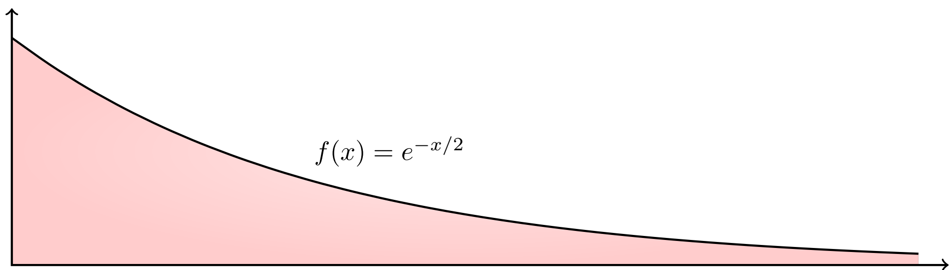
Section 8.8: Improper Integrals

Switching up the Limits of Integration: Up until now, we have required two properties of *definite* integral:

1. the domain of integration, $[a, b]$, is finite
2. the range of the integrand is finite on this domain.

We will now see what happens if we allow the domain or range to be infinite!

Infinite Limits of Integration: Let's consider the infinite region (unbounded on the right) that lies under the curve $y = e^{-x/2}$ in the first quadrant.



First, we examine what the area looks like over finite intervals. That is, we integrate over $[0, b]$.

$$A(b) := \int_0^b e^{-x/2}\,dx = -2e^{-x/2}\Big|_0^b = -2e^{-b/2} - \left[-2e^{-0/2}\right] = 2\left(1 - e^{-b/2}\right).$$

Now we have an expression for the area over a finite integral, we can let $b \longrightarrow \infty$ by calculating the limit of this expression.

$$A = \lim_{b\to\infty} A(b) = \lim_{b\to\infty} 2\left(1 - e^{-b/2}\right) = 2\,(1-0) = 2.$$

So,

$$\int_0^\infty e^{-x/2}\,dx = \lim_{b\to\infty}\int_0^b e^{-x/2}\,dx = 2.$$

So this is how we deal with infinite limits of integration - with a limit! Remember those?

Definition: Integrals with infinite limits of integration are called **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a,\infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty,b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to\infty}\int_{-a}^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty,\infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$

where c is any real number.

In each case, if the limit is finite we sat that the improper integral ______converges______ and that the limit is the ______value______ of the improper integral. If the limit fails to exist, the improper integral ______diverges______

Any of the integrals in the above definition can be interpreted as an area if $f(x)\ge 0$ on the interval of integration. If $f(x)\ge 0$ and the improper integral diverges, we say the area under the curve is **infinite**.

Example 1: Evaluate

$$\int_1^\infty \frac{\ln(x)}{x^2}\,dx.$$

$$u=\ln(x) \qquad dv=\frac{1}{x^2}\,dx$$

$$du=\frac{1}{x}\,dx \qquad v=-\frac{1}{x}$$

$$\begin{aligned}\int_1^b \frac{\ln(x)}{x^2}\,dx &= -\frac{\ln(x)}{x}\Big|_1^b - \int_1^b -\frac{1}{x^2}\,dx\\ &= -\frac{\ln(x)}{x}-\frac{1}{x}\Big|_1^b\\ &= -\frac{\ln(b)}{b}-\frac{1}{b}-\left[-\frac{\ln(1)}{1}-\frac{1}{1}\right]\\ &= -\frac{\ln(b)}{b}-\frac{1}{b}+1\end{aligned}$$

Now we take a limit,

$$\int_1^\infty \frac{\ln(x)}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln(x)}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{\ln(b)}{b}-\frac{1}{b}+1\right] = \lim_{b\to\infty}\left[-\frac{\ln(b)}{b}\right]-0+1 \overset{\text{L'H}}{=} \lim_{b\to\infty}\left[-\frac{1/b}{1}\right]+1=0+1=\boxed{1}$$

L'Hôpital's Rule Suppose that $f(a)=g(a)=0$, that $f(x)$ and $g(x)$ are differentiable on an open interval I containing a and that $g'(x)\neq 0$ on I if $x\neq a$. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

assuming that the limit on the left and right both exist.

Example 2: Evaluate

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$$

According to part 3 of our definition, we can choose any real number c and split this integral into two integrals and then apply parts 1 and 2 to each piece. Let's choose $c = 0$ and write

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx.$$

Now we will evaluate each piece separately.

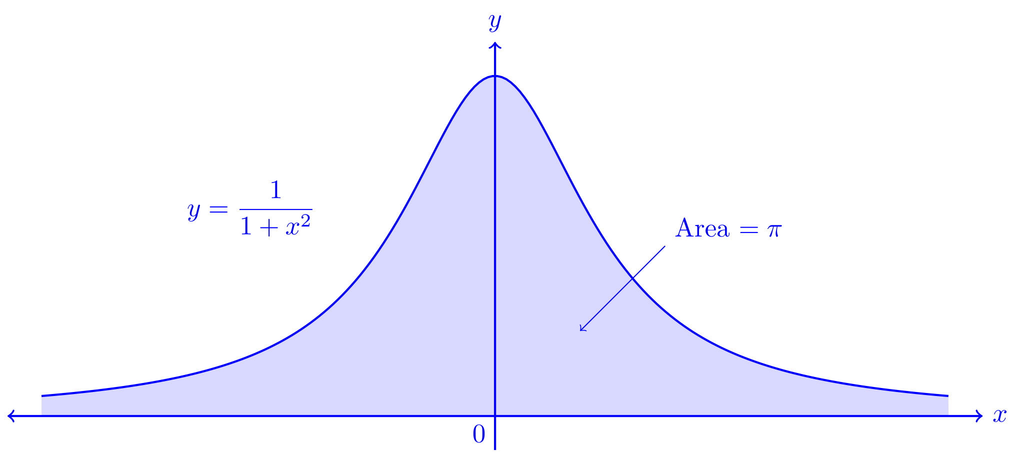
$$\begin{aligned}\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx &= \lim_{a\to-\infty}\int_{a}^{0} \frac{1}{1+x^2}\,dx \\ &= \lim_{a\to-\infty} \tan^{-1}(x)\Big|_{1}^{0} \\ &= \lim_{a\to-\infty} \tan^{-1}(0) - \tan^{-1}(a) \\ &= \lim_{a\to-\infty} -\tan^{-1}(a) \\ &= \frac{\pi}{2},\end{aligned}$$

$$\begin{aligned}\int_{0}^{\infty} \frac{1}{1+x^2}\,dx &= \lim_{b\to\infty}\int_{0}^{b} \frac{1}{1+x^2}\,dx \\ &= \lim_{b\to\infty} \tan^{-1}(x)\Big|_{0}^{b} \\ &= \lim_{b\to\infty} \tan^{-1}(b) - \tan^{-1}(0) \\ &= \lim_{b\to\infty} \tan^{-1}(b) \\ &= \frac{\pi}{2}.\end{aligned}$$

So,

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \boxed{\pi}$$

Since $1/(1+x^2) > 0$ on $\mathbb{R}$, the improper integral can be interpreted as the (finite) area between the curve and the x-axis.



A Special Example: For what values of p does the integral

$$\int_1^{\infty} \frac{1}{x^p}\,dx$$

converge? When the integral does converge, what is its value?

We split this investigation into two cases; when $p \neq 1$ and when $p = 1$.

If $p \neq 1$:

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^p}\,dx &= \lim_{b\to\infty} \int_1^b x^{-p}\,dx \\ &= \lim_{b\to\infty} \frac{x^{-p+1}}{-p+1}\bigg|_1^b \\ &= \lim_{b\to\infty} \frac{1}{1-p}\cdot\frac{1}{x^{p-1}}\bigg|_1^b \\ &= \lim_{b\to\infty}\left[\frac{1}{1-p}\left(\frac{1}{b^{p-1}}-1\right)\right] = \begin{cases}\frac{1}{p-1}, & p>1\\ \infty, & p<1.\end{cases}\end{aligned}$$

If $p = 1$:

$$\begin{aligned}\int_1^{\infty} \frac{1}{x}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx \\ &= \lim_{b\to\infty} \ln(x)\bigg|_1^b \\ &= \lim_{b\to\infty}\left[\ln(b)-\ln(1)\right] \\ &= \lim_{b\to\infty}\ln(b) = \infty\end{aligned}$$

Combining these two results we have

$$\boxed{\int_1^{\infty}\frac{1}{x^p}\,dx = \begin{cases}\frac{1}{p-1}, & p>1\\ \infty, & p\leq 1\end{cases}}$$

Integrands with Vertical Asymptotes: Another type of improper integral that can arise is when the integrand has a vertical asymptote (infinite discontinuity) at a limit of integration or at a point on the interval of integration. We apply a similar technique as in the previous examples of integrating over an altered interval before obtaining the integral we want by taking limits.

Example 4: Investigate the convergence of

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx.$$

First we find the integral over the region $[a, 1]$ where $0 < a \leq 1$.

$$\int_a^1 \frac{1}{\sqrt{x}}\,dx = \int_a^1 x^{-1/2}\,dx = 2x^{1/2}\bigg|_a^1 = 2\sqrt{x}\bigg|_a^1 = 2-2\sqrt{a} = 2(1-\sqrt{a}).$$

Then we find the limit as $a \to 0^+$:

$$\lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} 2\left(1-\sqrt{a}\right) = 2.$$

Therefore,

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}\,dx = \boxed{2}$$

Definition: Integrals of functions that become infinite at a point within the interval of integration are called **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^a f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite we sat that the improper integral ____converges____ and that the limit is the ____value____ of the improper integral. If the limit fails to exist, the improper integral ____diverges____

Example 5: Investigate the convergence of

$$\int_0^1 \frac{1}{1-x}\,dx.$$

$$\begin{aligned}
\int_0^1 \frac{1}{1-x}\,dx &= \lim_{b\to 1^-} \int_0^b \frac{1}{1-x}\,dx \\
&= \lim_{b\to 1^-} -\int_0^b \frac{1}{x-1}\,dx \\
&= \lim_{b\to 1^-} -\ln|x-1|\Big|_0^b \\
&= \lim_{b\to 1^-} -\ln(x-1)\Big|_0^b \\
&= \lim_{b\to 1^-} -\ln(1-b) \\
&= -(-\infty) \\
&= \boxed{\infty}
\end{aligned}$$

Tests for Convergence: When we cannot evaluate an improper integral directly, we try to determine whether it converges of diverges. If the integral diverges, we are done. If it converges we can use numerical methods to approximate its value. The principal tests for convergence or divergence are the Direct Comparison Test and the Limit Comparison Test.

Direct Comparison Test for Integrals: If $0 \leq f(x) \leq g(x)$ on the interval $(a, \infty]$, where $a \in \mathbb{R}$, then,

1. If $\displaystyle\int_a^\infty g(x)\,dx$ converges, then so does $\displaystyle\int_a^\infty f(x)\,dx$.
2. If $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then so does $\displaystyle\int_a^\infty g(x)\,dx$.

Why does this make sense?

1. If the area under the curve of $g(x)$ is *finite* and $f(x)$ is bounded above by $g(x)$ (and below by 0), then the area under the curve of $f(x)$ must be *less than or equal to* the area under the curve of $g(x)$. A positive number less that a *finite* number is also *finite*.
2. If the area under the curve of $f(x)$ is *infinite* and $g(x)$ is bounded below by $f(x)$, then the area under the curve of $g(x)$ must be *"less than or equal to"* the area under the curve of $g(x)$. Since there is no finite number "greater than" infinity, the area under $g(x)$ must also be *infinite*.

Example 6: Determine if the following integral is convergent or divergent.

$$\int_2^\infty \frac{\cos^2(x)}{x^2}\,dx.$$

We want to find a function $g(x)$ such that for some $a \in \mathbb{R}$, $f(x) = \dfrac{\cos^2(x)}{x^2} \leq g(x)$ or $f(x) = \dfrac{\cos^2(x)}{x^2} \geq g(x)$ for all $x \geq a$. One way we can do this is by finding *bounds* for $f(x)$. Since $0 \leq \cos^2(x) \leq 1$ for all x,

$$\frac{\cos^2(x)}{x^2} \leq \frac{1}{x^2}.$$

So then we can use $g(x) := \dfrac{1}{x^2}$. So,

$$0 \leq \int_2^\infty \frac{\cos^2(x)}{x^2}\,dx \leq \int_2^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_2^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left(-\frac{1}{b} - \left(-\frac{1}{2}\right)\right) = \frac{1}{2}.$$

So $\displaystyle\int_2^\infty \frac{\cos^2(x)}{x^2}\,dx$ converges.

Example 7: Determine if the following integral is convergent of divergent.

$$\int_3^\infty \frac{1}{x - e^{-x}}\,dx.$$

Since $x \geq x - e^{-x}$, $f(x) := \dfrac{1}{x} \leq \dfrac{1}{x - e^{-1}} =: g(x)$ for all $x \geq 3$. So,

$$0 \leq \int_3^\infty f(x)\,dx \leq \int_3^\infty g(x)\,dx.$$

By the Direct Comparison Test then, $\displaystyle\int_3^\infty \frac{1}{x - e^{-x}}\,dx$ *diverges* since $\displaystyle\int_3^\infty \frac{1}{x}\,dx$ diverges.

Limit Comparison Test for Integrals: If the positive functions $f(x)$ and $g(x)$ are continuous on $[a, \infty)$, and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

both converge or diverge.

Why does this make sense? The convergence is really only dependent on the "tail" of the integral. That is, the convergence is dictated by what happens "at infinity." If for sufficiently large values of x, $f(x) \approx Lg(x)$ and one of the two integrals converges, then the other one should also converge, since it is only off by "about a scalar multiple." The same goes for diverging, if one diverges, then multiplying it by a positive number won't suddenly make it converge, so the other one should also diverge.

Example 8: Show that

$$\int_1^\infty \frac{1}{1+x^2}\,dx$$

converges.

Let $f(x) := \dfrac{1}{1+x^2}$ and $g(x) := \dfrac{1}{x^2}$. Then,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2}{1+x^2} = \lim_{x\to\infty} \frac{1+x^2-1}{1+x^2} = \lim_{x\to\infty} \left(1 - \frac{1}{1+x^2}\right) = 1.$$

So, by the Limit Comparison Test, the integral $\displaystyle\int_1^\infty \frac{1}{1+x^2}\,dx$ *converges.*

Example 9: Show that

$$\int_1^\infty \frac{1-e^{-x}}{x}\,dx$$

dinverges.

Let $f(x) := \dfrac{1-e^{-x}}{x}$ and $g(x) := \dfrac{1}{x}$. Then,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \left(1 - e^{-x}\right) = 1.$$

So, by the Limit Comparison Test, the integral $\displaystyle\int_1^\infty \frac{1-e^{-x}}{x}\,dx$ *diverges.*