Commonly Used Taylor Series

SERIES			WHEN IS VALID/TRUE
$\dfrac{1}{1-x}$	$=$	$1 + x + x^2 + x^3 + x^4 + \ldots$	NOTE THIS IS THE GEOMETRIC SERIES. JUST THINK OF x AS r
	$=$	$\displaystyle\sum_{n=0}^{\infty} x^n$	$x \in (-1,1)$
e^x	$=$	$1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \ldots$	SO: $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$ $e^{(17x)} = \sum_{n=0}^{\infty} \frac{(17x)^n}{n!} = \displaystyle\sum_{n=0}^{\infty} \frac{17^n x^n}{n!}$
	$=$	$\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$x \in \mathbb{R}$
$\cos x$	$=$	$1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dfrac{x^8}{8!} - \ldots$	NOTE $y = \cos x$ IS AN <u>EVEN</u> FUNCTION (I.E., $\cos(-x) = +\cos(x)$) AND THE TAYLOR SERIS OF $y = \cos x$ HAS ONLY <u>EVEN</u> POWERS.
	$=$	$\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$x \in \mathbb{R}$
$\sin x$	$=$	$x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dfrac{x^9}{9!} - \ldots$	NOTE $y = \sin x$ IS AN <u>ODD</u> FUNCTION (I.E., $\sin(-x) = -\sin(x)$) AND THE TAYLOR SERIS OF $y = \sin x$ HAS ONLY <u>ODD</u> POWERS.
	$=$	$\displaystyle\sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^{2n-1}}{(2n-1)!} \overset{\text{or}}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$x \in \mathbb{R}$
$\ln(1+x)$	$=$	$x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dfrac{x^5}{5} - \ldots$	QUESTION: IS $y = \ln(1+x)$ EVEN, ODD, OR NEITHER?
	$=$	$\displaystyle\sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^n}{n} \overset{\text{or}}{=} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$	$x \in (-1,1]$
$\tan^{-1} x$	$=$	$x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dfrac{x^9}{9} - \ldots$	QUESTION: IS $y = \arctan(x)$ EVEN, ODD, OR NEITHER?
	$=$	$\displaystyle\sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^{2n-1}}{2n-1} \overset{\text{or}}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$x \in [-1,1]$

Fix an interval I in the real line (e.g., I might be $(-17, 19)$) and let x_0 be a point in I, i.e.,

$$x_0 \in I \ .$$

Next consider a function, whose domain is I,

$$f \colon I \to \mathbb{R}$$

and whose derivatives $f^{(n)} \colon I \to \mathbb{R}$ exist on the interval I for $n = 1, 2, 3, \ldots, N$.

Definition 1. The <u>*N^{th}-order Taylor polynomial*</u> for $y = f(x)$ at x_0 is:

$$p_N(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x - x_0)^N \ , \qquad \text{(open form)}$$

which can also be written as (recall that $0! = 1$)

$$p_N(x) = \frac{f^{(0)}(x_0)}{0!} + \frac{f^{(1)}(x_0)}{1!}(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x - x_0)^N \qquad \hookleftarrow \text{a finite sum, i.e. the sum stops} \ .$$

Formula (open form) is in <u>open form</u>. It can also be written in <u>closed form</u>, by using sigma notation, as

$$p_N(x) \ = \ \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!} \ (x - x_0)^n \ . \qquad \text{(closed form)}$$

So $y = p_N(x)$ is a polynomial of degree at most N and it has the form

$$p_N(x) \ = \ \sum_{n=0}^{N} c_n \ (x - x_0)^n \qquad \text{where the } \underline{\text{constants}} \qquad c_n \ = \ \frac{f^{(n)}(x_0)}{n!}$$

are specially chosen so that derivatives match up at x_0, i.e. the constants c_n's are chosen so that:

$$\begin{aligned} p_N(x_0) \ &= \ f(x_0) \\ p_N^{(1)}(x_0) \ &= \ f^{(1)}(x_0) \\ p_N^{(2)}(x_0) \ &= \ f^{(2)}(x_0) \\ &\ \vdots \\ p_N^{(N)}(x_0) \ &= \ f^{(N)}(x_0) \ . \end{aligned}$$

The constant c_n is the <u>n^{th} Taylor coefficient</u> of $y = f(x)$ about x_0. The <u>*N^{th}-order Maclaurin polynomial*</u> for $y = f(x)$ is just the N^{th}-order Taylor polynomial for $y = f(x)$ at $x_0 = 0$ and so it is

$$p_N(x) \ = \ \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} \ x^n \ .$$

Definition 2. [1] The <u>*Taylor series*</u> for $y = f(x)$ at x_0 is the power series:

$$P_\infty(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \ldots \qquad \text{(open form)}$$

which can also be written as

$$P_\infty(x) = \frac{f^{(0)}(x_0)}{0!} + \frac{f^{(1)}(x_0)}{1!}(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \ldots \qquad \hookleftarrow \text{the sum keeps on going and going.}$$

The Taylor series can also be written in closed form, by using sigma notation, as

$$P_\infty(x) \ = \ \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} \ (x - x_0)^n \ . \qquad \text{(closed form)}$$

The <u>*Maclaurin series*</u> for $y = f(x)$ is just the Taylor series for $y = f(x)$ at $x_0 = 0$.

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[1] Here we are assuming that the derivatives $y = f^{(n)}(x)$ exist for each x in the interval I and for each $n \in \mathbb{N} \equiv \{1, 2, 3, 4, 5, \ldots\}$.

Easier Question 3. For what values of x does the power (a.k.a. Taylor) series

$$P_\infty(x) \;=\; \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}\,(x-x_0)^n \tag{1}$$

converge? The Root or Ratio test usually answers this question for us.

Big Question 4. If the power/Taylor series in formula (1) does indeed converge at a point x, i.e., if

$$\lim_{N\to\infty} P_N(x) \qquad \text{exists} ,$$

does the series converge to what we would want it to converge to, i.e., does

$$\lim_{N\to\infty} P_N(x) \;\stackrel{?}{=}\; f(x) \tag{2}$$

in short, we are asking

$$f(x) \;=\; \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}\,(x-x_0)^n \;\; ?$$

The question in (2) is going to take some thought.

Definition 5. The <u>N^{th}*-order Remainder term*</u> for $y = f(x)$ at x_0 is:

$$R_N(x) \;\stackrel{\text{def}}{=}\; f(x) - P_N(x)$$

where $y = P_N(x)$ is the N^{th}-order Taylor polynomial for $y = f(x)$ at x_0.

So

$$f(x) \;=\; P_N(x) + R_N(x) \tag{3}$$

that is

$$f(x) \;\approx\; P_N(x) \qquad \text{within an error of} \quad R_N(x) \; .$$

The question is

$$f(x) \;\stackrel{??}{=} P_\infty(x) \qquad \text{i.e.,} \qquad f(x) \;\stackrel{??}{=} \lim_{N\to\infty} P_N(x) \; ,$$

where $y = P_\infty(x)$ is the Taylor series of $y = f(x)$ at x_0.
Well, let's think about what needs to be for $f(x) \stackrel{??}{=} P_\infty(x)$, i.e., <u>for f to equal to its Taylor series</u>.

Notice 6. Taking the $\lim_{N\to\infty}$ of both sides in equation (3), we see that

$$f(x) \;=\; \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}\,(x-x_0)^n \qquad\qquad \hookleftarrow \text{the sum keeps on going and going} \; .$$

if and only if

$$\lim_{N\to\infty} R_N(x) \;=\; 0 \; .$$

Recall 7. $\lim_{N\to\infty} R_N(x) \;=\; 0$ if and only if $\lim_{N\to\infty} |R_N(x)| \;=\; 0$.

<u>**Answer to the Big Question 4.**</u> So we know see that the following are equivalent,
(i.e. (4) holds $\Longleftrightarrow$ (5) holds $\Longleftrightarrow$ (6) holds).

$$f(x) \;=\; \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}\,(x-x_0)^n \; . \tag{4}$$

$$\lim_{N\to\infty} R_N(x) \;=\; 0 \; . \tag{5}$$

$$\lim_{N\to\infty} |R_N(x)| \;=\; 0 \; . \tag{6}$$

So, to show that a function is equal to its Taylor series, we basically want to show that (6) holds true.
How to do this? Well, this is where Mr. Taylor comes to the rescue![2]

[2]According to Mr. Taylor, his Remainder Theorem (see next page) was motivated by coffeehouse conversations about works of Newton on planetary motion and works of Halley (of *Halley's comet*) on roots of polynomials.

$\boxed{\text{Taylor's Remainder Theorem}}$

Version 1: for a fixed point $x \in I$ and a fixed $N \in \mathbb{N}$. [3]
There exists c between x and x_0 so that

$$R_N(x) \quad \overset{\text{def}}{=} \quad f(x) - P_N(x) \quad \overset{\text{theorem}}{=} \quad \frac{f^{(N+1)}(c)}{(N+1)!} \, (x - x_0)^{(N+1)} \; . \tag{7}$$

So either $x \le c \le x_0$ or $x_0 \le c \le x$. So we do not know exactly what c is but atleast we know that c is between x and x_0 and so $c \in I$.
Remark: This is a $\boxed{\text{Big Theorem}}$ by Taylor. See the book for the proof. The proof uses the Mean Value Theorem.
Note that formula (7) implies that

$$|R_N(x)| \quad = \quad \frac{\left|f^{(N+1)}(c)\right|}{(N+1)!} \; |x - x_0|^{(N+1)} \; . \tag{8}$$

Version 2: for the whole interval I and a fixed $N \in \mathbb{N}$.[3]
Assume we can find M so that

$$\text{the maximum of } \left|f^{(N+1)}(x)\right| \text{ on the interval } I \quad \le \; M \; ,$$

i.e.,

$$\max_{c \in I} \left|f^{(N+1)}(c)\right| \; \le \; M \; .$$

Then

$$|R_N(x)| \; \le \; \frac{M}{(N+1)!} \; |x - x_0|^{N+1} \tag{9}$$

for each $x \in I$.
Remark: This follows from formula (8).

Version 3: for the whole interval I and all $N \in \mathbb{N}$. [4]
Now assume that we can find a sequence $\{M_N\}_{N=1}^{\infty}$ so that

$$\max_{c \in I} \left|f^{(N+1)}(c)\right| \; \le \; M_N$$

for each $N \in \mathbb{N}$ and also so that

$$\lim_{N \to \infty} \frac{M_N}{(N+1)!} \; |x - x_0|^{N+1} \; = \; 0$$

for each $x \in I$. Then, by formula (9) and the Squeeze Theorem,

$$\lim_{N \to \infty} |R_N(x)| \; = \; 0$$

for each $x \in I$. Thus, by So ,

$$f(x) \; = \; \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} \; (x - x_0)^n$$

for each $x \in I$.

[3] Here we assume that the $(N+1)$-derivative of $y = f(x)$, i.e. $y = f^{(N+1)}(x)$, exists for each $x \in I$.
[4] Here we assume that $y = f^{(N)}(x)$, exists for each $x \in I$ and each $N \in \mathbb{N}$.