A Distribution Problem for Powerfree Values of Irreducible Polynomials

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Theorem 2: *Let $k \geq (\sqrt{2}-1/2)g$. Given any $\gamma > 0$, there exists a $\delta = \delta(\gamma) > 0$ such that*

$$\sum_{\substack{s_{n+1} \leq X \\ s_{n+1}-s_n \leq X^{\delta}}} (s_{n+1}-s_n)^{\gamma} \sim B(\gamma, f, k) X,$$

for some constant $B(\gamma, f, k)$ depending only on γ, $f(x)$, and k.

Definition 1: Given $f(x) \in \mathbb{Z}[x]$, let $s_n = s_n(f)$ be the nth positive integer m such that $f(m)$ is k-free. Let

$$L(h) = L(h, X) = L(h, X, f) = |\left\{ n \in \mathbb{Z}^{+} : h < s_{n+1} - s_n \leq 2h, X/2 < s_{n+1} \leq X \right\}|.$$

$$L(2h) \ll \frac{X}{h^{\gamma+\varepsilon}} \tag{2}$$

Comment: (2) & Theorem 3 $\Longrightarrow$ Theorems 1 & 2 (sort-of).

Notation: $D = R(f, f')$, the resultant of f and f'
$z = 4g$
$\mathcal{A} = \{p : p|D \text{ or } p \leq z\}$
$Q = \prod_{p \in \mathcal{A}} p^k$
$H = \dfrac{h \log h}{8gQ}$
$\rho(q) = \rho(f, q) = |\{a \in \mathbb{Z}_q : f(a) \equiv 0 \pmod{q}\}|$
$T \in \mathcal{T} = \{2^j H : j = 0, 1, \ldots, J\}$

Lemma 4: *Let*

$$F(n) = F(n, h, f) = \sum_{n < m \leq n+h} \sum_{\substack{p > H \\ p^k | f(m)}} 1.$$

If there are no integers m in $(n, n+h]$ such that $f(m)$ is k-free, then

$$F(n) \geq \frac{h}{4Q}.$$

Lemma 5: *For r a positive integer, define*

$$M_r = M_r(h, X, f) = \sum_{X/2 < n \leq X} F^r(n).$$

Then

$$L(2h) \ll_r \frac{M_r}{h^{r+1}} + 1.$$

Definition 2: Let $S_r(T)$ be the number of $2r$-tuples $(p_1, \ldots, p_r, m_1, \ldots, m_r)$ with $T < p_1 < p_2 < \cdots < p_r \leq 2T$ such that, for each $j \in \{1, 2, \ldots, r\}$,

$$|m_1 - m_j| < h, \quad \frac{X}{2} < m_j \leq X + h \leq 2X, \quad \text{and} \quad p_j^k | f(m_j).$$

Notation: $G(n, T) = G(n, T, f) = \displaystyle\sum_{n < m \leq n+h} \sum_{\substack{p^k | f(m) \\ T < p \leq 2T}} 1$

Lemma 6: *Let r be a positive integer. If $r = 1$, then*

$$\sum_{X/2<n\le X} \binom{G(n,T)}{r} \le (2g)^r h S_r(T).$$

If $r > 1$, then

$$\sum_{\substack{X/2<n\le X \\ G(n,T)\ge 2gr}} \binom{G(n,T)}{r} \le (2g)^r h S_r(T).$$

Lemma 7: *If $T \ge H$ and $r \in \mathbb{Z}^+$, then*

$$S_r(T) \ll \frac{h^{r-1}X}{T^{(k-1)r}\log^r T} + \frac{h^{r-1}T^r}{\log^r T}.$$

Lemma 8: *For $\mathcal{T}$ as previously defined,*

$$\sum_{\substack{T\in\mathcal{T} \\ U<T\le V}} T^a \ll \begin{cases} U^a & \text{if } a < 0 \\ V^a & \text{if } a > 0. \end{cases}$$

Lemma 9: *Let $k \ge (\sqrt{2} - 1/2)g$. Let $\varepsilon \in (0, (k-1)/k]$ such that*

$$\varepsilon < 1 - \frac{8g(g-1)}{(2k+g)^2 - 4}.$$

If $X^{1-\varepsilon} < T \ll X^{g/k}$, then there is a $\xi = \xi(\varepsilon) > 0$ such that

$$S_1(T) \ll X^{1-\xi}.$$

For Proof of Theorem 2:

$j \ge \max\{\gamma + \varepsilon', 2\}$ (ε' to be chosen depending on ε)
$h \le X^{\delta'}$ (δ' depends on ε', j, k, and γ)
$F_i(n) = \sum_{\theta_i} G(n,T)$ for $i \in \{1,2,3,4,5\}$
$j_1 = j_2 = j_3 = j$ and $j_4 = j_5 = 1$

Five Cases:

Let θ_1 represent the case that $T \le X^{1-\varepsilon}$ and $G(n,T) \le 2gj$.
Let θ_2 represent the case that $T > X^{1-\varepsilon}$ and $G(n,T) \le 2gj$.
Let θ_3 represent the case that $T \le X^{1/(kj)}$ and $G(n,T) > 2gj$.
Let θ_4 represent the case that $X^{1/(kj)} < T \le X^{1-\varepsilon}$ and $G(n,T) > 2gj$.
Let θ_5 represent the case that $T > X^{1-\varepsilon}$ and $G(n,T) > 2gj$.

$$L(2h) \ll 1 + \sum_{i=1}^{5} \frac{1}{h^{j_i+1}} \sum_{X/2<n\le X} F_i^{j_i}(n) \tag{5}$$