Seminar Notes 02/28/05

Subject Matter: *On rational values of $\phi(n!)/m!$ and $\sigma(n!)/m!$*

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Theorem 1: *Let f denote one of the arithmetic functions ϕ, σ and τ, and let k be a fixed positive integer. Then there are finitely many positive integers a, b, n, and m such that*

$$b \cdot f(n!) = a \cdot m!, \quad \gcd(a,b) = 1 \quad \textit{and} \quad \omega(ab) \le k. \tag{1}$$

Corollary 1: *Let n be a positive integer, and let q be a prime. Then $\nu_q(n!) = \dfrac{n}{q-1} + O\left(\dfrac{\log n}{\log q}\right)$.*

Corollary 2: *Let q be a prime, and let a and N be integers with $N > 2$. If $q|\Phi_N(a)$, then either $q \equiv 1 \pmod{N}$ or we have that both q is the largest prime factor of N and $q^2 \nmid \Phi_N(a)$.*

Lemma 6: *For n sufficiently large, $\phi(n) > \dfrac{n}{2\log\log n}$.*

Lemma 7: *Let k be a positive integer. There are positive numbers c_k and n_k such that if $n \ge n_k$, then either n has $\ge k+1$ odd prime factors that are $\le \log n$ or $\phi(n) \ge c_k n$.*

Lemma 9: *Fix primes q_1 and q_2 and a number $\varepsilon > 0$. Let n_1 and n_2 be sufficiently large integers. Then*

$$\gcd\left(q_1^{n_1} - 1, q_2^{n_2} - 1\right) < \max\{q_1^{\varepsilon n_1}, q_2^{\varepsilon n_2}\}.$$

Lemma 11: *Let q be a fixed odd prime. There is a constant $n_0 = n_0(q)$ such that if $n \ge n_0$, then*

$$\frac{n}{3(q-1)\log n} \le \nu_q\big(\sigma(n!)\big) \le \frac{25n}{\log n}.$$

Proof of Theorem 1 for $f = \sigma$:

- Note $\Phi_N(a) \ge (a-1)^{\phi(N)} \ge a^{\phi(N)/2}$ where N and $a \ge 3$ are positive integers.
- Need only show n is bounded. Fix k, and assume n is large and m, a and b satisfy (1).
- Fix q among the first $k+2$ odd primes that do not divide ab.
- Deduce $m \le (50(q-1)n)/\log n$ from Corollary 1 and Lemma 11.
- Let $q_1 < q_2 < \cdots < q_{k+1}$ be $k+1$ primes not dividing ab, and set $n_j = \nu_{q_j}\big(n!\big) + 1$.
- Case (i): $\exists j \in \{1, 2, \ldots, k+1\}$ such that n_j has $\ge k+1$ odd prime factors that are $\le \log n$.
- Call them $d_1, d_2, \ldots, d_{k+1}$ and define $m_i = n_j/d_i$. Lemma 6 implies $\Phi_{m_i}(q_j) \ge q_j^{m_i/(4\log\log m_i)}$.
- Use $c_i' n/\log n \le m_i \le n$, deduce $\Phi_{m_i}(q_j) \ge q_j^{c_i' n/(4\log n \log\log n)}$.
- From Corollary 2, $\exists D_i$ such that $p|\Phi_{m_i}(q_j)/D_i$ implies $p \equiv 1 \pmod{m_i}$.
- Deduce $\Phi_{m_i}(q_j)/D_i$ and $\Phi_{m_{i'}}(q_j)/D_{i'}$ are relatively prime if $i \ne i'$, and fix i so that $\gcd\big(a, \Phi_{m_i}(q_j)/D_i\big) = 1$.
- The product of the primes (with multiplicity) dividing $m!$ and $\equiv 1 \pmod{m_i}$ is bounded. Complete Case (i).
- Case (ii): $\forall j \in \{1, 2, \ldots, k+1\}$, n_j has $\le k$ odd prime factors $\le \log n$.

- Lemma 7 implies $\Phi_{n_j}(q_j) \geq q_j^{c_k n_j/2}$.
- Corollary 1 implies $\max_{1\leq i\leq k+1}\{n_i\} \leq q_{k+1} \min_{1\leq i\leq k+1}\{n_i\}$.
- Apply Lemma 9 with $\eta = \log q_{k+1}/\log q_1$ and $\varepsilon = c_k/(4kq_{k+1}\eta)$. Deduce

$$\begin{aligned} A_j &= \gcd\left(\Phi_{n_j}(q_j), \prod_{\substack{1\leq i\leq k+1\\ i\neq j}} \Phi_{n_i}(q_i)\right) \leq \gcd\left(q_j^{n_j}-1, \prod_{\substack{1\leq i\leq k+1\\ i\neq j}} \big(q_i^{n_i}-1\big)\right) \\ &\leq \big(\max_{1\leq i\leq k+1}\{q_i^{n_i}\}\big)^{\varepsilon k} \leq q_{k+1}^{\varepsilon k\max\{n_i\}} \leq q_j^{\varepsilon\eta k\max\{n_i\}} \leq q_j^{q_{k+1}\varepsilon\eta k n_j} = q_j^{c_k n_j/4} \leq \Phi_{n_j}(q_j)^{1/2}. \end{aligned}$$

- If $p|\Phi_{n_j}(q_j)/A_j$, then $\nu_p\big(\Phi_{n_j}(q_j)\big) > \nu_p\big(\Phi_{n_i}(q_i)\big)$ for $i\neq j$, so $\Phi_{n_j}(q_j)/A_j$ are pairwise relatively prime.
- Fix j with $\gcd(a, \Phi_{n_j}(q_j)/A_j) = 1$. Lemma 7 and Corollary 1 imply $\Phi_{n_j}(q_j)/A_j \geq 3^{c_k'' n}$.
- Corollary 2, Corollary 1 and the bound on m give a contradiction. Case (ii) is complete.