A Distribution Problem for Powerfree Values of Irreducible Polynomials

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Notation: $f(x) \in \mathbb{Z}[x]$ is irreducible
$g = \deg f$
$k \in \mathbb{Z}$, $k \geq 2$
$\gcd_{m \in \mathbb{Z}}(f(m))$ is k-free
s_n is the nth integer $m > 0$ for which $f(m)$ is k-free

Theorem 1. *Let $g \geq 2$, and let $k \geq (\sqrt{2} - 1/2)g$. Let*

$$\phi_1 = \frac{(2s+g)(k-s) - g(g-1)}{(2s+g)(k-s) + g(2s+1)},$$

where

$$s = \begin{cases} 1 & \text{if } 2 \leq g \leq 4 \\ \left[(\sqrt{2}-1)g/2\right] & \text{if } g \geq 5. \end{cases}$$

Let

$$\phi_2 = \begin{cases} \dfrac{8g(g-1)}{(2k+g)^2 - 4} & \text{if } (\sqrt{2} - 1/2)g \leq k \leq g \\ \dfrac{g}{2k - g + r} & \text{if } k \geq g + 1, \end{cases}$$

where r is the largest positive integer such that $r(r-1) < 2g$. Then $\phi_1 > 0$, $\phi_2 > 0$, and if

$$0 \leq \gamma < \min\left\{1 + \frac{\phi_1}{\phi_2}, k\right\},$$

then

$$\sum_{s_n \leq X} (s_{n+1} - s_n)^\gamma \sim B(\gamma, f, k)X,$$

for some constant $B(\gamma, f, k)$ depending only on γ, $f(x)$, and k.

Theorem 2. *Let $k \geq (\sqrt{2} - 1/2)g$. Given any $\gamma > 0$, there exists a $\delta = \delta(\gamma) > 0$ such that*

$$\sum_{\substack{s_n \leq X \\ s_{n+1} - s_n \leq X^\delta}} (s_{n+1} - s_n)^\gamma \sim B(\gamma, f, k)X,$$

for some constant $B(\gamma, f, k)$ depending only on γ, $f(x)$, and k.

Theorem 3. *For a fixed positive integer d, set*

$$N_d(X) = |\{m \in \mathbb{Z}^+ : m \le X, f(m) \text{ and } f(m+d) \text{ are } k\text{-free}, \\ f(m+1), f(m+2), \dots, f(m+d-1) \text{ are not } k\text{-free}\}|.$$

Suppose that for some positive integer j, $s_{j+1} - s_j = d$. Then there is a constant $c_d > 0$, depending on d, for which

$$N_d(X) \sim c_d X.$$

Definition 1. Given $f(x) \in \mathbb{Z}[x]$, let $s_n = s_n(f)$ be the nth positive integer m such that $f(m)$ is k-free. Let

$$L(h) = L(h, X) = L(h, X, f) = \left|\left\{n \in \mathbb{Z}^+ : h < s_{n+1} - s_n \le 2h, X/2 < s_n \le X\right\}\right|.$$

$$L(2h) \ll \frac{X}{h^{\gamma+\varepsilon}} \tag{2}$$

Comment: (2) & Theorem 3 $\Longrightarrow$ Theorems 1 & 2.

Lemma 1. *There exists an integer a such that if $\mathcal{B} = \{y \in (n, n+h] : y \equiv a \pmod{Q}\}$, then*

$$\sum_{p \le H} \sum_{\substack{m \in \mathcal{B} \\ p^k | f(m)}} 1 \le \sum_{\substack{z < p \le H \\ p \nmid D}} \left(\frac{h}{Qp^k} + 1\right) \rho(p^k).$$

Lemma 2. *If $p \nmid D$, then $\rho(p^k) \le g$ for every positive integer k.*

Lemma 3. *Given the notation above, $\sum_{z<p\le H, p\nmid D} \rho(p^k)/p^k \le 1/4$.*

Lemma 4. *Let*

$$F(n) = F(n, h, f) = \sum_{n < m \le n+h} \sum_{\substack{p > H \\ p^k | f(m)}} 1.$$

If there are no integers m in $(n, n+h]$ such that $f(m)$ is k-free, then

$$F(n) \ge \frac{h}{4Q}.$$