A **Trig Substitution** often works when the integrands involves:

$$a^2 - u^2 \qquad \text{or} \qquad a^2 + u^2 \qquad \text{or} \qquad u^2 - a^2 \ . \tag{1}$$

Here, $a > 0$ is a positive constant and $u$ is the variable.

| **SUMMARY CHART** | | | | | | |
|---|---|---|---|---|---|---|
| | **recall** | **integrand has** | **trig. sub. working form** | **trig. sub. reality form** | **restrictions on $u$** | **restrictions on $\theta$** |
| (1) | $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + c$ | $a^2 - u^2$ | $u = a\sin\theta$ | $\theta = \arcsin\left(\frac{u}{a}\right)$ | $\left\lvert\frac{u}{a}\right\rvert \le 1$ | $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ $(\star_1)$ |
| (2) | $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c$ | $a^2 + u^2$ | $u = a\tan\theta$ | $\theta = \arctan\left(\frac{u}{a}\right)$ | none | $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ $(\star_2)$ |
| (3) | $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\ \sec^{-1}\frac{\lvert u\rvert}{a} + c$ | $u^2 - a^2$ | $u = a\sec\theta$ | $\theta = \operatorname{arcsec}\left(\frac{u}{a}\right)$ | $1 \le \left\lvert\frac{u}{a}\right\rvert$ | |
| $(3^+)$ | if $u$ is positive | | | | $a \le u$ | $0 \le \theta < \frac{\pi}{2}$ $(\star_{3^+})$ |
| $(3^-)$ | if $u$ is negative | | | | $u \le -a$ | $\frac{\pi}{2} < \theta \le \pi$ $(\star_{3^-})$ |

**Example**: Evaluate the integral $\boxed{\int \frac{1}{(4x^2+9)^2}\ dx}$ .

Solution. Note that the integrand $\frac{1}{(4x^2+9)^2}$ involves an expression of a form in (1) since

$$4x^2 + 9 = (2x)^2 + (3)^2 = u^2 + a^2 \qquad \text{where } u = 2x \text{ and } a = 3.$$

As suggested in the Summary Chart, we should try the substitution $u = a\tan\theta$. So we try the substitution

$$2x = 3\tan\theta\ .$$

Thus $2\,dx = 3\sec^2\theta\,d\theta$ and

$$4x^2 + 9 \ = \ (2x)^2 + 3^2 \ = \ (3\tan\theta)^2 + 3^2 \ = \ 3^2\tan^2\theta + 3^2 \ = \ 3^2\left[\tan^2\theta + 1\right] \overset{(*)}{=} 3^2\sec^2\theta\ ,$$

where at $(*)$ we used that $1 + \tan^2\theta = \sec^2\theta$. Now substituting into the original integral we get the following.

$$\int \frac{dx}{\left[4x^2+9\right]^2} \ = \ \int \frac{\frac{3}{2}\sec^2\theta\ d\theta}{\left[3^2\sec^2\theta\right]^2} \ = \ \frac{3}{2}\,\frac{1}{3^4}\int\frac{\sec^2\theta\ d\theta}{\sec^4\theta} \ = \ \frac{1}{2\cdot 3^3}\int\frac{d\theta}{\sec^2\theta} \ = \ \frac{1}{2\cdot 3^3}\int\cos^2\theta\ d\theta$$

now use a half $\angle$ formula: $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$.

$$= \frac{1}{2\cdot 3^3}\,\frac{1}{2}\int\left(1+\cos(2\theta)\right)\ d\theta$$

$$= \frac{1}{2^2\cdot 3^3}\left[\int 1\ d\theta + \int\cos(2\theta)\ d\theta\right]$$

$$= \frac{1}{2^2\cdot 3^3}\left[\int 1\ d\theta + \frac{1}{2}\int\cos(2\theta)\ (2d\theta)\right]$$

$$= \frac{1}{2^2\cdot 3^3}\left[\theta + \frac{1}{2}\sin(2\theta)\right] + C$$

we want our answer in terms of $x$ and we know a relation between $x$ and $\theta$, namely $\tan\theta = \frac{2x}{3}$. But here we have a trig function evaluated at $2\theta$, namely, $\sin(2\theta)$. So we use the double $\angle$ formula $\sin(2\theta) = 2\sin\theta\cos\theta$.

$$= \frac{1}{2^2\cdot 3^3}\left[\theta + \frac{1}{2}\ 2\sin\theta\ \cos\theta\right] + C$$

$$= \frac{1}{2^2\cdot 3^3}\ \theta \ + \ \frac{1}{2^2\cdot 3^3}\sin\theta\ \cos\theta + C\ .$$

This is where we finished the last class and we will pick up here the next class.