) = \boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}\,(x-x_0)^n}$$

f. We know that $f(x) = P_N(x) + R_N(x)$. Taylor's BIG Theorem tells us that, for each $x \in I$,

$$R_N(x) = \boxed{\frac{f^{(N+1)}(c)}{(N+1)!}\,(x-x_0)^{(N+1)}} \text{ for some } c \boxed{\text{between } x \text{ and } x_0}.$$

g. A Maclaurin series is a Taylor series with the center specifically specified as $x_0 = \boxed{0}$.

0C. Commonly Used Taylor Series
Fill in the blank boxes with the choices a – ℓ, which are provided below.
You may use a choice more than once or not at all.
Sample questions, which are needed later in the exam, are already done for you.

a. $\sum_{n=0}^{\infty} x^n$ **d.** $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ **g.** $x \in \mathbb{R}$ **j.** $(-1, 1]$

b. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ **e.** $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ **h.** $(-1, 1)$ **k.** $[-1, 1)$

c. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$ **f.** $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ **i.** $[-1, 1]$ **ℓ.** none of the others

sample. A power series expansion for $y = \frac{1}{1-x}$ is $\boxed{\text{a}}$ and is valid precisely when $\boxed{\text{h}}$.

sample. A power series expansion for $y = e^x$ is $\boxed{\text{b}}$ and is valid precisely when $\boxed{\text{g}}$.

0.1. A power series expansion for $y = \cos x$ is $\boxed{\text{d}}$ and is valid precisely when $\boxed{\text{g}}$.

0.2. A power series expansion for $y = \sin x$ is $\boxed{\text{e}}$ and is valid precisely when $\boxed{\text{g}}$.

0.3. A power series expansion for $y = \ln(1 + x)$ is $\boxed{\text{c}}$ and is valid precisely when $\boxed{\text{j}}$.

0.4. A power series expansion for $y = \tan^{-1} x$ is $\boxed{\text{f}}$ and is valid precisely when $\boxed{\text{i}}$.

0D. Parametric Curves In this part, fill in the 3 boxes. Consider the curve $\mathcal{C}$ parameterized by

$$\begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned}$$

for $a \leq t \leq b$.

1) Express $\frac{dy}{dx}$ in terms of derivatives with respect to t. Answer: $\frac{dy}{dx} = \boxed{\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}}$

2) The tangent line to $\mathcal{C}$ when $t = t_0$ is $y = mx + b$ where m is $\boxed{\frac{dy}{dx}}$ evaluated at $t = t_0$.

3) The arc length of $\mathcal{C}$, expressed as on integral with respect to t, is

$$\text{Arc Length} = \boxed{\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt}$$

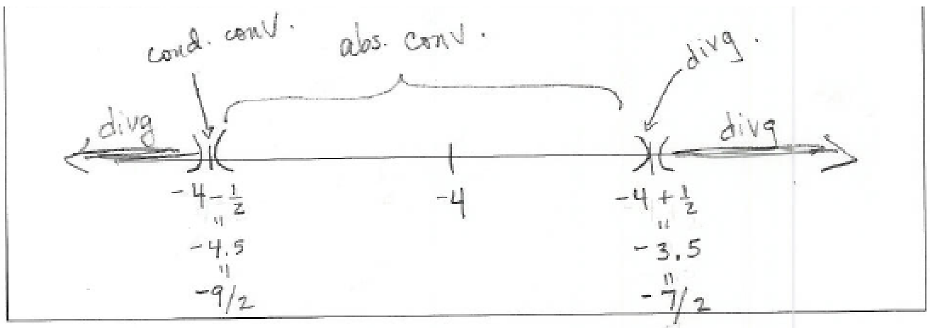
Work Out: problems 1–3.

1. Consider the formal power series

$$\sum_{n=1}^{\infty} \frac{(2x+8)^n}{n} \,.$$

As we did in class, in the box below draw a diagram indicating for which x's this series is: absolutely convergent, conditionally convergent, and divergent. Of course, indicate your reasoning. Don't forget to check the endpoints, if there are any.

To find center: $\frac{(2x+8)^n}{n} = \frac{2^n\left(x - {}^-4\right)^n}{n}$ so center is at -4



Ratio Test for abs. conv

$$\rho = \lim_{n\to\infty} \left| \frac{2^{n+1}(x+4)^{n+1}}{n+1} \cdot \frac{n}{2^n (x+4)^n} \right| = 2\,|x+4| \lim_{n\to\infty} \frac{n}{n+1}$$

$$= 2\,|x - {}^-4| < 1 \iff |x - {}^-4| < \frac{1}{2}$$

Check endpoints

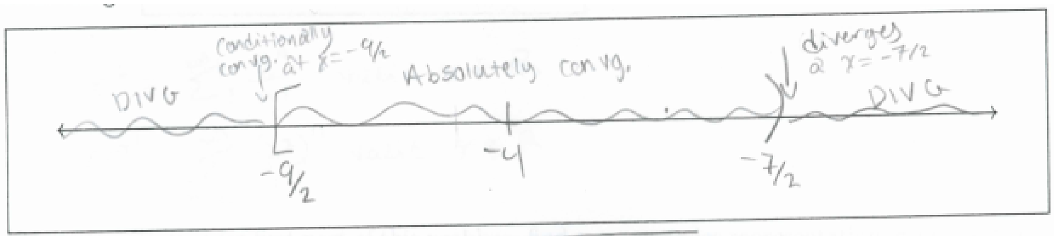
$x = {}^-7/2$: $\sum \frac{(2x+8)^n}{n} = \sum \frac{1}{n}$ ← divges, harmonic series or p-series, $p = 1 \leq 1$

$x = -9/2$ $\sum \frac{(2x+8)^n}{n} = \sum \frac{(-1)^n}{n}$ ← cond. conv.

- $\sum \frac{1}{n}$ divg
- $\sum \frac{(-1)^n}{n}$ conv by AST

Nest page has anoher sample solution.

Another solution, this one using the root test.



$$\sum_{n=1}^{\infty} \frac{(2x+8)^n}{n} = \sum_{n=1}^{\infty} \underbrace{\frac{2(x-(-4))^n}{n}}_{=a_n} \qquad \boxed{x_0 = -4}$$

root Test $\rho: \lim_{n\to\infty} |a_n|^{1/n} \overset{\lim_{n\to\infty}}{=} \left|\frac{(2(x-(-4)))^n}{n}\right|^{1/n} \overset{\lim_{n\to\infty}}{=} \left|\frac{(2(x-(-4)))}{\sqrt[n]{n}}\right|$

$$2(x-(-4)) \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = |2(x-(-4))|\cdot 1 < 1 \qquad \boxed{R=\frac{1}{2}}$$

$$|x-(-4)| < \frac{1}{2}$$

Interval of conv: $(x_0 - R, x_0 + R)$

$$\left(-4-\frac{1}{2}, -4+\frac{1}{2}\right) = \left(-\frac{8}{2}-\frac{1}{2}, -\frac{8}{2}+\frac{1}{2}\right) = \left(-\frac{9}{2}, -\frac{7}{2}\right)$$

Check endpoints

$x: -\frac{9}{2}$ $\quad \sum_{n=1}^{\infty} \frac{(\cancel{2}(-\frac{9}{\cancel{2}})+8)^n}{n} = \sum_{n=1}^{\infty} \frac{(-9+8)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \sum_{n=1}^{\infty} (-1)^n\left(\frac{1}{n}\right)$ $\leftarrow u_n$

AST
- $u_n > 0$ ✓
- u_n decrease ✓
  - $u_{n+1} < u_n$ ✓
- $\lim_{n\to\infty} u_n = 0$ ✓

$\sum (-1)^n u_n$ converges ✓

∘ the series conditionally conv. @ $x=-\frac{9}{2}$

$x: -\frac{7}{2}$ $\quad \sum_{n=1}^{\infty} \frac{(2(-7/2)+8)^n}{n} = \sum_{n=1}^{\infty} \frac{(-7+8)^n}{n} = \sum_{n=1}^{\infty} \frac{(1)^n}{n} = \sum_{n=1}^{\infty} (1)^n\left(\frac{1}{n}\right)$

$\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)$ ∘ p-series with $p=1$, when $p \le 1$, the series divg. so the series divg at $x=-\frac{7}{2}$

2. Hint: $e^{x^2} = e^{\left(x^2\right)}$.

2.1. Using a *Commonly Used Taylor Series* (see probelm **0C**), find a power series representation, centered about $x_0 = 0$, for the function

$$f\left(x\right) = e^{x^2} \tag{2.1}$$

and say when it is valid.
Express your series in **CLOSED** form (i.e., with a $\sum$-sign and without $\ldots$).

Soln: $e^{x^2} = \boxed{\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}}$, valid when $x \in \mathbb{R}$ also correct $x \in (-\infty, \infty)$

A *Commonly Used Taylor Series*: $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$, valid for $t \in \mathbb{R}$. Now let $t = x^2$. Note $\left(x^2\right)^n = x^{2n}$.

2.2. Using your solution for first part of this problem, find a power series representation, centered about $x_0 = 0$, for

$$\int e^{x^2}\, dx. \tag{2.2}$$

Express your series in **CLOSED** form (i.e., with a $\sum$-sign and without $\ldots$).
Note: we cannot express $\int e^{x^2}\, dx$. as an elementary function (loosely speaking, you cannot integrate e^{x^2}).

Soln: $\int e^{x^2}\, dx = \boxed{\sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!\,(2n+1)} \qquad + \text{C}}$

$$\begin{aligned}
\int e^{x^2}\, dx &= \int \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \right) dx \\
&= \sum_{n=0}^{\infty} \left(\int \frac{x^{2n}}{n!}\, dx \right) \\
&= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!\,(2n+1)} + C.
\end{aligned}$$

3. Hint: $\sqrt{e} = e^{\frac{1}{2}}$.

3.1. Using a *Commonly Used Taylor Series* (see probelm **0C**), express the number

$$\sqrt{e} \tag{3.1}$$

as a numerical series. Express your series in **CLOSED** form (i.e., with a $\sum$-sign and without $\ldots$).

Soln: $\sqrt{e} = \boxed{\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{n!} \text{, also correct } \sum_{n=0}^{\infty} \frac{1}{n!2^n}}$

$f(x) = e^t$ $\qquad e^x = \sum \frac{x^n}{n!}$ $\qquad e^{1/2} = \sum \frac{(1/2)^n}{n!}$

A *Commonly Used Taylor Series*: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, valid for $x \in \mathbb{R}$. Now let $x = \frac{1}{2}$.

3.2. In the first part of this problem, you found a_n's so that $\sqrt{e} = \sum_{n=0}^{\infty} a_n$. Now estimate the error in approximating $\sqrt{e}$ by the partial sum $\sum_{n=0}^{2} a_n$ of your infinite series $\sum_{n=0}^{\infty} a_n$ in Part **3.1**.

answer: $\left| \sqrt{e} - \sum_{n=0}^{2} a_n \right| \leq \frac{\sqrt{e}}{3!\ 2^3}$

let $x_0 = 0$

$(N=2)$ $\qquad R_N = \frac{f^{(N+1)}(c)}{(N+1)!}(x-x_0)^{N+1} = \frac{f^3(c)}{3!}x^3 = \frac{e^c x^3}{3!}$

Let $x = \frac{1}{2}$

$\frac{e^c\left(\frac{1}{2}\right)^3}{3!} = \frac{e^c\left(\frac{1}{8}\right)}{3!} < \frac{e^{\frac{1}{2}}\left(\frac{1}{8}\right)}{3!}$

$0 \leq c \leq \frac{1}{2}$

Problem from the nice read *How to Ace Calculus* (p. 59). Let $f(x) = e^x$ and $x_0 = 0$. Note $f^{(n)} = e^x$ for each $n \in \mathbb{N}$. Write $f(x) = P_2(x) + R_2(x)$ where $y = P_2(x)$ be the 2nd-order Taylor polynomial and $y = R_2(x)$ be the 2nd-order Taylor remainder of $f(x) = e^x$ about x_0. Letting $x = \frac{1}{2}$ we get

$$\left| \sqrt{e} - \sum_{n=0}^{2} a_n \right| = \left| e^{\frac{1}{2}} - P_2\left(\frac{1}{2}\right) \right| = \left| R_2\left(\frac{1}{2}\right) \right| \overset{\text{Taylor Remainder}}{\underset{\text{Theorem}}{=}} \left| \frac{f^{(3)}(c)}{3!}\left(\frac{1}{2} - 0\right)^3 \right| = \frac{e^c}{3!2^3}$$

for some c between $\frac{1}{2}$ and 0. Since $0 \leq c \leq \frac{1}{2}$ and $f(x) = e^x$ is an increasing function, $e^c \leq e^{\frac{1}{2}}$. So

$$\frac{e^c}{3!2^3} \leq \frac{e^{\frac{1}{2}}}{3!2^3} \overset{\text{i.e.}}{=} \frac{\sqrt{e}}{3!2^3}.$$

Note we cannot use the AST remainder theorem since we do not have an Alternating Series.

Multiple Choice: problems 4-13.

4. Let the function $y = f(x)$ have a power series power series representation $\sum_{n=0}^{\infty} c_n x^n$, which is valid in some interval $(-R, R)$ where $R > 0$.

4soln. If a function can be represented by a power series centered at 0 on some interval $(-R, R)$, with $R > 0$, then that power series must be the Taylor series centered at 0. So $c_0 = \frac{f^{(0)}(0)}{0!} = f(0)$.

5. Let the function $y = f(x)$ have a power series power series representation $\sum_{n=0}^{\infty} a_n x^n$, which is valid in some interval J containing 0 and the raduis of J striclty positive. Consider the two statements:
(A) If $y = f(x)$ is an even function (i.e., $f(-x) = f(x)$), then $a_1 = a_3 = a_5 = \cdots = 0$.
(B) If $y = f(x)$ is an odd function (i.e., $f(-x) = -f(x)$), then $a_0 = a_2 = a_4 = \cdots = 0$.

5soln. Both (A) and (B) are true.

►. **Problems 4 and 5 were meant to help you with Problem 0C.** ☺☺☺

*2.

Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for all x in an open interval $(-R, R)$.

a. Show that if f is even, then $a_1 = a_3 = a_5 = \cdots = 0$, i.e., the Taylor series for f at $x = 0$ contains only even powers of x.

b. Show that if f is odd, then $a_0 = a_2 = a_4 = \cdots = 0$, i.e., the Taylor series for f at $x = 0$ contains only odd powers of x.

It is known that all power series that converge to a function f(x) on an interval $(-R,R)$ are the same. This is a key property of power series that will be needed to complete this proof.

a. If f(x) is even, then $f(-x) =$ (1) ________.

Substitute $-x$ for x in the series $\sum_{n=0}^{\infty} a_n x^n$. What are the coefficients of the resulting power series for odd n?

The coefficients for odd n are (2) ________.

How does this show that the Taylor series for an even function f at $x = 0$ contains only even powers of x?

- (●) **A. The coefficients of the odd-n terms in the series for $f(-x)$ must equal both a_n and $-a_n$. The only solution to $a_n = -a_n$ is $a_n = 0$.**
- ( ) **B.** The coefficients of the odd-n terms in the series for $f(-x)$ must equal both a_n and $2a_n$. The only solution to $a_n = 2a_n$ is $a_n = 0$.
- ( ) **C.** The substitution of $-x$ resulted in a coefficient of 0 for all odd n, so the statement has been proven.
- ( ) **D.** The coefficients of the odd-n terms in the series for $f(-x)$ must equal both a_n and $\frac{1}{2}a_n$. The only solution to $a_n = \frac{1}{2}a_n$ is $a_n = 0$.

b. If f(x) is odd, then $f(-x) =$ (3) ________.

Substitute $-x$ for x in the series $\sum_{n=0}^{\infty} a_n x^n$. What are the coefficients of the resulting power series for even n?

The coefficients for even n are (4) ________.

How does this show that the Taylor series for an odd function f at $x = 0$ contains only odd powers of x?

- ( ) **A.** The substitution of $-x$ resulted in a coefficient of 0 for all even n, so the statement has been proven.
- ( ) **B.** The coefficients of the even-n terms in the series for $f(-x)$ must equal both a_n and $\frac{1}{2}a_n$. The only solution to $a_n = \frac{1}{2}a_n$ is $a_n = 0$.
- ( ) **C.** The coefficients of the even-n terms in the series for $f(-x)$ must equal both a_n and $2a_n$. The only solution to $a_n = 2a_n$ is $a_n = 0$.
- (●) **D. The coefficients of the even-n terms in the series for $f(-x)$ must equal both a_n and $-a_n$. The only solution to $a_n = -a_n$ is $a_n = 0$.**

(1) (●) **f(x)** ( ) $-f(x)$

(2) ( ) a_n (●) **$-a_n$** ( ) 0 ( ) $2a_n$ ( ) $\frac{1}{2}a_n$

(3) ( ) f(x) (●) **$-f(x)$**

(4) (●) **a_n** ( ) $-a_n$ ( ) 0 ( ) n ( ) $\frac{1}{2}a_n$

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6. Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{10^n}$$

Recall that the interval of convergence is the set of x's for which the power series converges, either absolutely or conditionally.

6soln. The interval of convergence is $(-8, 12)$.

$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1 \Rightarrow \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{10^{n+1}}\cdot\frac{10^n}{(x-2)^n}\right|<1 \Rightarrow \frac{|x-2|}{10}<1 \Rightarrow |x-2|<10 \Rightarrow -10<x-2<10 \Rightarrow -8<x<12$; when $x=-8$ we have $\sum_{n=1}^{\infty}(-1)^n$, a divergent series; when $x=12$ we have $\sum_{n=1}^{\infty}1$, a divergent series

(a) the radius is 10; the interval of convergence is $-8<x<12$

(b) the interval of absolute convergence is $-8<x<12$

(c) there are no values for which the series converges conditionally

7. Find the 3rd order Taylor polynomial, about the center $x_0 = 1$, for the function $f(x) = x^5 - x^2 + 5$.

7soln. The computations below show that the 3rd order Taylor polynomial, about the center $x_0 = 1$, for the function $f(x) = x^5 - x^2 + 5$ is $p_3(x) = 5 + 3(x-1) + 9(x-1)^2 + 10(x-1)^3$.

we were given $x_0 = 1$			
n	$f^{(n)}(x)$	$f^{(n)}(x_0)$	$\frac{f^{(n)}(x_0)}{n!}$
0	$x^5 - x^2 + 5$	5	$\frac{5}{0!} = \frac{5}{1} = 5$
1	$5x^4 - 2x$	$5 - 2 = 3$	$\frac{3}{1!} = \frac{3}{1} = 3$
2	$5 \cdot 4x^3 - 2$	$20 - 2 = 18$	$\frac{18}{2!} = \frac{18}{2} = 9$
3	$5 \cdot 4 \cdot 3x^2$	$(5)(4)(3)$	$\frac{(5)(4)(3)}{3!} = \frac{(5)(4)(3)}{(3)(2)} = \frac{(5)(4)}{2} = 10$

8. Using the geometric series, find a power series representation about (i.e., centered at) $x_0 = 5$ for the function

$$g(x) = \frac{3}{x-2}$$

and indicate when the representation is valid.

8soln.

$g(x) = \frac{3}{x-2} = \frac{3}{3-[-(x-5)]} = \frac{1}{1-\left[-\left(\frac{x-5}{3}\right)\right]} = \sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n (x-5)^n$, which converges for

$\left|\frac{x-5}{3}\right| < 1$ or $2 < x < 8$.

9. Using a known (commonly used) Taylor series, find the Taylor series for

$$f(x) = \frac{1}{(1-x)^4}$$

about the center $x_0 = 0$ which is valid for $|x| < 1$. Hint. Start with the Taylor series expansion

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad \text{valid for } |x| < 1$$

and differentiate (as many times as needed). Be careful and don't forget the chain rule:

$$D_x(1-x)^{-1} = (-1)(1-x)^{-2}\, D_x(1-x) = (-1)(1-x)^{-2}\,(-1) = (1-x)^{-2}\ .$$

9soln.

Start with Geometric Series and take Derivatives as many times as need.

Geometric Series is valid when $|x|<1$ so resulting power series expansions will also be valid when $|x|<1$.

Geometric Series $\Rightarrow$ $(1-x)^{-1} = \sum_{k=0}^{\infty} x^k \overset{D_x}{\Longrightarrow} (1-x)^{-2} = \sum_{k=1}^{\infty} k\, x^{k-1}$

$\overset{D_x}{\Longrightarrow} 2(1-x)^{-3} = \sum_{k=2}^{\infty} k(k-1)\, x^{k-2} \overset{D_x}{\Longrightarrow} 2\cdot 3(1-x)^{-4} = \sum_{k=3}^{\infty} k(k-1)(k-2)\, x^{k-3}$

So

$$(1-x)^{-4} = \sum_{k=3}^{\infty} \frac{k(k-1)(k-2)}{6}\, x^{k-3} = \sum_{n=0}^{\infty} \frac{(n+3)(n+2)(n+1)}{6}\, x^n$$

$\boxed{\text{let } k-3=n} \Rightarrow k = n+3$

10. Find a parameterization for the line segment from $(-1, 2)$ to $(10, -6)$ for $0 \le t \le 1$.

10soln. ans: $x = -1 + 11t$ and $y = 2 - 8t$

$$x(t) = -1 + \left(10 - \left({}^{-}1\right)\right) t = -1 + 11t$$
$$y(t) = 2 + \left({}^{-}6 - 2\right) t \qquad = 2 - 8t.$$

11. A parametrization of a circle with center at $(0, 0)$ and radius 1, which is traced out twice in the clockwise direction is

11soln. $x(t) = \cos t$ and $y(t) = -\sin t$ for $0 \le t \le 4\pi$. Note $[x(t)]^2 + [y(t)]^2 = 1$ so the puffo is running around a circle with center $(0, 0)$ and radius 1. The negative on the y makes the tracing go clockwise while $0 \le t \le 4\pi$ traces the circle twice.

12. Find an equation for the line tangent to the curve parameterized by

$$x = 2t^2 + 3$$
$$y = t^4$$

at the point defined by the value $t = -1$.

12soln.

$$t=-1 \Rightarrow x=5,\ \ y=1;\ \frac{dx}{dt}=4t,\ \ \frac{dy}{dt}=4t^3 \Rightarrow \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{4t^3}{4t}=t^2 \Rightarrow \left.\frac{dy}{dx}\right|_{t=-1}=(-1)^2=1;\ \text{tangent line is}$$

$$y-1=1\cdot(x-5)\ \text{ or }\ y=x-4;$$

13. Find the Cartesian coordinates of the point with polar coordinates

$$\left(-3, \frac{5\pi}{6}\right)$$

13soln.

$$x=-3\cos\frac{5\pi}{6}=\frac{3\sqrt{3}}{2},\ y=-3\sin\frac{5\pi}{6}=-\frac{3}{2} \Rightarrow \text{ Cartesian coordinates are } \left(\frac{3\sqrt{3}}{2},-\frac{3}{2}\right)$$