A Distribution Problem for Powerfree Values of Irreducible Polynomials

by B. Beasley and M. Filaseta

Notation: $f(x) \in \mathbb{Z}[x]$ is irreducible
$g = \deg f$
$k \in \mathbb{Z}$, $k \geq 2$
$\gcd_{m \in \mathbb{Z}}(f(m))$ is k-free
s_n is the nth integer $m > 0$ for which $f(m)$ is k-free

Theorem 2. *Let $k \geq (\sqrt{2} - 1/2)g$. Given any $\gamma > 0$, there exists a $\delta = \delta(\gamma) > 0$ such that*

$$\sum_{\substack{s_{n+1} \leq X \\ s_{n+1} - s_n \leq X^\delta}} (s_{n+1} - s_n)^\gamma \sim B(\gamma, f, k)X,$$

for some constant $B(\gamma, f, k)$ depending only on γ, $f(x)$, and k.

Definition 1. Given $f(x) \in \mathbb{Z}[x]$, let $s_n = s_n(f)$ be the nth positive integer m such that $f(m)$ is k-free. Let

$$L(h) = L(h, X) = L(h, X, f) = \left| \left\{ n \in \mathbb{Z}^+ : h < s_{n+1} - s_n \leq 2h, X/2 < s_n \leq X \right\} \right|.$$

$$L(2h) \ll \frac{X}{h^{\gamma+\varepsilon}} \tag{2}$$

Comment: (2) & Theorem 3 $\Longrightarrow$ Theorems 1 & 2 (sort-of).

Notation: $D = R(f, f')$, the resultant of f and f'
$z = 4g$
$\mathcal{A} = \{p : p|D \text{ or } p \leq z\}$
$Q = \prod_{p \in \mathcal{A}} p^k$
$H = \dfrac{h \log h}{8gQ}$
$\rho(q) = \rho(f, q) = |\{a \in \mathbb{Z}_q : f(a) \equiv 0 \pmod{q}\}|$

Lemma 1. *There exists an integer a such that if $\mathcal{B} = \{y \in (n, n+h] : y \equiv a \pmod{Q}\}$, then*

$$\sum_{p \leq H} \sum_{\substack{m \in \mathcal{B} \\ p^k | f(m)}} 1 \leq \sum_{\substack{z < p \leq H \\ p \nmid D}} \left(\frac{h}{Qp^k} + 1 \right) \rho(p^k).$$

Lemma 2. *If $p \nmid D$, then $\rho(p^k) \le g$ for every positive integer k.*

Lemma 3. *Given the notation above, $\sum_{z<p\le H, p\nmid D} \rho(p^k)/p^k \le 1/4$.*

Lemma 4. *Let*

$$F(n) = F(n,h,f) = \sum_{n<m\le n+h} \sum_{\substack{p>H \\ p^k | f(m)}} 1.$$

If there are no integers m in $(n, n+h]$ such that $f(m)$ is k-free, then

$$F(n) \ge \frac{h}{4Q}.$$

Lemma 5. *For r a positive integer, define*

$$M_r = M_r(h, X, f) = \sum_{X/2<n\le X} F^r(n).$$

Then

$$L(2h) \ll_r \frac{M_r}{h^{r+1}} + 1.$$

Definition 2. Let $S_r(T)$ be the number of $2r$-tuples $(p_1, \ldots, p_r, m_1, \ldots, m_r)$ with $T < p_1 < p_2 < \cdots < p_r \le 2T$ such that, for each $j \in \{1, 2, \ldots, r\}$,

$$|m_1 - m_j| < h, \quad \frac{X}{2} < m_j \le X + h \le 2X, \quad \text{and} \quad p_j^k | f(m_j).$$

Notation: $G(n,T) = G(n,T,f) = \sum_{n<m\le n+h} \sum_{\substack{p^k | f(m) \\ T<p\le 2T}} 1$

Lemma 6. *Let r be a positive integer. If $r = 1$, then*

$$\sum_{X/2<n\le X} \binom{G(n,T)}{r} \le (2g)^r h S_r(T).$$

If $r > 1$, then

$$\sum_{\substack{X/2<n\le X \\ G(n,T)\ge 2gr}} \binom{G(n,T)}{r} \le (2g)^r h S_r(T).$$