Notes for Seminar:
*The Odd Covering Problem and Its Relatives, Part III*

**Lemma 6:** Suppose $f(x)x^a+1$ is divisible by $\Phi_m(x)$ for some positive integer $m$. Then $f(x)x^n+1$ is divisible by $\Phi_m(x)$ if and only if $n \equiv a \pmod{m}$.

**Proof:** Let $F(x) = f(x)x^n+1$. If $n \equiv a \pmod{m}$, then clearly $F(\zeta_m)=0$ so that $F(x)$ is divisible by $\Phi_m(x)$. If $F(x)$ is divisible by $\Phi_m(x)$, the equality

$$0 = \zeta_m^{n-a}\big(f(\zeta_m)\zeta_m^a+1\big) - F(\zeta_m) = \zeta_m^{n-a}-1$$

implies $n \equiv a \pmod{m}$.

**Comment:** Note that if $f(x)x^n+1$ is divisible by $g(x)$ for some irreducible $g(x) \in \mathbb{Z}[x]$ and for at least two different nonnegative integers $n$, then $g(x) = \Phi_m(x)$ for some $m$.

**Lemma 7:** Let $m$ be an integer $>1$. Then $\Phi_m(1) = \begin{cases} p & \text{if } m = p^r \text{ for some } r \in \mathbb{Z}^+ \\ 1 & \text{otherwise} \end{cases}$.

**Proof:** Clearly, $\Phi_p(1) = p$. If $m = p^r k$ with $k$ and $r$ positive integers such that $p \nmid k$, then Lemma 2 implies $\Phi_m(1) = \Phi_{pk}(1^{p^{r-1}}) = \Phi_{pk}(1)$. The lemma follows if $k=1$. If $k>1$, then applying Lemma 2 again we obtain $\Phi_m(1) = \Phi_{pk}(1) = \Phi_k(1^p)/\Phi_k(1) = 1$.

**Lemma 8:** Let $m$ and $\ell$ be integers with $m \geq 1$ and $\ell \geq 0$. For $\alpha \in \mathbb{Q}(\zeta_m)$, let $N(\alpha) = N_{\mathbb{Q}(\zeta_m)/\mathbb{Q}}(\alpha)$ denote the norm of $\alpha$. Then $N\big(\zeta_m^\ell - 1\big)$ is divisible by a prime $p$ if and only if $m/\gcd(\ell,m)$ is a power of $p$.

**Proof:** Apply Lemma 7 and use that $N\big(\zeta_m^\ell-1\big) = \pm\Phi_{m/\gcd(\ell,m)}(1)^{\phi(m)/\phi(m/\gcd(\ell,m))}$.

**Comment:** We only need the "only if" part of Lemma 8 which follows from $N\big(\zeta_m^\ell-1\big)$ dividing a power of $\Phi_{m/\gcd(\ell,m)}(1)$.

**Main Lemma:** Let $f(x) \in \mathbb{Z}[x]$, and suppose $n$ is sufficiently large (depending on $f$). Then the non-reciprocal part of $f(x)x^n+1$ is irreducible or identically $\pm 1$ unless one of the following holds:

(i) $-f(x)$ is a $p$th power for some prime $p$ dividing $n$.
(ii) $f(x)$ is 4 times a 4th power and $n$ is divisible by 4.

**Proof of Theorem Assuming Main Lemma:** We suppose (as we may) that $f(0) \neq 0$. Since $x^{2^t}+1 = \Phi_{2^{t+1}}(x)$ is irreducible for every $t \in \mathbb{Z}^+$, we deduce $f(x) \not\equiv 1$. Let $\tilde{f}(x) = x^{\deg f} f(1/x)$. Then each reciprocal factor $g(x)$ of $F(x) = f(x)x^n+1$ divides

$$f(x)\widetilde{F}(x) - x^{\deg f}F(x) = f(x)\big(x^{n+\deg f}+\tilde{f}(x)\big) - x^{\deg f}\big(f(x)x^n+1\big) = f(x)\tilde{f}(x) - x^{\deg f}.$$

In particular, there is a finite list of irreducible reciprocal factors that can divide $f(x)x^n+1$ as $n$ varies. Each reciprocal non-cyclotomic irreducible factor divides at most one polynomial of the form $f(x)x^n+1$. By the Main Lemma, we deduce that there are $\Phi_{m_1}(x), \dots, \Phi_{m_r}(x)$ such that if $n$ is sufficiently large and both (i) and (ii) do not hold, then $\Phi_{m_j}(x) \big| \big(f(x)x^n+1\big)$ for some $j$. Note that (ii) does not hold since otherwise $f(x)x^n+1$ could not be divisible by a cycltomic polynomial (if $\Phi_m(x)$ were a factor, then $f(\zeta_m)\zeta_m^n = -1$, contradicting that the left side has even norm and the right side has odd norm) so that $f(x)x^n+1$ is irreducible whenever $4 \nmid n$ and $n$ is sufficiently large. We may suppose that there is an $a_j$ such that $\Phi_{m_j}(x) | \big(f(x)x^{a_j}+1\big)$. Let $\mathcal{P}$ denote the set of primes $p$ for which $f(x)$ is minus a $p$th power. We remove from consideration any $m_j$ divisible by a $p \in \mathcal{P}$ (but abusing notation we keep the range of subscripts). Then Lemmas 5 and 6 imply that the congruences

$$x \equiv 0 \pmod{p} \quad \text{for } p \in \mathcal{P} \qquad \text{and} \qquad x \equiv a_j \pmod{m_j} \quad \text{for } j \in \{1,2,\ldots,r\}$$

cover the integers.

**Claim:** Suppose $m_j = p^t m_0$ and $m_i = p^s m_0$, where $p$ is prime, $m_0$ is an integer $> 1$ such that $p \nmid m_0$, and $t$ and $s$ are integers with $t > s \geq 0$. Then $a_j \equiv a_i \pmod{m_0}$.

Take $p = 2$ in the Claim. We replace $x \equiv a_j \pmod{m_j}$ and $x \equiv a_i \pmod{m_i}$ with $x \equiv a_j \pmod{m_0}$. If for some $j$ there is no $i$ as above, we still replace $x \equiv a_j \pmod{m_j}$ with $x \equiv a_j \pmod{m_0}$. Then we are left with a covering with moduli that are distinct odd numbers together with possibly powers of 2. Observe that $\sum_{j=1}^{\infty} 1/2^j = 1$ implies that there is an $a \in \mathbb{Z}$ and a $k \in \mathbb{Z}^+$ such that no integer satifying $x \equiv a \pmod{2^k}$ satisfies one of the congruences in our covering with moduli a power of 2. Denote by $x \equiv a'_j \pmod{m'_j}$ the congruences with $m'_j$ odd. Let $u$ and $v$ be integers such that

$$2^k u + v\Big(\prod m'_j\Big) = 1.$$

For any $n \in \mathbb{Z}$, consider the number $m = a + 2^k u(n-a)$. Then $m \equiv n \pmod{m'_j}$ for every $m'_j$ and $m \equiv a \pmod{2^k}$. It follows that $n \equiv m \equiv a'_j \pmod{m'_j}$ for some $m'_j$. Therefore, every $n \in \mathbb{Z}$ satisfies one of the congruences $x \equiv a'_j \pmod{m'_j}$. So these congruences form an odd covering of the integers.