Notes for Seminar:
The Odd Covering Problem and Its Relatives, Part I

Definition: A *covering of the integers* is a system of congruences $x \equiv a_j \pmod{m_j}$ such that every integer satisfies at least one of the congruences.

Examples:

$x \equiv 0 \pmod 2$	$x \equiv 0 \pmod 2$	$x \equiv 0 \pmod 2$	$x \equiv 0 \pmod 2$
$x \equiv 1 \pmod 2$	$x \equiv 1 \pmod 4$	$x \equiv 2 \pmod 3$	$x \equiv 0 \pmod 3$
	$x \equiv 3 \pmod 8$	$x \equiv 1 \pmod 4$	$x \equiv 1 \pmod 4$
	$x \equiv 7 \pmod{16}$	$x \equiv 1 \pmod 6$	$x \equiv 3 \pmod 8$
	$\vdots$	$x \equiv 3 \pmod{12}$	$x \equiv 7 \pmod{12}$
			$x \equiv 23 \pmod{24}$

Open Problem 1: For every $c > 0$, does there exist a finite covering with distinct moduli and with the minimum modulus $> c$? (Erdős \$1000)

Open Problem 2 ("Odd Covering" Problem): Does there exist a finite covering with distinct odd moduli > 1? (Erdős \$25 for "No", Selfridge \$2000 for construction)

Polignac's Conjecture: For every odd integer $k > 1$, there is a prime p and an integer n such that $k = 2^n + p$.

Comments: The Prime Number Theorem suggests this is reasonable (but 127 is the smallest counterexample and 905 is the smallest composite counterexample). The last covering example above gives an Erdős' proof that for a positive proportion of k, the conjecture does not hold. (Take $k \equiv 1 \pmod 2$, $k \equiv 1 \pmod 3$, $k \equiv 1 \pmod 7$, $k \equiv 2 \pmod 5$, $k \equiv 8 \pmod{17}$, $k \equiv 11 \pmod{13}$, $k \equiv 121 \pmod{241}$, and $k \equiv 3 \pmod{31}$.)

Sierpinski's Theorem: A positive proportion of integers ℓ satisfy $\ell \times 2^n + 1$ is composite for all nonnegative integers n.

Comments: It is unknown what the smallest such ℓ is. It is probably $\ell = 78557$ (due to Selfridge). Schinzel noted Sierpinski's Theorem follows from the above solution to Polignac's Conjecture. Take $\ell = -k$. For each $n \geq 0$, consider $p \in \{3, 5, 7, 13, 17, 241\}$ such that $\ell + 2^{239n} \equiv 0 \pmod p$. Then $\ell + 2^{239n} \equiv \ell + 2^{-n} \pmod p$ so that $\ell \times 2^n + 1 \equiv 0 \pmod p$.

The Analogous Polynomial Problem: Find $f(x) \in \mathbb{Z}[x]$ with $f(1) \neq -1$ such that $f(x)x^n + 1$ (or $x^n + f(x)$ provided also $f(0) \neq 0$) is reducible for all $n \geq 0$.

Schinzel's Example: If $f(x) = 5x^9 + 6x^8 + 3x^6 + 8x^5 + 9x^3 + 6x^2 + 8x + 3$, then $f(x)x^n + 12$ is reducible for all $n \geq 0$.

Comments: This follows from the third covering example above. If $n \equiv 0 \pmod 2$, then $f(x)x^n + 12 \equiv 0 \pmod{x+1}$; if $n \equiv 2 \pmod 3$, then $f(x)x^n + 12 \equiv 0 \pmod{x^2+x+1}$; if $n \equiv 1 \pmod 4$, then $f(x)x^n + 12 \equiv 0 \pmod{x^2+1}$; and so on. The dual role of 12 here might be misleading. One can find an example of an $f(x) \in \mathbb{Z}^+[x]$ with 12 replaced by 4.

Schinzel's Theorem: If there is an $f(x)$ as in the analogous polynomial problem, then there is an odd covering of the integers.

Turan's Conjecture: There is a constant C such that if $f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$, then there is an irreducible $g(x) = \sum_{j=0}^{n} b_j x^j \in \mathbb{Z}[x]$ such that $\sum_{j=0}^{n} |a_j - b_j| \leq C$.

Comments: Schinzel's Theorem in the case of $x^n + f(x)$ addresses this conjecture. He further considered polynomials of the form $x^m \pm x^n + f(x)$ and showed that one may take $C = 3$ if one allows $\deg g(x)$ to exceed $\deg f(x)$.