On the Factorization of Lacunary Polynomials

by Michael Filaseta

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Includes Joint Work With

Douglas Meade, Robert Murphy, & Andrzej Schinzel

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- $f(x)$ *reciprocal* means $\tilde{f}(x) = \pm f(x)$
- the *non-reciprocal part of* $f(x)$ is $f(x)$ removed of its irreducible reciprocal factors (sort of)

BASIC QUESTION 1: Are lacunary polynomials easier to factor than non-lacunary polynomials?

LJUNGGREN'S IDEA: Assume $f(x)$ has more than one non-reciprocal factor. Then $f(x) = u(x)v(x)$ for some $u(x)$ and $v(x)$ in $\mathbb{Z}[x]$ which are non-reciprocal.

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Compare the coefficients of $x^{\deg f}$ on the left and right. On the left it is $\|f\|^2$, and on the right it is $\|w\|^2$. Hence, $\|w\| = \|f\|$.

$$
\begin{aligned}
f(x) &= u(x)v(x) \\
w(x) &= u(x)\tilde{v}(x)
\end{aligned}
$$

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Three Properties of $w(x)$:

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Three Properties of $w(x)$:

- There exists $w(x)$ with $f(x)\tilde{f}(x) = w(x)\widetilde{w}(x)$, $w(x) \neq f(x)$, and $w(x) \neq \tilde{f}(x)$ if and only if the non-reciprocal part of $f(x)$ is reducible.

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- $\|w\| = \|f\|$
- If $f(x)$ is a $\mathbf{0, 1}$-polynomial, then $w(x)$ is also.

Example: Does

$$f(x) = 1 + x^{211} + x^{517} + x^{575} + x^{1245} + x^{1398}$$

have more than one non-reciprocal factor?

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$$\begin{aligned} f(x)\tilde{f}(x) = 1 &+ x^{153} + x^{211} + x^{364} + x^{517} \\ &+ x^{575} + x^{670} + x^{728} + x^{823} + x^{881} \\ &+ x^{1034} + x^{1092} + x^{1187} + x^{1245} \\ &+ x^{1340} + 6x^{1398} + \cdots \end{aligned}$$

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$$\begin{aligned} f\tilde{f} \longrightarrow [&0, 153, 211, 364, 517, 575, 670, 728, 823, 881, \\ &1034, 1092, 1187, 1245, 1340, 1398, \ldots] \end{aligned}$$

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$$w = 1 + \cdots + x^{1398}$$

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Question: How can we get the exponent 153 in $w\widetilde{w}$?

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$$w \longrightarrow [\mathbf{0}, \mathbf{153}, \ldots, \mathbf{1398}]$$

$$\widetilde{w} \longrightarrow [\mathbf{0}, \ldots, \mathbf{1245}, \mathbf{1398}]$$

$w\widetilde{w}$ (so far) $\longrightarrow [\mathbf{0}, \mathbf{153}, \mathbf{1245}, \mathbf{1398}, \mathbf{1551}, \mathbf{2643}, \mathbf{2796}]$

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$\boldsymbol{w\widetilde{w}}$ (so far) $\boldsymbol{\longrightarrow [0, 153, 1245, 1398, 1551, 2643, 2796]}$

Question: How can we get $\mathbf{211}$ in $\boldsymbol{w\widetilde{w}}$?

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Question: How can we get $\mathbf{364}$ in $w\widetilde{w}$?

$$
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\end{aligned}
$$

$$
w \longrightarrow [\mathbf{0, 153, 211, \ldots, 1398}]
$$

$$
\widetilde{w} \longrightarrow [\mathbf{0, \ldots, 1187, 1245, 1398}]
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$$f\tilde{f} \longrightarrow [\mathbf{0}, \mathbf{153}, \mathbf{211}, \mathbf{364}, \mathbf{517}, \mathbf{575}, \mathbf{670}, \mathbf{728}, \mathbf{823}, \mathbf{881}, \mathbf{1034}, \mathbf{1092}, \mathbf{1187}, \mathbf{1245}, \mathbf{1340}, \mathbf{1398}, \ldots]$$

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$$w\widetilde{w} \text{ (so far)} \longrightarrow [\mathbf{0, 153, 211, 364, 1034, 1187, 1245,}$$
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Note: Everything appears fine ... or does it?

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Problem: The coefficient of x^{1187} is $\mathbf{2}$ (since $\mathbf{0 + 1187 = 153 + 1034 = 1187}$).

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How to Proceed: Since $\mathbf{364}$ cannot be an exponent in w, we backtrack and consider $\mathbf{364}$ as an exponent in $\widetilde{w}$.

$$
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\end{aligned}
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\boldsymbol{w} \longrightarrow [\mathbf{0, 153, 211, \ldots, 1034, 1398}]
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$$w \longrightarrow [\mathbf{0, 153, 211, \ldots, 1034, 1398}]$$

$$\widetilde{w} \longrightarrow [\mathbf{0, 364, \ldots, 1187, 1245, 1398}]$$

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Note: Everything appears fine.

Question: How can we get $\mathbf{670}$ in $w\widetilde{w}$?

$$f\tilde{f} \longrightarrow [\mathbf{0, 153, 211, 364, 517, 575, 670, 728, 823, 881, 1034, 1092, 1187, 1245, 1340, 1398, \ldots}]$$

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$$\mathbf{939, 1034, 1187, 1245, 1340, 1398, \ldots}]$$

Problem: The coefficient of $x^{\mathbf{1034}}$ is $\mathbf{2}$

(since $\mathbf{1034 + 0 = 670 + 364 = 1034}$).

$$f\tilde{f} \longrightarrow [0, 153, 211, 364, 517, 575, 670, 728, 823, 881, 1034, 1092, 1187, 1245, 1340, 1398, \ldots]$$

$$w \longrightarrow [0, 153, 211, 670, 1034, 1398]$$

$$\widetilde{w} \longrightarrow [0, 364, 728, 1187, 1245, 1398]$$

$$w\widetilde{w} \longrightarrow [0, 153, 211, 364, 517, 575, 670, 728, 881, 939, 1034, 1187, 1245, 1340, 1398, \ldots]$$

Problem: The coefficient of x^{1034} is 2
(since $1034 + 0 = 670 + 364 = 1034$).
Also, the exponent 939 should not be in $w\widetilde{w}$.

$$f\tilde{f} \longrightarrow [\mathbf{0, 153, 211, 364, 517, 575, 670, 728, 823, 881,}$$
$$\mathbf{1034, 1092, 1187, 1245, 1340, 1398, \ldots}]$$

$$w \longrightarrow [\mathbf{0, 153, 211, 670, 1034, 1398}]$$

$$\widetilde{w} \longrightarrow [\mathbf{0, 364, 728, 1187, 1245, 1398}]$$

$$w\widetilde{w} \longrightarrow [\mathbf{0, 153, 211, 364, 517, 575, 670, 728, 881,}$$
$$\mathbf{939, 1034, 1187, 1245, 1340, 1398, \ldots}]$$

How to Proceed: Since $\mathbf{670}$ cannot be an exponent in w, we backtrack and consider $\mathbf{670}$ as an exponent in $\widetilde{w}$.

$$f\tilde{f} \longrightarrow [\mathbf{0, 153, 211, 364, 517, 575, 670, 728, 823, 881,} \\ \mathbf{1034, 1092, 1187, 1245, 1340, 1398, \ldots}]$$

$$\boldsymbol{w} \longrightarrow [\mathbf{0, 153, 211, 728, 1034, 1398}]$$

$$\widetilde{\boldsymbol{w}} \longrightarrow [\mathbf{0, 364, 670, 1187, 1245, 1398}]$$

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Note: We now have $f\tilde{f} = w\widetilde{w}$!!

$$f\tilde{f} \longrightarrow [0, 153, 211, 364, 517, 575, 670, 728, 823, 881, 1034, 1092, 1187, 1245, 1340, 1398, \dots]$$

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Note: We now have $f\tilde{f} = w\widetilde{w}$!!

How to Proceed: We check that $w(x) \neq f(x)$ and that $w(x) \neq \tilde{f}(x)$. Since $w(x) \neq f(x)$ and $w(x) \neq \tilde{f}(x)$, the non-reciprocal part of $f(x)$ is reducible.

Recall

$$f(x) = 1 + x^{211} + x^{517} + x^{575} + x^{1245} + x^{1398}.$$

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A factor of $f(x)$ is

$$\gcd(f(x), w(x)) = 1 + x^{211} - x^{364} + x^{881}.$$

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We have the factorization

$$f(x) = (1 + x^{211} - x^{364} + x^{881})(1 + x^{364} + x^{517}).$$

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MAPLE: `irreduc(f);`

took > 3 hours on an Ultra 5 Workstation

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Comment: $x^2 + 1$ is a factor of $f(x)$

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Should we be happy with an algorithm which determines whether such an n is prime in time that is of the order of a polynomial in $\log n$? Maybe not. What is the length of the input?

Similar Comment: Suppose we want to test a polynomial $f(x) \in \mathbb{Z}[x]$ for irreducibility. If the polynomial is lacunary, should we be content with an algorithm that runs in time that is polynomial in $\deg f$ (as well as the logarithm of its height)?

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POSSIBLE THEOREM: *There is an algorithm with the following property: Given a non-reciprocal $f(x) \in \mathbb{Z}[x]$ with N non-zero terms and height H, the algorithm determines whether $f(x)$ is irreducible in time*

$$c(N, H)(\log \deg f)^{c'(N)}$$

where $c(N, H)$ depends only on N and H and $c'(N)$ depends only on N.

BASIC QUESTION 2: Can we categorize the polynomials having small Euclidean norm that are reducible?

Selmer (1956): Investigated the irreducibility of

$$x^a \pm x^b \pm 1.$$

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$$x^a \pm x^b \pm 1 \quad \text{and} \quad x^a \pm x^b \pm x^c \pm 1.$$

Theorem: The non-cyclotomic parts of $x^a \pm x^b \pm 1$ (if $a > b > 0$) and of $x^a \pm x^b \pm x^c \pm 1$ (if $a > b > c > 0$) are always irreducible.

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Theorem: The non-cyclotomic parts of $x^a \pm x^b \pm 1$ (if $a > b > 0$) and of $x^a \pm x^b \pm x^c \pm 1$ (if $a > b > c > 0$) are always irreducible.

Mills (1985): Noted that

$$x^8 + x^7 + x - 1 = (x^2 + 1)(x^3 + x^2 - 1)(x^3 - x + 1).$$

$$x^8+x^7+x-1=(x^2+1)(x^3+x^2-1)(x^3-x+1)$$
$$x^8+x^4+x^2-1=(x^2+1)(x^3+x^2-1)(x^3-x^2+1)$$
$$x^8-x^6-x^4-1=(x^2+1)(x^3-x-1)(x^3-x+1)$$
$$x^8-x^7-x-1=(x^2+1)(x^3-x-1)(x^3-x^2+1)$$

$$x^8+x^7+x-1=(x^2+1)(x^3+x^2-1)(x^3-x+1)$$
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$$x^8-x^6-x^4-1=(x^2+1)(x^3-x-1)(x^3-x+1)$$
$$x^8-x^7-x-1=(x^2+1)(x^3-x-1)(x^3-x^2+1)$$

Call these “variations” of each other.

Mill's Theorem: Suppose

$$f(x) = x^a \pm x^b \pm 1 \quad \text{with } a > b > 0$$

or

$$f(x) = x^a \pm x^b \pm x^c \pm 1 \quad \text{with } a > b > c > 0.$$

Then the non-cyclotomic part of $f(x)$ is irreducible unless $f(x)$ is a variation of

$$\begin{aligned} &x^{8k} + x^{7k} + x^k - 1 \\ &\qquad = (x^{2k} + 1)(x^{3k} + x^{2k} - 1)(x^{3k} - x^k + 1). \end{aligned}$$

Theorem (Schinzel): Fix $a_0, \dots, a_r \in \mathbb{Z} - \{0\}$. Then it is possible to classify the polynomials of the form

$$a_r x^{d_r} + \cdots + a_1 x^{d_1} + a_0$$

that have reducible non-reciprocal part.

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that have reducible non-reciprocal part.

Theorem (Solan and F.): If $a > b > c > d > 0$, then the non-reciprocal part of $x^a + x^b + x^c + x^d + 1$ is irreducible.

Theorem: If $a > b > c > d > e > 0$, then the non-reciprocal part of

$$f(x) = x^a + x^b + x^c + x^d + x^e + 1$$

is irreducible unless $f(x)$ is a variation of

$$\begin{aligned} f(x) &= x^{5s+3t} + x^{4s+2t} + x^{2s+2t} + x^t + x^s + 1 \\ &= (x^{3s+2t} - x^{s+t} + x^t + 1)(x^{2s+t} + x^s + 1). \end{aligned}$$

Theorem: *If* $n > c > b > a > 0$, *then the non-reciprocal part of* $f(x) = x^n \pm x^c \pm x^b \pm x^a \pm 1$ *is irreducible unless* $f(x)$ *is a variation of one of the following:*

Theorem: *If $n > c > b > a > 0$, then the non-reciprocal part of $f(x) = x^n \pm x^c \pm x^b \pm x^a \pm 1$ is irreducible unless $f(x)$ is a variation of one of the following:*

$$x^{8t} - x^{7t} - x^{4t} + x^{2t} - 1 = (x^{3t} - x^t - 1)(x^{5t} - x^{4t} + x^{3t} - x^t + 1)$$
$$x^{8t} - x^{6t} + x^{4t} - x^t - 1 = (x^{3t} - x^{2t} + 1)(x^{5t} + x^{4t} - x^{2t} - x^t - 1)$$
$$x^{9t} - x^{7t} + x^{6t} - x^t - 1 = (x^{3t} - x^{2t} + 1)(x^{6t} + x^{5t} - x^{2t} - x^t - 1)$$
$$x^{10t} - x^{7t} - x^{6t} - x^{4t} - 1 = (x^{3t} - x^t - 1)(x^{7t} + x^{5t} + x^{2t} - x^t + 1)$$
$$x^{10t} - x^{9t} + x^{8t} - x^t - 1 = (x^{3t} - x^{2t} + 1)(x^{7t} + x^{5t} - x^{2t} - x^t - 1)$$
$$x^{10t} - x^{6t} - x^{5t} + x^{4t} - 1 = (x^{5t} - x^{4t} + x^{3t} - x^t + 1)(x^{5t} + x^{4t} - x^{2t} - x^t - 1)$$
$$x^{10t} - x^{9t} - x^{6t} + x^{3t} - 1 = (x^{3t} - x^t - 1)(x^{7t} - x^{6t} + x^{5t} - x^{3t} + x^{2t} - x^t + 1)$$
$$x^{10t} + x^{7t} + x^{4t} - x^t - 1 = (x^{3t} - x^{2t} + 1)(x^{7t} + x^{6t} + x^{5t} + x^{4t} - x^{2t} - x^t - 1)$$

$$x^{11t} - x^{8t} - x^{6t} - x^{5t} - 1 = (x^{4t} - x^{t} + 1)(x^{7t} - x^{3t} - x^{2t} - x^{t} - 1)$$
$$x^{11t} + x^{8t} + x^{6t} - x^{t} - 1 = (x^{3t} - x^{2t} + 1)(x^{8t} + x^{7t} + x^{6t} + x^{5t} - x^{2t} - x^{t} - 1)$$
$$x^{13t} - x^{11t} - x^{9t} - x^{4t} - 1 = (x^{3t} - x^{t} - 1)(x^{10t} + x^{7t} - x^{6t} + x^{5t} + x^{2t} - x^{t} + 1)$$
$$x^{13t} - x^{11t} + x^{10t} - x^{2t} - 1 = (x^{5t} - x^{4t} + x^{2t} - x^{t} + 1)(x^{8t} + x^{7t} - x^{2t} - x^{t} - 1)$$
$$x^{14t} - x^{11t} + x^{9t} - x^{3t} - 1 = (x^{7t} - x^{6t} + x^{3t} - x^{t} + 1) \times (x^{7t} + x^{6t} + x^{5t} - x^{3t} - x^{2t} - x^{t} - 1)$$
$$x^{14t} - x^{9t} - x^{8t} + x^{7t} - 1 = (x^{7t} - x^{6t} + x^{5t} - x^{3t} + x^{2t} - x^{t} + 1) \times (x^{7t} + x^{6t} - x^{4t} - x^{t} - 1)$$
$$x^{2t+u} - x^{t+2u} + x^{2u} - x^{t} - 1 = (x^{t} - x^{u} + 1)(x^{t+u} - x^{u} - 1)$$
$$x^{5t+2u} - x^{4t+2u} - x^{t+u} - x^{t} - 1 = (x^{2t+u} - x^{t+u} - 1)(x^{3t+u} + x^{t} + 1)$$
$$x^{5t+3u} - x^{4t+2u} - x^{t+u} - x^{t} - 1 = (x^{2t+u} - x^{t} - 1)(x^{3t+2u} + x^{t+u} + 1)$$
$$\vdots \qquad \vdots \qquad \vdots$$

How to Prove Such Theorems:

$$f = x^n - x^c - x^b + x^a + 1$$

$$f = x^n - x^c - x^b + x^a + 1$$
$$\tilde{f} = x^n + x^{n-a} - x^{n-b} - x^{n-c} + 1$$

$$f = x^n - x^c - x^b + x^a + 1$$
$$\tilde{f} = x^n + x^{n-a} - x^{n-b} - x^{n-c} + 1$$

$$f\tilde{f} = 1 + x^a - x^b - x^c - x^{n-c} - x^{n-b} + x^{n-a} - x^{n+a-c} - x^{n+a-b} + x^{n+b-c} + \cdots$$

$$
\begin{aligned}
f &= x^n - x^c - x^b + x^a + 1 \\
\tilde{f} &= x^n + x^{n-a} - x^{n-b} - x^{n-c} + 1
\end{aligned}
$$

$$
\begin{aligned}
f\tilde{f} = 1 + x^a - x^b - x^c - x^{n-c} - x^{n-b} + x^{n-a} \\
- x^{n+a-c} - x^{n+a-b} + x^{n+b-c} + \cdots
\end{aligned}
$$

$$
\begin{aligned}
w &= x^n + x^t - x^s - x^r + 1 \\
\widetilde{w} &= x^n - x^{n-r} - x^{n-s} + x^{n-t} + 1
\end{aligned}
$$

$$
\begin{aligned}
f &= x^n - x^c - x^b + x^a + 1 \\
\tilde{f} &= x^n + x^{n-a} - x^{n-b} - x^{n-c} + 1
\end{aligned}
$$

$$
\begin{aligned}
f\tilde{f} = 1 + x^a - x^b - x^c - x^{n-c} - x^{n-b} + x^{n-a} \\
- x^{n+a-c} - x^{n+a-b} + x^{n+b-c} + \cdots
\end{aligned}
$$

$$
\begin{aligned}
w &= x^n + x^t - x^s - x^r + 1 \\
\widetilde{w} &= x^n - x^{n-r} - x^{n-s} + x^{n-t} + 1
\end{aligned}
$$

$$
\begin{aligned}
w\tilde{w} = 1 - x^r - x^s + x^t + x^{n-t} - x^{n-s} - x^{n-r} \\
- x^{n+r-t} + x^{n+r-s} - x^{n+s-t} + \cdots
\end{aligned}
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$$
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f\tilde{f} = 1 + x^{a} - x^{b} - x^{c} - x^{n-c} - x^{n-b} + x^{n-a} \\
- x^{n+a-c} - x^{n+a-b} + x^{n+b-c} + \cdots
\end{aligned}
$$

$$
\begin{aligned}
w\tilde{w} = 1 - x^{r} - x^{s} + x^{t} + x^{n-t} - x^{n-s} - x^{n-r} \\
- x^{n+r-t} + x^{n+r-s} - x^{n+s-t} + \cdots
\end{aligned}
$$

$$f\tilde{f} = 1+x^a - x^b - x^c - x^{n-c} - x^{n-b} + x^{n-a} - x^{n+a-c} - x^{n+a-b} + x^{n+b-c} + \cdots$$

$$w\tilde{w} = 1-x^r - x^s + x^t + x^{n-t} - x^{n-s} - x^{n-r} - x^{n+r-t} + x^{n+r-s} - x^{n+s-t} + \cdots$$

Basic Idea: We want to equate exponents. But there may be cancellation of terms.

$$
\begin{aligned}
&x^{a} + x^{n-a} + x^{n+b-c} + x^{r} + x^{s} \\
&\qquad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
&= x^{c} + x^{n-c} + x^{b} + x^{n-b} + x^{n+a-c} \\
&\qquad + x^{n+a-b} + x^{t} + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

$$
\begin{aligned}
&x^{a} + x^{n-a} + x^{n+b-c} + x^{r} + x^{s} \\
&\qquad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
&= x^{c} + x^{n-c} + x^{b} + x^{n-b} + x^{n+a-c} \\
&\qquad + x^{n+a-b} + x^{t} + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

Basic Idea: Solve the resulting systems of equations obtained by equating exponents.

$$\begin{aligned}
&x^{a}+x^{n-a}+x^{n+b-c}+x^{r}+x^{s}\\
&\qquad +x^{n-s}+x^{n-r}+x^{n+r-t}+x^{n+s-t}\\
&=x^{c}+x^{n-c}+x^{b}+x^{n-b}+x^{n+a-c}\\
&\qquad +x^{n+a-b}+x^{t}+x^{n-t}+x^{n+r-s}
\end{aligned}$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

$$
\begin{aligned}
&x^a + x^{n-a} + x^{n+b-c} + x^r + x^s \\
&\qquad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
&= x^c + x^{n-c} + x^b + x^{n-b} + x^{n+a-c} \\
&\qquad + x^{n+a-b} + x^t + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

$$0 < a < b < c < n$$

$$0 < r < s < t < n$$

$$
\begin{aligned}
& x^{a} + x^{n-a} + x^{n+b-c} + x^{r} + x^{s} \\
& \qquad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
= {} & x^{c} + x^{n-c} + x^{b} + x^{n-b} + x^{n+a-c} \\
& \qquad + x^{n+a-b} + x^{t} + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:
a

$$\begin{aligned} & x^{a} + x^{n-a} + x^{n+b-c} + x^{r} + x^{s} \\ & \quad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\ = {} & x^{c} + x^{n-c} + x^{b} + x^{n-b} + x^{n+a-c} \\ & \quad + x^{n+a-b} + x^{t} + x^{n-t} + x^{n+r-s} \end{aligned}$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:

$a, n-a$

$$\begin{aligned} & x^{a} + x^{n-a} + x^{n+b-c} + x^{r} + x^{s} \\ & \quad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\ = {} & x^{c} + x^{n-c} + x^{b} + x^{n-b} + x^{n+a-c} \\ & \quad + x^{n+a-b} + x^{t} + x^{n-t} + x^{n+r-s} \end{aligned}$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:

$a, n-a, n+b-c$

$$
\begin{aligned}
& x^{a} + x^{n-a} + x^{n+b-c} + x^{r} + x^{s} \\
& \quad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
= \; & x^{c} + x^{n-c} + x^{b} + x^{n-b} + x^{n+a-c} \\
& \quad + x^{n+a-b} + x^{t} + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:

$a, n-a, n+b-c, r$

$$
\begin{aligned}
&x^{a} + x^{n-a} + x^{n+b-c} + x^{r} + x^{s} \\
&\qquad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
&= x^{c} + x^{n-c} + x^{b} + x^{n-b} + x^{n+a-c} \\
&\qquad + x^{n+a-b} + x^{t} + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:

$a, n-a, n+b-c, r, n-s$

$$
\begin{aligned}
&x^{a} + x^{n-a} + x^{n+b-c} + x^{r} + x^{s} \\
&\qquad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
&= x^{c} + x^{n-c} + x^{b} + x^{n-b} + x^{n+a-c} \\
&\qquad + x^{n+a-b} + x^{t} + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:
$a, n-a, n+b-c, r, n-s, n+r-t$

$$\begin{aligned} & x^a + x^{n-a} + x^{n+b-c} + x^r + x^s \\ & \qquad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\ = {} & x^c + x^{n-c} + x^b + x^{n-b} + x^{n+a-c} \\ & \qquad + x^{n+a-b} + x^t + x^{n-t} + x^{n+r-s} \end{aligned}$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:

$a, n-a, n+b-c, r, n-s, n+r-t$

Possible Least Exponent on the Right:

$n-c$

$$
\begin{aligned}
&x^a + x^{n-a} + x^{n+b-c} + x^r + x^s \\
&\quad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
= &\; x^c + x^{n-c} + x^b + x^{n-b} + x^{n+a-c} \\
&\quad + x^{n+a-b} + x^t + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:

$a, n-a, n+b-c, r, n-s, n+r-t$

Possible Least Exponent on the Right:

$n-c, b$

$$
\begin{aligned}
& x^{a} + x^{n-a} + x^{n+b-c} + x^{r} + x^{s} \\
& \qquad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
&= x^{c} + x^{n-c} + x^{b} + x^{n-b} + x^{n+a-c} \\
& \qquad + x^{n+a-b} + x^{t} + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:

$a, n-a, n+b-c, r, n-s, n+r-t$

Possible Least Exponent on the Right:

$n-c, b, t$

$$
\begin{aligned}
&x^{a} + x^{n-a} + x^{n+b-c} + x^{r} + x^{s} \\
&\qquad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
&= x^{c} + x^{n-c} + x^{b} + x^{n-b} + x^{n+a-c} \\
&\qquad + x^{n+a-b} + x^{t} + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:

$a, n-a, n+b-c, r, n-s, n+r-t$

Possible Least Exponent on the Right:

$n-c, b, t, n-t$

$$
\begin{aligned}
&x^a + x^{n-a} + x^{n+b-c} + x^r + x^s \\
&\qquad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
&= x^c + x^{n-c} + x^b + x^{n-b} + x^{n+a-c} \\
&\qquad + x^{n+a-b} + x^t + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:
$a, n-a, n+b-c, r, n-s, n+r-t$

Possible Least Exponent on the Right:
$n-c, b, t, n-t$

$$
\begin{aligned}
&x^{a} + x^{n-a} + x^{n+b-c} + x^{r} + x^{s} \\
&\qquad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
&= x^{c} + x^{n-c} + x^{b} + x^{n-b} + x^{n+a-c} \\
&\qquad + x^{n+a-b} + x^{t} + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:

$a, n-a, n+b-c, r, n-s, n+r-t$

Possible Least Exponent on the Right:

$n-c, \quad t, n-t$

$$
\begin{aligned}
&x^a + x^{n-a} + x^{n+b-c} + x^r + x^s \\
&\qquad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
&= x^c + x^{n-c} + x^b + x^{n-b} + x^{n+a-c} \\
&\qquad + x^{n+a-b} + x^t + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:

$a, \qquad\qquad n+b-c, r, n-s, n+r-t$

Possible Least Exponent on the Right:

$n-c, \quad t, n-t$

$$
\begin{aligned}
&x^a + x^{n-a} + x^{n+b-c} + x^r + x^s \\
&\qquad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
&= x^c + x^{n-c} + x^b + x^{n-b} + x^{n+a-c} \\
&\qquad + x^{n+a-b} + x^t + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:

$a, \qquad r, n-s, n+r-t$

Possible Least Exponent on the Right:

$n-c, \quad t, n-t$

$$\begin{aligned}
&x^{a} + x^{n-a} + x^{n+b-c} + x^{r} + x^{s} \\
&\qquad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
&= x^{c} + x^{n-c} + x^{b} + x^{n-b} + x^{n+a-c} \\
&\qquad + x^{n+a-b} + x^{t} + x^{n-t} + x^{n+r-s}
\end{aligned}$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:

$a, \qquad\qquad r, n-s, n+r-t$

Possible Least Exponent on the Right:

$n-c, \qquad n-t$

$$
\begin{aligned}
& x^{a} + x^{n-a} + x^{n+b-c} + x^{r} + x^{s} \\
& \qquad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
= {} & x^{c} + x^{n-c} + x^{b} + x^{n-b} + x^{n+a-c} \\
& \qquad + x^{n+a-b} + x^{t} + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:

a, $\qquad$ r, $\qquad$ $n+r-t$

Possible Least Exponent on the Right:

$n-c$, $\qquad$ $n-t$

$$
\begin{aligned}
& x^a + x^{n-a} + x^{n+b-c} + x^r + x^s \\
& \qquad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
= {} & x^c + x^{n-c} + x^b + x^{n-b} + x^{n+a-c} \\
& \qquad + x^{n+a-b} + x^t + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:

a, $\qquad\qquad$ r

Possible Least Exponent on the Right:

$n-c$, $\qquad$ $n-t$

Possible Least Exponent on the Left:

a, $\qquad\qquad\qquad\qquad\qquad\qquad$ r

Possible Least Exponent on the Right:

$n - c$, $\qquad\qquad$ $n - t$

Possible Least Exponent on the Left:

$a,$ $\qquad\qquad$ r

Possible Least Exponent on the Right:

$n - c,$ $\qquad$ $n - t$

One of the Following Holds:

$$a = n - c$$
$$a = n - t$$
$$r = n - c$$
$$r = n - t$$