Seminar Notes 02/07/05

Subject Matter: *On rational values of* $\phi(n!)/m!$ *and* $\sigma(n!)/m!$

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Florian Luca's Result: Let f denote one of the arithmetic functions ϕ, σ and τ, and let r be a fixed rational number. Then there are finitely many pairs of positive integers n and m for which $f(n!)/m! = r$.

Theorem 1: *Let* f *denote one of the arithmetic functions* ϕ *and* σ*, and let* k *be a fixed positive integer. Then there are finitely many positive integers* a*,* b*,* n*, and* m *such that*

$$b \cdot f(n!) = a \cdot m!, \quad \gcd(a,b) = 1 \quad \textit{and} \quad \omega(ab) \leq k.$$

Theorem 2: *Let* $t > 0$*. Then there are finitely many positive integers* a*,* b*,* n*, and* m *such that*

$$b \cdot \phi(n!) = a \cdot m!, \quad \gcd(a,b) = 1 \quad \textit{and} \quad \omega(ab) \leq \log^t(nm).$$

Lemma 1: *Let* n *be a positive integer, and let* q *be a prime. Then*

$$\nu_q(n!) = \frac{n - s_q(n)}{q-1},$$

where $s_q(n)$ *is the sum of the base* q *digits of* n*.*

Corollary 1: *Let* n *be a positive integer, and let* q *be a prime. Then*

$$\nu_q(n!) = \frac{n}{q-1} + O\left(\frac{\log n}{\log q}\right),$$

where the implied constant is absolute.

Lemma 2: *Let* n *and* m *be integers. If* n *is sufficiently large and*

$$m \geq n + \frac{n}{\log n} + O\left(\frac{n}{\log^2 n}\right), \tag{$*$}$$

then the interval $(n, m]$ *contains* $\gg m/\log^2 m$ *prime numbers.*

Lemma 3: *Fix* $M > 0$*. Then there is an absolute constant* $c_1 > 0$ *and a constant* $c_2 = c_2(M) > 0$ *such that*

$$\pi(x; b, a) = \frac{\pi(x)}{\phi(b)} + E, \quad \textit{where} \quad |E| \leq c_2 x \exp\left(-c_1\sqrt{\log x}\right),$$

for $x \geq 2$ *and every choice of relatively prime integers* a *and* b *with* $1 \leq b \leq (\log x)^M$*.*

Proof of Theorem 2:

- It suffices to show n is bounded, so assume n is large and (a, b, n, m) is as in Theorem 2.
- Establish ($*$). To do this, take $M = t + 2$ and $q \leq \log^M n$ with $q \nmid ab$. Use Corollary 1 and Lemma 3 to obtain

  $$\nu_q(b \cdot \phi(n!)) \geq \nu_q(\phi(n!)) \geq \frac{n}{q-1} + \frac{n}{(q-1)\log n} + O\left(\frac{n}{q \log^2 n}\right).$$

  Combine this with an application of Corollary 1 to estimate $\nu_q(a \cdot m!)$.
- Apply Lemma 2.