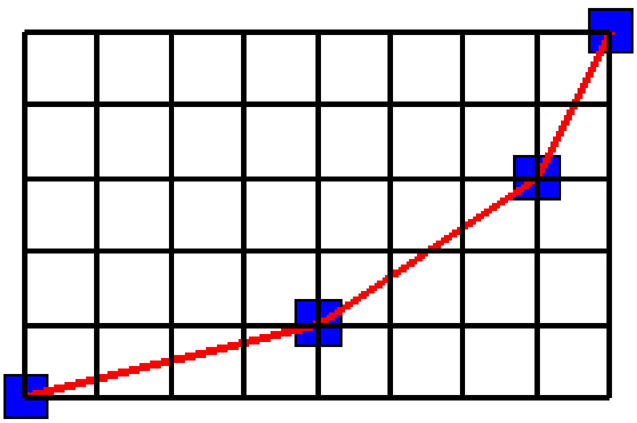
A Comment on a Third Irreducibility Theorem of I. Schur

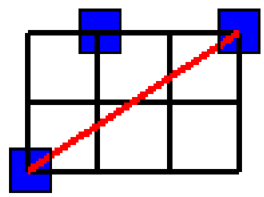
(joint work with Martha Allen)

Theorem: For $j \geq 0$, define $u_{2j} = 1 \times 3 \times 5 \times \cdots \times (2j-1)$. For $n > 1$, define $f(x) = \sum_{j=0}^{n} a_j x^{2j}/u_{2j}$, where $a_0, a_1, \ldots, a_n$ are arbitrary integers with $|a_0| = 1$. Then there is a finite set T of pairs (a_n, n) such that if $0 < |a_n| < 2n-1$, then $f(x)$ is irreducible unless $(a_n, n) \in T$.

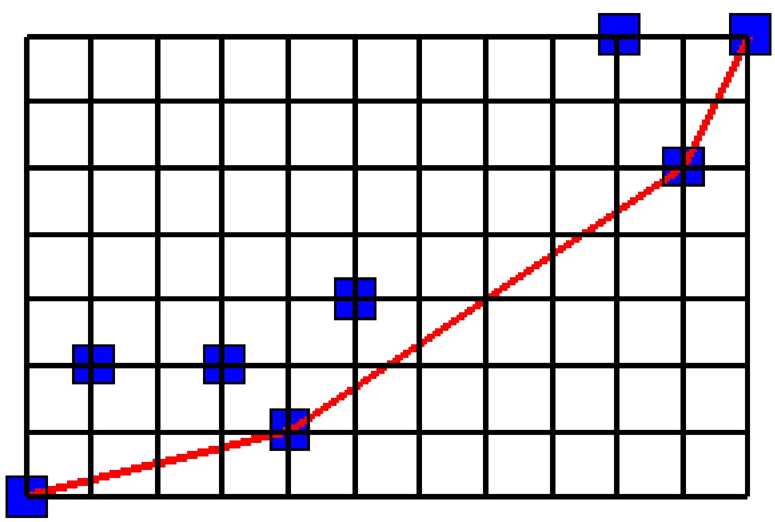
Lemma 1 (Dumas): The Newton polygon of $g(x)h(x)$ with respect to a prime is determined from the Newton polygons of $g(x)$ and of $h(x)$ with respect to the same prime as illustrated below.



NEWTON POLYGON OF $g(x)$



NEWTON POLYGON OF $h(x)$



NEWTON POLYGON OF $g(x)h(x)$

Lemma 2: Let $a_0, a_1, \ldots, a_n$ denote arbitrary integers with $|a_0| = 1$, and let $f(x) = \sum_{j=0}^{n} a_j x^{2j}/u_{2j}$. Let k be a positive odd integer $\leq n$. Suppose there exists a prime p and a positive integer r such that

(i) $p \geq k + 2$

(ii) $p^r | (2n-1)(2n-3) \cdots (2n-k)$

(iii) $p^r \nmid a_n$

Then $f(x)$ cannot have a factor of degree k and cannot have a factor of degree $k+1$.

Lemma 3: For $n \geq 3$ and $k \in [3, n]$,

$$\prod_{\substack{p^r | (2n-1)(2n-3)\cdots(2n-k) \\ p \geq k+2}} p^r > 2n - 1$$

unless one of the following holds:

(a) $k = 3$ and one of $\{2n-1, 2n-3\}$ is a power of 3

(b) $k = 5$ and one of $\{2n-1, 2n-3, 2n-5\}$ is a power of 3 and another is a power of 5

(c) $k = 7$ and one of $\{2n-1, 2n-3, 2n-5, 2n-7\}$ is a power of 3, another a power of 5 or 3 times a power of 5, and another is a power of 7 or 3 times a power of 7.

Lemma 4 (Lehmer): If $N > 1$ is odd and $N(N-2)$ is divisible only by primes ≤ 5, then $N \in \{3, 5, 27\}$. If $N > 1$ is odd and $N(N-2)$ is divisible only by primes ≤ 7, then $N \in \{3, 5, 7, 9, 27, 245\}$. If $N > 3$ is odd and $N(N-4)$ is divisible only by primes ≤ 5, then $N \in \{5, 9\}$.

Conclusion: If $k \leq n$ and $k = 5$, then $2n-3 \geq 7$ and $2n-1 \geq 9$. Hence, (b) occurs only when $n \in \{5, 14, 15\}$. If $k \leq n$ and $k = 7$, then $2n-5 \geq 9$. Observe that 239 and 241 are primes, $13|247$, and $83|249$. Also, 23, 29, and 31 are primes. Noting that in (c) two of $2n-1$, $2n-3$, $2n-5$, and $2n-7$ are consecutive odd numbers divisible only by primes ≤ 7, we deduce $n = 14$. Discuss the following cases using Maple.

Case 1: $n = 5$ ($f(x)$ has a factor of degree 5; $0 < |a_n| < 9$)

Case 2: $n = 15$ ($f(x)$ has a factor of degree 5 or 6; $0 < |a_n| < 29$)

Case 3: $n = 14$ ($f(x)$ has a factor with degree in $[5, 8]$; $0 < |a_n| < 27$)