Irreducibility of Classical Polynomials and their Generalizations

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by Michael Filaseta

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- $f(x)$ has degree at least 1,
- $f(x)$ does not factor as a product of two polynomials in $\mathbb{Q}[x]$ each of degree ≥ 1.

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  - wavelets

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- Legendre Polynomials (too hard)

The Laguerre Polynomials:

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$$L_n(x) = \frac{e^x}{n!}\frac{d^n(x^n e^{-x})}{dx^n} = \sum_{j=0}^{n} \frac{(-1)^j}{j!}\binom{n}{j} x^j$$

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Theorem 1 (I. Schur, 1929): Let n be a positive integer, and let $a_0, a_1, \cdots, a_n$ denote arbitrary integers with $|a_0| = |a_n| = 1$. Then

$$a_n\frac{x^n}{n!} + a_{n-1}\frac{x^{n-1}}{(n-1)!} + \cdots + a_1 x + a_0$$

is irreducible.

Theorem (1996): Let $a_0, a_1, \ldots, a_n$ denote arbitrary integers with $|a_0| = 1$, and let

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in which cases either $f(x)$ is irreducible or $f(x)$ is the product of two irreducible polynomials of equal degree. If $|a_n| = n$, then for some choice of $a_1, \ldots, a_{n-1} \in \mathbb{Z}$ and $a_0 = \pm 1$, we have that $f(x)$ is divisible by $x \pm 1$.

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$$L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!}\frac{d^n\left(x^{n+\alpha}e^{-x}\right)}{dx^n}$$
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$L_n^{(0)}(x) = L_n(x)$ (the Laguerre Polynomials)

$$L_n^{(1)}(x) = (n+1) \sum_{j=0}^{n} \binom{n}{j} \frac{(-x)^j}{(j+1)!}$$

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Theorem 2 (I. Schur): Let n be a positive integer, and let $a_0, a_1, \cdots, a_n$ denote arbitrary integers with $|a_0| = |a_n| = 1$. Then

$$a_n \frac{x^n}{(n+1)!} + a_{n-1}\frac{x^{n-1}}{n!} + \cdots + a_1 \frac{x}{2} + a_0$$

is irreducible (over the rationals) unless $n = 2^r - 1 > 1$ (when $x \pm 2$ can be a factor) or $n = 8$ (when a quadratic factor is possible).

Theorem (joint with M. Allen): For n an integer ≥ 1, define

$$f(x) = \sum_{j=0}^{n} a_j \frac{x^j}{(j+1)!}$$

where the a_j's are arbitrary integers with $|a_0| = 1$. Write

$$n + 1 = k' 2^u \quad \text{with } k' \text{ odd}$$

and

$$(n+1)n = k'' 2^v 3^w \quad \text{with } \gcd(k'', 6) = 1.$$

If

$$0 < |a_n| < \min\{k', k''\},$$

then $f(x)$ is irreducible.

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$$L_2^{(2)}(x) = \frac{1}{2}(x-2)(x-6)$$

$$L_2^{(23)}(x) = \frac{1}{2}(x-20)(x-30)$$

$$L_4^{(23)}(x) = \frac{1}{24}(x-30)(x^3-78x^2+1872x-14040)$$

$$L_4^{(12/5)}(x) = \frac{1}{15000}(25x^2-420x+1224)(25x^2-220x+264)$$

$$L_5^{(39/5)}(x) = \frac{-1}{375000}(5x-84)\big(625x^4-29500x^3 + 448400x^2-2662080x+5233536\big)$$

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Theorem (joint with T.-Y. Lam): Let α be a rational number which is not a negative integer. Then for all but finitely many positive integers n, the polynomial $L_n^{(\alpha)}(x)$ is irreducible over the rationals.

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A Special Case: $\alpha = n$

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- Schur showed $\displaystyle\sum_{j=0}^{n} \frac{x^j}{j!}$ has Galois group A_n if $4|n$.
- Schur did not find a sequence of polynomials having Galois group A_n with $n \equiv 2 \pmod{4}$.

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Comment: Gow also showed that $L_n^{(n)}(x)$ is irreducible if

- $n = 2p^k$ where $k \in \mathbb{Z}^+$ and $p > 3$ is prime
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Conjecture: If $n > 2$, then $L_n^{(n)}(x)$ is irreducible.

Theorem (joint work with R. Williams): For almost all positive integers n the polynomial $L_n^{(n)}(x)$ is irreducible (and, hence, has Galois group A_n for almost all $n \equiv 2 \pmod 4$). More precisely, the number of $n \le t$ such that $L_n^{(n)}(x)$ is reducible is

$$\ll \exp\left(\frac{9\log(2t)}{\log\log(2t)}\right).$$

Furthermore, for all but finitely many n, $L_n^{(n)}(x)$ is either irreducible or $L_n^{(n)}(x)$ is the product of a linear polynomial times an irreducible polynomial of degree $n-1$.

Theorem (joint work with R. Williams): For all but

$$O\Big(\exp(9\log(2t)/\log\log(2t))\Big)$$

positive integers $n \le t$, the polynomial

$$f(x) = \sum_{j=0}^{n} a_j \binom{2n}{n-j} \frac{x^j}{j!}$$

is irreducible over the rationals for every choice of integers $a_0, a_1, \ldots, a_n$ with $|a_0| = |a_n| = 1$.

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Comment: The number of $n \le t$ for which $f(x)$ is reducible for some choice of a_j as above is

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> *There exist explicit numbers α and $\beta > 0$ such that, for $n \geq \alpha$,*

$$n(n+1) = 2^k 3^\ell m \implies m > n^\beta.$$

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The Ramanujan-Nagell equation

$$x^2 + 7 = 2^n$$

has as its only solutions $(\pm x, n)$ in

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Moreover, there exist explicit numbers α and $\beta > 0$ such that, for $x \geq \alpha$,

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$$\begin{aligned} H_n(x) &= (-1)^n e^{x^2/2} \frac{d^n\left(e^{-x^2/2}\right)}{dx^n} \\ &= \sum_{j=0}^{[n/2]} (-1)^n \binom{n}{2j} u_{2j}\, x^{m-2j} \end{aligned}$$

where

$$u_{2j} = (2j-1)(2j-3)\cdots 3 \cdot 1$$

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Theorem 3 (I. Schur, 1929): For $n > 1$ and arbitrary integers a_j with $|a_0| = |a_n| = 1$, the polynomial

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Theorem 4 (I. Schur, 1929): For $n \geq 1$ and arbitrary integers a_j with $|a_0| = |a_n| = 1$, the polynomial

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is irreducible unless $2n$ is of the form $3^u - 1$ with $u > 1$.

Email from Mark Kon:

Given a function $f(x)$, its wavelet transform consists of the family of functions $g(2^j x) * f(x)$, where g is the gaussian function, and j is an integer. The question was: if we know the zeroes of the second derivatives of this family of functions (over all j), can we recover f? ... The problem reduces to showing that none of these polynomials [certain Hermite polynomials] has zeroes (aside from the trivial one at the origin) which coincides with a zero of another one. So the bottom line is that the conjecture that f is uniquely recoverable follows from the non-overlapping of the zeroes of the Hermite polynomials.

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$$p(x) = (n-1)\big(x^{n+1} - 1\big) - (n+1)\big(x^n - x\big).$$

Then $p(x)$ *is* $(x-1)^3$ *times an irreducible polynomial if* n *is even and* $(x-1)^3(x+1)$ *times an irreducible polynomial if* n *is odd.*

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Joint Work With A. Borisov, T.-Y. Lam, O. Trifonov:

> *True for all but* $O(t^{4/5+\varepsilon})$ *values of* $n \leq t$.

Theorem 3 (I. Schur, 1929): For $n > 1$ and arbitrary integers a_j with $|a_0| = |a_n| = 1$, the polynomial

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Theorem (joint with M. Allen): For $n > 1$ and arbitrary integers a_j with $|a_0| = 1$ and

$$0 < |a_n| < 2n - 1,$$

the polynomial $f(x)$ above is irreducible for all but finitely many pairs (a_n, n).

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is irreducible unless $2n$ is of the form $3^u - 1$ with $u > 1$.

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where the a_j's are arbitrary integers with $|a_0| = 1$. Write

$$2n + 1 = k'3^u \quad \text{with} \quad 3 \nmid k'$$

and

$$(2n+1)(2n-1) = k''3^v5^w \quad \text{with} \quad (k'', 15) = 1.$$

If

$$0 < |a_n| < \min\{k', k''\},$$

then $f(x)$ is irreducible for all but finitely many pairs (a_n, n).

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- O. Trifonov and M.F. have now shown that all Bessel polynomials are irreducible.

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- Analysis to show that the conditions hold; usually this involves cases to eliminate possible factors depending on the size of their degrees.
  - small gaps between primes for large degrees
  - results about products of consecutive integers having some large prime factors for intermediate degrees
  - Thue type diophantine equations for small degrees

- Newton polygons are used to show that if certain conditions on divisibility by primes holds, then $f(x)$ is irreducible.

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A result of M.G. Dumas (in 1906) eliminates possible degrees for the factors of a polynomial using information about the divisibility of the coefficients by a given prime p (forming Newton polygons with respect to p).

- Newton polygons are used to show that if certain conditions on divisibility by primes holds, then $f(x)$ is irreducible.

> "Two such factorization schemes with a common, non-trivial factorization, will be called *compatible*. Otherwise, we call them incompatible. It is clear that if one can exhibit two incompatible factorization schemes, one thereby will have proved the irreducibility of the polynomial considered."
>
> Emil Grosswald
> Bessel Polynomials
> Lecture Notes Series

- Newton polygons are used to show that if certain conditions on divisibility by primes holds, then $f(x)$ is irreducible.

Idea: To consider factorization schemes using many primes and show that they are incompatible. For a polynomial of degree n and a $k \in [1, n/2]$, find a prime p such that the Newton polygon with respect to p does not allow for a factor of $f(x)$ to have degree k.

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Example:

For $1 \leq k \leq n/2$, show

$$\prod_{\substack{p^r \| (2n-1)(2n-3)\cdots(2n-2k+1) \\ p \geq 2k+1}} p^r > 2n-1.$$

The Generalized Bernoulli Polynomials:

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Theorem (joint with A. Adelberg): A positive proportion of the polynomials $B_n^{(n)}(x)$ are Eisenstein (and, hence, irreducible). More precisely, if the number of $n \le t$ for which $B_n^{(n)}(x)$ is Eisenstein is $\mathcal{B}(t)$, then

$$\mathcal{B}(t) > t/5 \qquad \text{for } t \text{ sufficiently large.}$$