NOT MY TITLE

How many positive integers $n \leq 1000$ are such that 2^n begins with the digit 1?

NOT MY TITLE

How many positive integers $n \leq 1000$ are such that 2^n begins with the digit 1?

$$\mathbf{301}$$

NOT MY TITLE

How many positive integers $n \leq 1000$ are such that 2^n begins with the digit 1?

$$301$$

$$\log_{10} 2 = 0.30102999566398\ldots$$

x	$\#\{n \leq x : 2^n$ begins with $1\}$
$\mathbf{10}$	$\mathbf{3}$

x	$\#\{n \leq x : 2^n$ begins with $1\}$
$\mathbf{10}$	$\mathbf{3}$
$\mathbf{100}$	$\mathbf{30}$
$\mathbf{1000}$	$\mathbf{301}$

x	$\#\{n \leq x : 2^n$ begins with $1\}$
10	3
100	30
1000	301
10000	3010
100000	30102
1000000	301029
10000000	3010299

x	$\#\{n \leq x : 2^n$ begins with $1\}$
$\mathbf{10}$	$\mathbf{3}$
$\mathbf{100}$	$\mathbf{30}$
$\mathbf{1000}$	$\mathbf{301}$
$\mathbf{10000}$	$\mathbf{3010}$
$\mathbf{100000}$	$\mathbf{30102}$
$\mathbf{1000000}$	$\mathbf{301029}$
$\mathbf{10000000}$	$\mathbf{3010299}$

$$\log_{10} 2 = 0.30102999566398\ldots$$

x	$\#\{n \leq x : 2^n$ begins with $1\}$
$\mathbf{10}$	$\mathbf{3}$
$\mathbf{100}$	$\mathbf{30}$
$\mathbf{1000}$	$\mathbf{301}$
$\mathbf{10000}$	$\mathbf{3010}$
$\mathbf{100000}$	$\mathbf{30102}$
$\mathbf{1000000}$	$\mathbf{301029}$
$\mathbf{10000000}$	$\mathbf{3010299}$

To Ponder: Prove the pattern continues.

Applications of Padé Approximations of $(1-z)^k$ to Number Theory

by Michael Filaseta

University of South Carolina

General Areas of Applications:

General Areas of Applications:

- irrationality measures

General Areas of Applications:

- irrationality measures
- diophantine equations

General Areas of Applications:

- irrationality measures
- diophantine equations
- Waring's problem

General Areas of Applications:

- irrationality measures
- diophantine equations
- Waring's problem
- the factorization of $n(n+1)$

General Areas of Applications:

- irrationality measures
- diophantine equations
- Waring's problem
- the factorization of $n(n+1)$
- Galois groups of classical polynomials

General Areas of Applications:

- irrationality measures
- diophantine equations
- Waring's problem
- the factorization of $n(n+1)$
- Galois groups of classical polynomials
- the Ramanujan-Nagell equation

General Areas of Applications:

- irrationality measures
- diophantine equations
- Waring's problem
- the factorization of $n(n+1)$
- Galois groups of classical polynomials
- the Ramanujan-Nagell equation
- k-free numbers in short intervals

General Areas of Applications:

- irrationality measures
- diophantine equations
- Waring's problem
- the factorization of $n(n+1)$
- Galois groups of classical polynomials
- the Ramanujan-Nagell equation
- k-free numbers in short intervals
- k-free values of polynomials and binary forms

General Areas of Applications:

- irrationality measures
- diophantine equations
- Waring's problem
- the factorization of $n(n+1)$
- Galois groups of classical polynomials
- the Ramanujan-Nagell equation
- k-free numbers in short intervals
- k-free values of polynomials and binary forms
- the abc-conjecture

What are the Padé approximations of $(1-z)^k$?

"Man, did you scare me! For a second there I thought you were my math teacher!"

What are the Padé approximations of $(1-z)^k$?

"Man did you scare me! For a second there I thought you were my Math teacher!"

What are the Padé approximations of $(1-z)^k$?

"Man did you scare me! For a second there I thought you were my Math teacher!"

What are the Padé approximations of $(1-z)^k$?

Answer: Rational functions that give good approximations to $(1-z)^k$ near the origin.

What are the Padé approximations of $(1-z)^k$?

Answer: Rational functions that give good approximations to $(1-z)^k$ near the origin.

What are the Padé approximations of e^z?

Answer: Rational functions that give good approximations to e^z near the origin.

What are the Padé approximations of $(1-z)^k$?

Answer: Rational functions that give good approximations to $(1-z)^k$ near the origin.

What are the Padé approximations of $(1-z)^k$?

Answer: Rational functions that give good approximations to $(1-z)^k$ near the origin.

What are the Padé approximations of $(1-z)^k$?

Answer: Rational functions that give good approximations to $(1-z)^k$ near the origin.

Important Equation:

What are the Padé approximations of $(1-z)^k$?

Answer: Rational functions that give good approximations to $(1-z)^k$ near the origin.

Important Equation:

$$(1-z)^k \approx \frac{P(z)}{Q(z)}$$

What are the Padé approximations of $(1-z)^k$?

Answer: Rational functions that give good approximations to $(1-z)^k$ near the origin.

Important Equation:

$$(1-z)^k = \frac{P(z)}{Q(z)} - z^m R(z)$$

What are the Padé approximations of $(1-z)^k$?

Answer: Rational functions that give good approximations to $(1-z)^k$ near the origin.

Important Equation:

degree $< k$ (usually)

$$(1-z)^k = \frac{P(z)}{Q(z)} - z^m R(z)$$

What are the Padé approximations of $(1-z)^k$?

Answer: Rational functions that give good approximations to $(1-z)^k$ near the origin.

Important Equation:

$$P - (1-z)^k Q = z^m E$$

What are the Padé approximations of $(1-z)^k$?

Answer: Rational functions that give good approximations to $(1-z)^k$ near the origin.

Important Equation:

$$P_r - (1-z)^k Q_r = z^m E_r$$

What are the Padé approximations of $(1-z)^k$?

Answer: Rational functions that give good approximations to $(1-z)^k$ near the origin.

Important Equation:

$$P_r - (1-z)^k Q_r = z^{2r+1} E_r$$

What are the Padé approximations of $(1-z)^k$?

Answer: Rational functions that give good approximations to $(1-z)^k$ near the origin.

Important Equation:

$$P_r - (1-z)^k Q_r = z^{2r+1} E_r$$

What are the Padé approximations of $(1-z)^k$?

Answer: Rational functions that give good approximations to $(1-z)^k$ near the origin.

Important Equation:

$$P_r - (1-z)^k Q_r = z^{2r+1} E_r$$

What are the Padé approximations of $(1-z)^k$?

Answer: Rational functions that give good approximations to $(1-z)^k$ near the origin.

Important Equation:

$$P_r - (1-z)^k Q_r = z^{2r+1} E_r$$

$$\deg P_r = \deg Q_r = r < k, \quad \deg E_r = k - r - 1$$

Some Properties of the Polynomials:

(i) $P_r(z)$, $(-z)^k Q_r(z)$, and $z^{2r+1}E_r(z)$ satisfy

$$z(z-1)y'' + \big(2r(1-z)-(k-1)z\big)y' + r(k+r)y = 0.$$

(ii) $$Q_r(z) = \sum_{j=0}^{r} \binom{2r-j}{r}\binom{k-r+j-1}{j} z^j$$

(iii) $$Q_r(z) = \frac{(k+r)!}{(k-r-1)!\,r!\,r!}\int_0^1 (1-t)^r t^{k-r-1}(1-t+zt)^r \,\mathrm{d}t$$

(iv) $$P_r(z)Q_{r+1}(z) - Q_r(z)P_{r+1}(z) = cz^{2r+1}$$

$$\boldsymbol{P_r - (1 - z)^k Q_r = z^{2r+1} E_r}$$

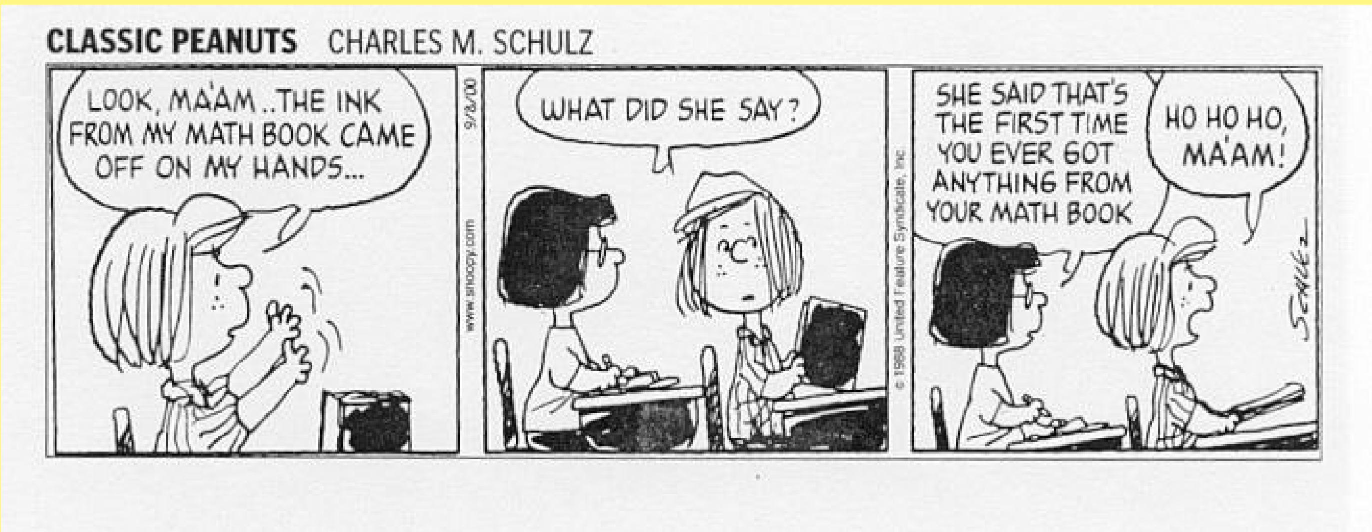
$$P_r - (1-z)^k Q_r = z^{2r+1} E_r$$

WARNING: In the applications you are about to see, the true identies used have been changed.

$$P_r - (1-z)^k Q_r = z^{2r+1} E_r$$

WARNING: In the applications you are about to see, the true identies used have been changed. They have been changed to conform to the identity above. The identity above gives a result of the type wanted. Typically, a closer analysis of these polynomials or even a variant of the polynomials is used to obtain the currently best known results in the applications.

Irrationality measures:



Irrationality measures:

Theorem (Liouville): Fix $\alpha \in \mathbb{R} - \mathbb{Q}$ with α algebraic and of degree n. Then there is a constant $C = C(\alpha) > 0$ such that

$$\left|\alpha - \frac{a}{b}\right| > \frac{C}{b^n}$$

where a and b with $b > 0$ are arbitrary integers.

Irrationality measures:

Theorem (Liouville): Fix $\alpha \in \mathbb{R} - \mathbb{Q}$ with α algebraic and of degree n. Then there is a constant $C = C(\alpha) > 0$ such that

$$\left|\alpha - \frac{a}{b}\right| > \frac{C}{b^{n}}$$

where a and b with $b > 0$ are arbitrary integers.

Irrationality measures:

Theorem (Roth): Fix $\varepsilon > 0$ and $\alpha \in \mathbb{R} - \mathbb{Q}$ with α algebraic. Then there is a constant $C = C(\alpha, \varepsilon) > 0$ such that

$$\left|\alpha - \frac{a}{b}\right| > \frac{C}{b^{2+\varepsilon}}$$

where a and b with $b > 0$ are arbitrary integers.

Irrationality measures:

Theorem (Roth): Fix $\varepsilon > 0$ and $\alpha \in \mathbb{R} - \mathbb{Q}$ with α algebraic. Then there is a constant $C = C(\alpha, \varepsilon) > 0$ such that

$$\left|\alpha - \frac{a}{b}\right| > \frac{C}{b^{2+\varepsilon}}$$

where a and b with $b > 0$ are arbitrary integers.

Comment: Liouville's result is effective; Roth's is not.

Irrationality measures:

Theorem (Baker): For a and b integers with $b > 0$,

$$\left|\sqrt[3]{2} - \frac{a}{b}\right| > \frac{C}{b^{2.955}}$$

where $C = 10^{-6}$.

Irrationality measures:

Theorem (Baker): For a & b integers with $b > 0$,

$$\left|\sqrt[3]{2} - \frac{a}{b}\right| > \frac{1}{10^6 b^{2.955}}.$$

Irrationality measures:

Theorem (Chudnovsky): For a & b integers with $b > 0$,

$$\left|\sqrt[3]{2} - \frac{a}{b}\right| > \frac{1}{c \cdot b^{2.43}}.$$

Irrationality measures:

Theorem (Bennett): For a & b integers with $b > 0$,

$$\left| \sqrt[3]{2} - \frac{a}{b} \right| > \frac{1}{c \cdot b^{2.47}}.$$

Irrationality measures:

Theorem (Bennett): For a & b integers with $b > 0$,

$$\left|\sqrt[3]{2} - \frac{a}{b}\right| > \frac{1}{4 \cdot b^{2.47}}.$$

Irrationality measures:

Theorem (Bennett): For a & b integers with $b > 0$,

$$\left|\sqrt[3]{2} - \frac{a}{b}\right| > \frac{1}{4 \cdot b^{2.47}}.$$

Comment: Similar explicit estimates have also been made for certain other cube roots.

The Basic Approach:

The Basic Approach:

$$P_r - (1 - z)^k Q_r = z^{2r+1} E_r$$

The Basic Approach:

$$P_r - (\,1 - z\,)^{1/3} Q_r = z^{2r+1} E_r$$

The Basic Approach:

$$\mathbf{P_r - (\,1 - \underset{\substack{\uparrow \\ 3/128}}{z}\,)^{1/3} Q_r = z^{2r+1} E_r}$$

The Basic Approach:

$$P_r - (125/128)^{1/3} Q_r = z^{2r+1} E_r$$

The Basic Approach:

$$P_r - (125/128)^{1/3} Q_r = z^{2r+1} E_r$$

Rearrange and Normalize to Integers

The Basic Approach:

$$P_r - (125/128)^{1/3} Q_r = z^{2r+1} E_r$$

Rearrange and Normalize to Integers

$$\sqrt[3]{2}\, b_r - a_r = \text{small}$$

The Basic Approach:

$$P_r - (125/128)^{1/3} Q_r = z^{2r+1} E_r$$

Rearrange and Normalize to Integers

$$\sqrt[3]{2}\, b_r - a_r = \text{small}_r$$

The Basic Approach:

$$P_r - (125/128)^{1/3} Q_r = z^{2r+1} E_r$$

Rearrange and Normalize to Integers

$$\sqrt[3]{2}\, b_r - a_r = \text{small}_r$$

$$\left| \sqrt[3]{2} - \frac{a_r}{b_r} \right| = \text{small}_r$$

The Basic Approach:

$$P_r - (125/128)^{1/3} Q_r = z^{2r+1} E_r$$

Rearrange and Normalize to Integers

$$\sqrt[3]{2}\, b_r - a_r = \text{small}_r$$

$$\left| \sqrt[3]{2} - \frac{a_r}{b_r} \right| = \text{small}_r$$

The Basic Approach:

$$P_r - (125/128)^{1/3} Q_r = z^{2r+1} E_r$$

Rearrange and Normalize to Integers

$$\sqrt[3]{2}\, b_r - a_r = \text{small}_r$$

$$\left| \sqrt[3]{2} - \frac{a_r}{b_r} \right| = \text{small}_r$$

Wait!!

The Basic Approach:

$$\boldsymbol{P_r - (125/128)^{1/3} Q_r = z^{2r+1} E_r}$$

Rearrange and Normalize to Integers

$$\boldsymbol{\sqrt[3]{2}\, b_r - a_r =} \text{ small}_{\boldsymbol{r}}$$

$$\boldsymbol{\left| \sqrt[3]{2} - \frac{a_r}{b_r} \right| =} \text{ small}_r$$

Wait!! I thought we wanted that LARGE!!

The Basic Approach:

$$\left| \sqrt[3]{2} - \frac{a_r}{b_r} \right| = \text{small}_r$$

The Basic Approach:

$$\left| \sqrt[3]{2} - \frac{a_r}{b_r} \right| = \text{small}_r$$

What's small_r?

The Basic Approach:

$$\left| \sqrt[3]{2} - \frac{a_r}{b_r} \right| = \text{small}_r$$

What's small_r? Let b be a positive integer.

The Basic Approach:

$$\left|\sqrt[3]{2} - \frac{a_r}{b_r}\right| = \text{small}_r$$

What's $\textbf{small}_r$? Let b be a positive integer. Choosing r right, one can obtain

$$\text{small}_r < \frac{1}{2b\,b_r} \quad \text{and} \quad b_r < cb^{1.47}.$$

The Basic Approach:

$$\left|\sqrt[3]{2} - \frac{a_r}{b_r}\right| = \text{small}_r$$

What's small$_r$? Let b be a positive integer. Choosing r right, one can obtain

$$\text{small}_r < \frac{1}{2b\,b_r} \quad \text{and} \quad b_r < cb^{1.47}.$$

$$\left|\sqrt[3]{2} - \frac{a}{b}\right| \geq$$

The Basic Approach:

$$\left|\sqrt[3]{2} - \frac{a_r}{b_r}\right| = \text{small}_r$$

What's small_r? Let b be a positive integer. Choosing r right, one can obtain

$$\text{small}_r < \frac{1}{2b\,b_r} \quad \text{and} \quad b_r < cb^{1.47}.$$

$$\left|\sqrt[3]{2} - \frac{a}{b}\right| \geq \left|\frac{a_r}{b_r} - \frac{a}{b}\right| - \left|\sqrt[3]{2} - \frac{a_r}{b_r}\right| >$$

The Basic Approach:

$$\left| \sqrt[3]{2} - \frac{a_r}{b_r} \right| = \text{small}_r$$

What's small$_r$? Let b be a positive integer. Choosing r right, one can obtain

$$\text{small}_r < \frac{1}{2b\, b_r} \quad \text{and} \quad b_r < cb^{1.47}.$$

$$\left| \sqrt[3]{2} - \frac{a}{b} \right| \geq \left| \frac{a_r}{b_r} - \frac{a}{b} \right| - \left| \sqrt[3]{2} - \frac{a_r}{b_r} \right| >$$

The Basic Approach:

$$\left|\sqrt[3]{2} - \frac{a_r}{b_r}\right| = \text{small}_r$$

What's small_r? Let b be a positive integer. Choosing r right, one can obtain

$$\text{small}_r < \frac{1}{2b\,b_r} \quad \text{and} \quad b_r < cb^{1.47}.$$

$$\left|\sqrt[3]{2} - \frac{a}{b}\right| \geq \left|\frac{a_r}{b_r} - \frac{a}{b}\right| - \left|\sqrt[3]{2} - \frac{a_r}{b_r}\right| > \frac{1}{b\,b_r}$$

The Basic Approach:

$$\left| \sqrt[3]{2} - \frac{a_r}{b_r} \right| = \text{small}_r$$

What's $\textbf{small}_r$? Let b be a positive integer. Choosing r right, one can obtain

$$\text{small}_r < \frac{1}{2b\, b_r} \quad \text{and} \quad b_r < c b^{1.47}.$$

$$\left| \sqrt[3]{2} - \frac{a}{b} \right| \geq \left| \frac{a_r}{b_r} - \frac{a}{b} \right| - \left| \sqrt[3]{2} - \frac{a_r}{b_r} \right| > \frac{1}{b\, b_r}$$

The Basic Approach:

$$\left| \sqrt[3]{2} - \frac{a_r}{b_r} \right| = \text{small}_r$$

What's small$_r$? Let b be a positive integer. Choosing r right, one can obtain

$$\text{small}_r < \frac{1}{2b\,b_r} \quad \text{and} \quad b_r < cb^{1.47}.$$

$$\left| \sqrt[3]{2} - \frac{a}{b} \right| \geq \left| \frac{a_r}{b_r} - \frac{a}{b} \right| - \left| \sqrt[3]{2} - \frac{a_r}{b_r} \right| > \frac{1}{b\,b_r}$$

The Basic Approach:

$$\left|\sqrt[3]{2} - \frac{a_r}{b_r}\right| = \text{small}_r$$

What's small$_r$? Let b be a positive integer. Choosing r right, one can obtain

$$\text{small}_r < \frac{1}{2b\,b_r} \quad \text{and} \quad b_r < cb^{1.47}.$$

$$\left|\sqrt[3]{2} - \frac{a}{b}\right| \geq \left|\frac{a_r}{b_r} - \frac{a}{b}\right| - \left|\sqrt[3]{2} - \frac{a_r}{b_r}\right| > \frac{1}{b\,b_r}$$

The Basic Approach:

$$\left|\sqrt[3]{2} - \frac{a_r}{b_r}\right| = \text{small}_r$$

What's small$_r$? Let b be a positive integer. Choosing r right, one can obtain

$$\text{small}_r < \frac{1}{2b\,b_r} \quad \text{and} \quad b_r < cb^{1.47}.$$

$$\left|\sqrt[3]{2} - \frac{a}{b}\right| \geq \left|\frac{a_r}{b_r} - \frac{a}{b}\right| - \left|\sqrt[3]{2} - \frac{a_r}{b_r}\right| > \frac{1}{b\,b_r} -$$

The Basic Approach:

$$\left| \sqrt[3]{2} - \frac{a_r}{b_r} \right| = \text{small}_r$$

What's small_r? Let b be a positive integer. Choosing r right, one can obtain

$$\text{small}_r < \frac{1}{2b\,b_r} \quad \text{and} \quad b_r < cb^{1.47}.$$

$$\left| \sqrt[3]{2} - \frac{a}{b} \right| \geq \left| \frac{a_r}{b_r} - \frac{a}{b} \right| - \left| \sqrt[3]{2} - \frac{a_r}{b_r} \right| > \frac{1}{b\,b_r} - \frac{1}{2b\,b_r}$$

The Basic Approach:

$$\left| \sqrt[3]{2} - \frac{a_r}{b_r} \right| = \text{small}_r$$

What's small_r? Let b be a positive integer. Choosing r right, one can obtain

$$\text{small}_r < \frac{1}{2b\,b_r} \quad \text{and} \quad b_r < cb^{1.47}.$$

$$\left| \sqrt[3]{2} - \frac{a}{b} \right| \ge \left| \frac{a_r}{b_r} - \frac{a}{b} \right| - \left| \sqrt[3]{2} - \frac{a_r}{b_r} \right| > \frac{1}{2b\,b_r}$$

The Basic Approach:

$$\left|\sqrt[3]{2}-\frac{a_r}{b_r}\right| = \text{small}_r$$

What's $\textbf{small}_r$**?** Let b be a positive integer. Choosing r right, one can obtain

$$\text{small}_r < \frac{1}{2b\,b_r} \quad \text{and} \quad b_r < cb^{1.47}.$$

$$\left|\sqrt[3]{2}-\frac{a}{b}\right| \geq \left|\frac{a_r}{b_r}-\frac{a}{b}\right| - \left|\sqrt[3]{2}-\frac{a_r}{b_r}\right| > \frac{1}{2b\,b_r}$$

The Basic Approach:

$$\left| \sqrt[3]{2} - \frac{a_r}{b_r} \right| = \text{small}_r$$

What's small$_r$? Let b be a positive integer. Choosing r right, one can obtain

$$\text{small}_r < \frac{1}{2b\, b_r} \quad \text{and} \quad b_r < cb^{1.47}.$$

$$\left| \sqrt[3]{2} - \frac{a}{b} \right| \geq \left| \frac{a_r}{b_r} - \frac{a}{b} \right| - \left| \sqrt[3]{2} - \frac{a_r}{b_r} \right| > \frac{1}{2c\, b^{2.47}}$$

The Basic Approach:

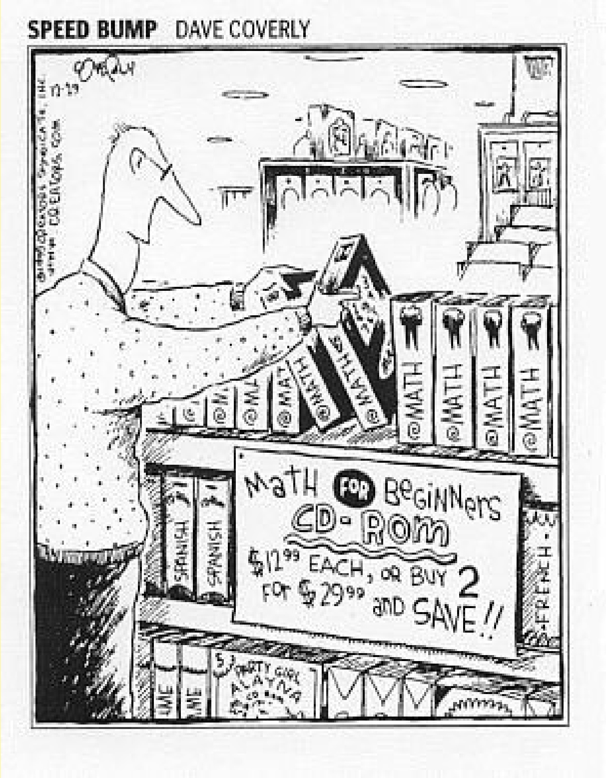
$$\left|\sqrt[3]{2} - \frac{a_r}{b_r}\right| = \text{small}_r$$

What's small$_r$? Let b be a positive integer. Choosing r right, one can obtain

$$\text{small}_r < \frac{1}{2b\,b_r} \quad \text{and} \quad b_r < cb^{1.47}.$$

$$\left|\sqrt[3]{2} - \frac{a}{b}\right| \geq \left|\frac{a_r}{b_r} - \frac{a}{b}\right| - \left|\sqrt[3]{2} - \frac{a_r}{b_r}\right| > \frac{1}{4 \cdot b^{2.47}}$$

Diophantine equations:



Diophantine equations:

Theorem (Bennett): For a and b integers with $b > 0$,

$$\left|\sqrt[3]{2} - \frac{a}{b}\right| > \frac{1}{4 \cdot b^{2.47}}.$$

Diophantine equations:

Theorem (Bennett): For a and b integers with $b \neq 0$,

$$\left|\sqrt[3]{2} - \frac{a}{b}\right| > \frac{1}{4 \cdot b^{2.47}}.$$

Diophantine equations:

Theorem (Bennett): For a and b integers with $b \neq 0$,

$$\left|\sqrt[3]{2} - \frac{a}{b}\right| > \frac{1}{4 \cdot |b|^{2.47}}.$$

Diophantine equations:

Theorem (Bennett): For a and b integers with $b \neq 0$,

$$\left|\sqrt[3]{2} - \frac{a}{b}\right| > \frac{1}{4 \cdot |b|^{2.5}}.$$

Diophantine equations:

Theorem (Bennett): For a and b integers with $b \neq 0$,

$$\left|\sqrt[3]{2} - \frac{a}{b}\right| > \frac{1}{4 \cdot |b|^{2.5}}.$$

$$x^3 - 2y^3 = n$$

Diophantine equations:

Theorem (Bennett): For a and b integers with $b \neq 0$,

$$\left|\sqrt[3]{2} - \frac{a}{b}\right| > \frac{1}{4 \cdot |b|^{2.5}}.$$

$$x^3 - 2y^3 = n, \quad y \neq 0$$

Diophantine equations:

Theorem (Bennett): For a and b integers with $b \neq 0$,

$$\left|\sqrt[3]{2} - \frac{a}{b}\right| > \frac{1}{4 \cdot |b|^{2.5}}.$$

$$x^3 - 2y^3 = n, \quad y \neq 0$$

$$\left|\sqrt[3]{2} - \frac{x}{y}\right| \left|\sqrt[3]{2}e^{2\pi \mathrm{i}/3} - \frac{x}{y}\right| \left|\sqrt[3]{2}e^{4\pi \mathrm{i}/3} - \frac{x}{y}\right| = \frac{|n|}{|y|^3}$$

Diophantine equations:

Theorem (Bennett): For a and b integers with $b \neq 0$,

$$\left|\sqrt[3]{2} - \frac{a}{b}\right| > \frac{1}{4 \cdot |b|^{2.5}}.$$

$$x^3 - 2y^3 = n, \quad y \neq 0$$

$$\left|\sqrt[3]{2} - \frac{x}{y}\right| \left|\sqrt[3]{2}e^{2\pi \mathrm{i}/3} - \frac{x}{y}\right| \left|\sqrt[3]{2}e^{4\pi \mathrm{i}/3} - \frac{x}{y}\right| = \frac{|n|}{|y|^3}$$

Diophantine equations:

Theorem (Bennett): For a and b integers with $b \neq 0$,

$$\left|\sqrt[3]{2} - \frac{a}{b}\right| > \frac{1}{4 \cdot |b|^{2.5}}.$$

$$x^3 - 2y^3 = n, \quad y \neq 0$$

$$\left|\sqrt[3]{2} - \frac{x}{y}\right| \left|\sqrt[3]{2} e^{2\pi \mathrm{i}/3} - \frac{x}{y}\right| \left|\sqrt[3]{2} e^{4\pi \mathrm{i}/3} - \frac{x}{y}\right| = \frac{|n|}{|y|^3}$$

$$\left|\sqrt[3]{2} - \frac{x}{y}\right| < \frac{|n|}{|y|^3}$$

Diophantine equations:

Theorem (Bennett): For a and b integers with $b \neq 0$,

$$\left|\sqrt[3]{2} - \frac{a}{b}\right| > \frac{1}{4 \cdot |b|^{2.5}}.$$

$$x^3 - 2y^3 = n, \quad y \neq 0$$

$$\left|\sqrt[3]{2} - \frac{x}{y}\right| \left|\sqrt[3]{2}e^{2\pi i/3} - \frac{x}{y}\right| \left|\sqrt[3]{2}e^{4\pi i/3} - \frac{x}{y}\right| = \frac{|n|}{|y|^3}$$

$$\frac{1}{4\,|y|^{2.5}} < \left|\sqrt[3]{2} - \frac{x}{y}\right| < \frac{|n|}{|y|^3}$$

Diophantine equations:

$$x^3 - 2y^3 = n, \quad y \neq 0$$

$$\left|\sqrt[3]{2} - \frac{x}{y}\right| \left|\sqrt[3]{2}e^{2\pi \mathrm{i}/3} - \frac{x}{y}\right| \left|\sqrt[3]{2}e^{4\pi \mathrm{i}/3} - \frac{x}{y}\right| = \frac{|n|}{|y|^3}$$

$$\frac{1}{4\,|y|^{2.5}} < \left|\sqrt[3]{2} - \frac{x}{y}\right| < \frac{|n|}{|y|^3}$$

Diophantine equations:

$$x^3 - 2y^3 = n, \quad y \neq 0$$

$$\left|\sqrt[3]{2} - \frac{x}{y}\right| \left|\sqrt[3]{2}e^{2\pi \mathrm{i}/3} - \frac{x}{y}\right| \left|\sqrt[3]{2}e^{4\pi \mathrm{i}/3} - \frac{x}{y}\right| = \frac{|n|}{|y|^3}$$

$$\frac{1}{4\,|y|^{2.5}} < \left|\sqrt[3]{2} - \frac{x}{y}\right| < \frac{|n|}{|y|^3}$$

$$|y|^{1/2} < 4|n|$$

Diophantine equations:

$$x^3 - 2y^3 = n, \quad y \neq 0$$

$$\left|\sqrt[3]{2} - \frac{x}{y}\right| \left|\sqrt[3]{2} e^{2\pi i/3} - \frac{x}{y}\right| \left|\sqrt[3]{2} e^{4\pi i/3} - \frac{x}{y}\right| = \frac{|n|}{|y|^3}$$

$$\frac{1}{4\,|y|^{2.5}} < \left|\sqrt[3]{2} - \frac{x}{y}\right| < \frac{|n|}{|y|^3}$$

$$|y|^{1/2} < 4|n| \implies |y| < 16n^2$$

Diophantine equations:

$$x^3 - 2y^3 = n, \quad y \neq 0$$

$$\left|\sqrt[3]{2} - \frac{x}{y}\right| \left|\sqrt[3]{2}e^{2\pi i/3} - \frac{x}{y}\right| \left|\sqrt[3]{2}e^{4\pi i/3} - \frac{x}{y}\right| = \frac{|n|}{|y|^3}$$

$$\frac{1}{4\,|y|^{2.5}} < \left|\sqrt[3]{2} - \frac{x}{y}\right| < \frac{|n|}{|y|^3}$$

$$|y|^{1/2} < 4|n| \implies |y| < 16n^2$$

Diophantine equations:

Theorem: Let n be a non-zero integer. If x and y are integers satisfying $x^3 - 2y^3 = n$, then $|y| < 16n^2$.

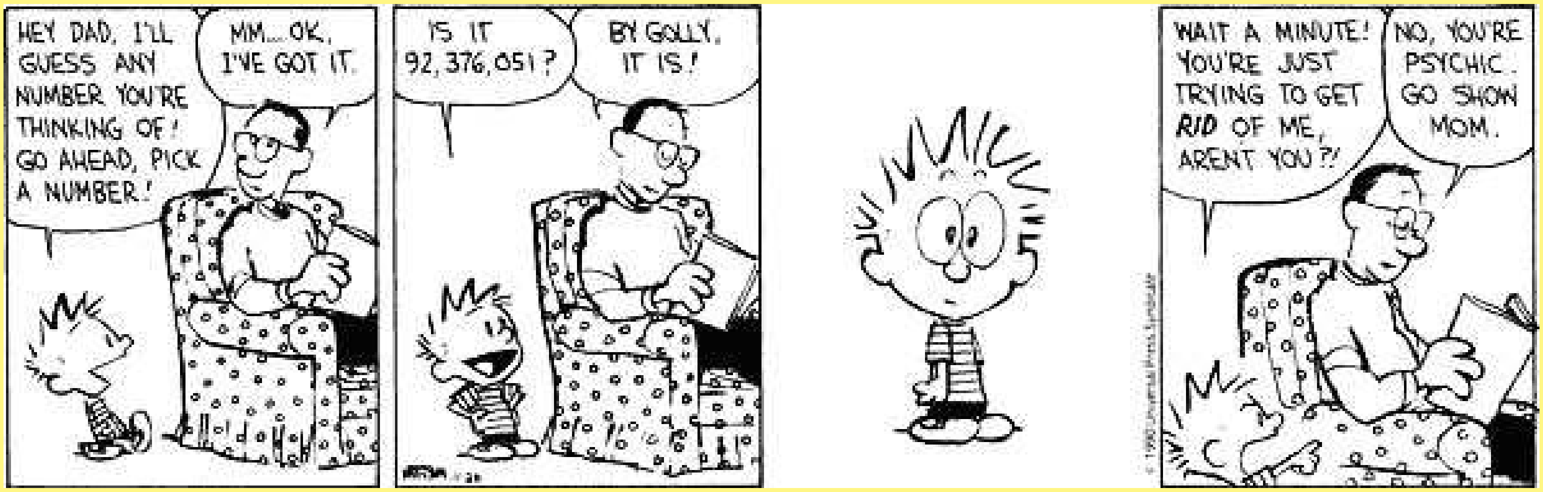
Diophantine equations:

Theorem (Bennett): If a, b, and n are integers with $ab \neq 0$ and $n \geq 3$, then the equation

$$|ax^n + by^n| = 1$$

has at most 1 solution in positive integers x and y.

Waring’s Problem:



Waring's Problem:

Waring's Problem: Let k be an integer ≥ 2. Then there exists a number s such that every natural number is a sum of s k^{th} powers.

Waring's Problem:

Waring's Problem: Let k be an integer ≥ 2. Then there exists a number s such that every natural number is a sum of s k^{th} powers. If $g(k)$ is the least such s, what is $g(k)$?

Waring's Problem:

Waring's Problem: Let k be an integer ≥ 2. Then there exists a number s such that every natural number is a sum of s k^{th} powers. If $g(k)$ is the least such s, what is $g(k)$?

Known: (i) $g(k) = 2^k + \left[\left(\frac{3}{2}\right)^k\right] - 2$

Waring's Problem:

Waring's Problem: Let $\boldsymbol{k}$ be an integer $\geq \mathbf{2}$. Then there exists a number $\boldsymbol{s}$ such that every natural number is a sum of $\boldsymbol{s}$ $\boldsymbol{k}^{\text{th}}$ powers. If $\boldsymbol{g(k)}$ is the least such $\boldsymbol{s}$, what is $\boldsymbol{g(k)}$?

Known: (i) $\boldsymbol{g(k) = 2^k + \left[\left(\frac{3}{2}\right)^k\right] - 2}$

(ii) no one knows how to prove (i)

Waring's Problem:

Waring's Problem: Let k be an integer ≥ 2. Then there exists a number s such that every natural number is a sum of s k^{th} powers. If $g(k)$ is the least such s, what is $g(k)$?

Known: (i) $g(k) = 2^k + \left[\left(\frac{3}{2}\right)^k\right] - 2$

(ii) no one knows how to prove (i)

(iii) (i) holds if $\left\|\left(\frac{3}{2}\right)^k\right\| > 0.75^k$

Waring's Problem:

Waring's Problem: Let k be an integer ≥ 2. Then there exists a number s such that every natural number is a sum of s k^{th} powers. If $g(k)$ is the least such s, what is $g(k)$?

Known: (i) $g(k) = 2^k + \left[\left(\frac{3}{2}\right)^k \right] - 2$

(ii) no one knows how to prove (i)

(iii) (i) holds if $\left\| \left(\frac{3}{2}\right)^k \right\| > 0.75^k$

(iv) (iii) holds if $k > 8$

Waring's Problem:

Waring's Problem: Let k be an integer ≥ 2. Then there exists a number s such that every natural number is a sum of s k^{th} powers. If $g(k)$ is the least such s, what is $g(k)$?

Known: (i) $g(k) = 2^k + \left[\left(\frac{3}{2}\right)^k \right] - 2$

(ii) no one knows how to prove (i)

(iii) (i) holds if $\left\| \left(\frac{3}{2}\right)^k \right\| > 0.75^k$

(iv) (iii) holds if $k > 8$

(v) no one knows how to prove (iv)

Waring's Problem:

Known: (i) $g(k) = 2^k + \left[\left(\frac{3}{2}\right)^k\right] - 2$

(ii) No one knows how to prove (i).

(iii) (i) holds if $\left\|\left(\frac{3}{2}\right)^k\right\| > 0.75^k$

Waring's Problem:

Known: (i) $g(k) = 2^k + \left[\left(\frac{3}{2}\right)^k\right] - 2$

(ii) No one knows how to prove (i).

(iii) (i) holds if $\left\|\left(\frac{3}{2}\right)^k\right\| > 0.75^k$

Theorem (Beukers): If $k > 4$, then

$$\left\|\left(\frac{3}{2}\right)^k\right\| > 0.5358^k.$$

Waring's Problem:

Known: (i) $g(k) = 2^k + \left[\left(\frac{3}{2}\right)^k \right] - 2$

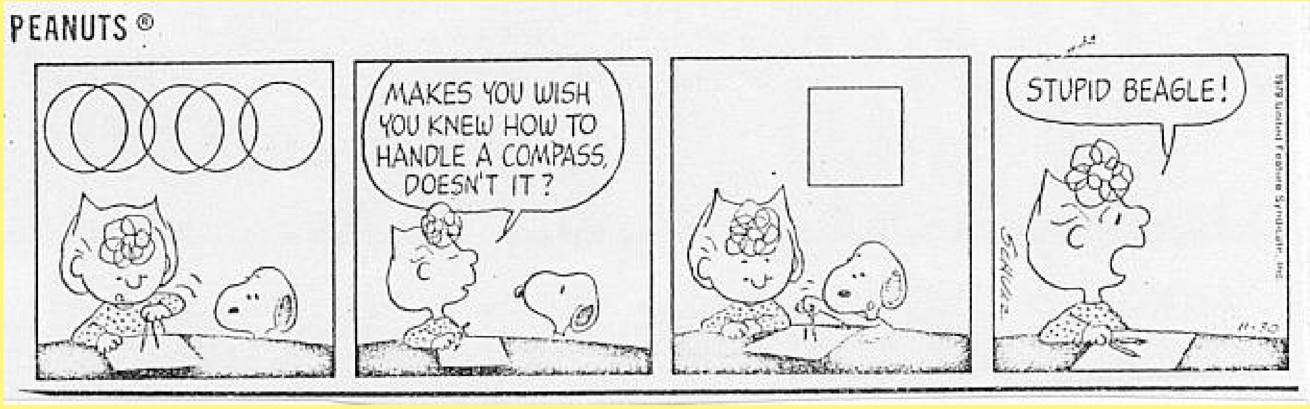
(ii) No one knows how to prove (i).

(iii) (i) holds if $\left\| \left(\frac{3}{2}\right)^k \right\| > 0.75^k$

Theorem (Dubitskas): If $k > 4$, then

$$\left\| \left(\frac{3}{2}\right)^k \right\| > 0.5767^k.$$

The factorization of $n(n + 1)$:



The factorization of $n(n+1)$:

Well-Known: The largest prime factor of $n(n+1)$ tends to infinity with n.

The factorization of $n(n+1)$:

Well-Known: The largest prime factor of $n(n+1)$ tends to infinity with n.

Let $p_1, p_2, \ldots, p_r$ be primes. There is an N such that if $n \geq N$ and

$$n(n+1) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} m$$

for some integer m, then $m > 1$.

Let $p_1, p_2, \ldots, p_r$ be primes. There is an N such that if $n \geq N$ and

$$n(n+1) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} m$$

for some integer m, then $m > 1$.

Let $p_1, p_2, \ldots, p_r$ be primes. There is an N such that if $n \geq N$ and

$$n(n+1) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} m$$

for some integer m, then $m > 1$.

Lehmer: Gave some explicit estimates:

Let $p_1, p_2, \ldots, p_r$ be primes. There is an N such that if $n \geq N$ and

$$n(n+1) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} m$$

for some integer m, then $m > 1$.

Lehmer: Gave some explicit estimates:

$n(n+1)$ divisible only by primes $\leq 11 \Longrightarrow n \leq$

Let $p_1, p_2, \ldots, p_r$ be primes. There is an N such that if $n \geq N$ and

$$n(n+1) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} m$$

for some integer m, then $m > 1$.

Lehmer: Gave some explicit estimates:

$n(n+1)$ divisible only by primes $\leq 11 \implies n \leq$

... only by primes $\leq 41 \implies n \leq$

Let $p_1, p_2, \ldots, p_r$ be primes. There is an N such that if $n \geq N$ and

$$n(n+1) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} m$$

for some integer m, then $m > 1$.

Lehmer: Gave some explicit estimates:

$n(n+1)$ divisible only by primes $\leq 11 \implies n \leq 9800$

... only by primes $\leq 41 \implies n \leq$

Let $p_1, p_2, \ldots, p_r$ be primes. There is an N such that if $n \geq N$ and

$$n(n+1) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} m$$

for some integer m, then $m > 1$.

Lehmer: Gave some explicit estimates:

$n(n+1)$ divisible only by primes $\leq 11 \implies n \leq 9800$

... only by primes $\leq 41 \implies n \leq 63927525375$

Let $p_1, p_2, \ldots, p_r$ be primes. There is an N such that if $n \geq N$ and

$$n(n+1) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} m$$

for some integer m, then $m > 1$.

Let $p_1, p_2, \ldots, p_r$ be primes. There is an N such that if $n \geq N$ and

$$n(n+1) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} m$$

for some integer m, then $m > 1$.

Let $p_1, p_2, \ldots, p_r$ be primes. There is an N such that if $n \geq N$ and

$$n(n+1) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} m$$

for some integer m, then $m > \mathbf{1}$.

Let $p_1, p_2, \ldots, p_r$ be primes. There is an N such that if $n \geq N$ and

$$n(n+1) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} m$$

for some integer m, then $m > n^{\theta}$.

Want: Let $p_1, p_2, \ldots, p_r$ be primes. There is an $N = N(\theta, p_1, \ldots, p_r)$ such that if $n \geq N$ and

$$n(n+1) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} m$$

for some integer m, then $m > n^\theta$.

Want: Let $p_1, p_2, \ldots, p_r$ be primes. There is an $N = N(\theta, p_1, \ldots, p_r)$ such that if $n \geq N$ and

$$n(n+1) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} m$$

for some integer m, then $m > n^{\theta}$.

abc-conjecture $\implies$ $\theta =$

Want: Let $p_1, p_2, \ldots, p_r$ be primes. There is an $N = N(\theta, p_1, \ldots, p_r)$ such that if $n \geq N$ and

$$n(n+1) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} m$$

for some integer m, then $m > n^{\theta}$.

$$abc\text{-conjecture} \implies \theta = 1 - \varepsilon$$

Want: Let $p_1, p_2, \ldots, p_r$ be primes. There is an $N = N(\theta, p_1, \ldots, p_r)$ such that if $n \geq N$ and

$$n(n+1) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} m$$

for some integer m, then $m > n^{\theta}$.

abc-conjecture $\implies$ $\theta = 1 - \varepsilon$

unconditionally one can obtain $\theta =$

Want: Let $p_1, p_2, \ldots, p_r$ be primes. There is an $N = N(\theta, p_1, \ldots, p_r)$ such that if $n \geq N$ and

$$n(n+1) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} m$$

for some integer m, then $m > n^{\theta}$.

abc-conjecture $\implies$ $\theta = 1 - \varepsilon$

unconditionally one can obtain $\theta = 1 - \varepsilon$

Want: Let $p_1, p_2, \ldots, p_r$ be primes. There is an $N = N(\theta, p_1, \ldots, p_r)$ such that if $n \geq N$ and

$$n(n+1) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} m$$

for some integer m, then $m > n^{\theta}$.

abc-conjecture $\implies$ $\theta = 1 - \varepsilon$

unconditionally one can obtain $\theta = 1 - \varepsilon$

(ineffective)

Want: Let $p_1, p_2, \ldots, p_r$ be primes. There is an $N = N(\theta, p_1, \ldots, p_r)$ such that if $n \geq N$ and

$$n(n+1) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} m$$

for some integer m, then $m > n^{\theta}$.

Effective Approach:

Want: Let $p_1, p_2, \ldots, p_r$ be primes. There is an $N = N(\theta, p_1, \ldots, p_r)$ such that if $n \geq N$ and

$$n(n+1) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} m$$

for some integer m, then $m > n^{\theta}$.

Effective Approach: (Linear Forms of Logarithms)

Want: Let $p_1, p_2, \ldots, p_r$ be primes. There is an $N = N(\theta, p_1, \ldots, p_r)$ such that if $n \geq N$ and

$$n(n+1) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} m$$

for some integer m, then $m > n^{\theta}$.

Effective Approach: (Linear Forms of Logarithms)

$$\theta = \frac{c}{\log \log n}$$

Want: Let $p_1, p_2, \ldots, p_r$ be primes. There is an $N = N(\theta, p_1, \ldots, p_r)$ such that if $n \geq N$ and

$$n(n+1) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} m$$

for some integer m, then $m > n^{\theta}$.

Effective Approach: (Linear Forms of Logarithms)

$$\theta = \frac{c}{\log \log n}$$

Problem: Can we narrow the gap between these ineffective and effective results?

Want: Let $p_1, p_2, \ldots, p_r$ be primes. There is an $N = N(\theta, p_1, \ldots, p_r)$ such that if $n \geq N$ and

$$n(n+1) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} m$$

for some integer m, then $m > n^{\theta}$.

Want: Let $p_1, p_2, \ldots, p_r$ be primes. There is an $N = N(\theta, p_1, \ldots, p_r)$ such that if $n \geq N$ and

$$n(n+1) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} m$$

for some integer m, then $m > n^{\theta}$.

Theorem (Bennett, F., Trifonov): If $n \geq 9$ and

$$n(n+1) = 2^k 3^\ell m,$$

then

$$m \geq$$

Want: Let $p_1, p_2, \ldots, p_r$ be primes. There is an $N = N(\theta, p_1, \ldots, p_r)$ such that if $n \geq N$ and

$$n(n+1) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} m$$

for some integer m, then $m > n^\theta$.

Theorem (Bennett, F., Trifonov): If $n \geq 9$ and

$$n(n+1) = 2^k 3^\ell m,$$

then

$$m \geq n^{1/4}.$$

Theorem (Bennett, F., Trifonov): The set of 4-tuples (k, ℓ, M_1, M_2) of positive integers with

$$0 < \left|3^k M_1 - 2^\ell M_2\right| \leq 100, \quad \gcd(6, M_1 M_2) = 1,$$

$$M_1 M_2 \leq \min\left\{3^k M_1, 2^\ell M_2\right\}^{0.25}$$

consists of 28 tuples of which 26 satisfy $k \leq 5$, $\ell \leq 8$ and $M_1 M_2 = 1$ and the remaining two are $(6, 7, 1, 5)$ and $(8, 15, 5, 1)$.

Conjecture: For $n > 512$,

$$n(n+1) = 2^u 3^v m \implies m > \sqrt{n}.$$

Conjecture: For $n > 512$,

$$n(n+1) = 2^u 3^v m \implies m > \sqrt{n}.$$

Comment: The conjecture has been verified for

$$512 < n \leq$$

Conjecture: For $n > 512$,

$$n(n+1) = 2^u 3^v m \implies m > \sqrt{n}.$$

Comment: The conjecture has been verified for

$$512 < n \leq 10^{1000}.$$

The Method:

The Method:

$$n(n+1) = 3^k 2^\ell m$$

The Method:

$$n(n+1) = 3^k 2^\ell m$$

$$3^k m_1 - 2^\ell m_2 = \pm 1$$

The Method:

$$n(n+1) = 3^k 2^\ell m$$

$$3^k m_1 - 2^\ell m_2 = \pm 1$$

Main Idea: Find “small” integers P, Q, and E such that

$$3^k P - 2^\ell Q = E.$$

The Method:

$$n(n+1) = 3^k 2^\ell m$$

$$3^k m_1 - 2^\ell m_2 = \pm 1$$

Main Idea: Find “small” integers P, Q, and E such that

$$3^k P - 2^\ell Q = E.$$

Then

$$3^k \left(Q m_1 - P m_2\right) = \pm Q - E m_2.$$

The Method:

$$n(n+1) = 3^k 2^\ell m$$

$$3^k m_1 - 2^\ell m_2 = \pm 1$$

Main Idea: Find "small" integers P, Q, and E such that

$$3^k P - 2^\ell Q = E.$$

Then

$$3^k \left(Q m_1 - P m_2\right) = \pm Q - E m_2.$$

Main Idea: Find “small” integers P, Q, and E such that

$$3^k P - 2^\ell Q = E$$

and

$$Q m_1 - P m_2 \neq 0.$$

Then

$$3^k \left(Q m_1 - P m_2 \right) = \pm Q - E m_2.$$

Main Idea: Find "small" integers P, Q, and E such that

$$3^k P - 2^\ell Q = E$$

and

$$Qm_1 - Pm_2 \neq 0.$$

Then

$$3^k \left(Qm_1 - Pm_2 \right) = \pm Q - Em_2.$$

Obtain an upper bound on 3^k.

Main Idea: Find "small" integers P, Q, and E such that

$$3^k P - 2^\ell Q = E$$

and

$$Qm_1 - Pm_2 \neq 0.$$

Then

$$3^k \left(Qm_1 - Pm_2\right) = \pm Q - Em_2.$$

Obtain an upper bound on 3^k. Since $3^k m_1 \geq n$, it follows that m_1 and $m = m_1 m_2$ are not small.

The “small” integers P, Q, and E are obtained through the use of Padé approximations for $(1 - z)^k$.

The “small” integers P, Q, and E are obtained through the use of Padé approximations for $(1-z)^k$.

More precisely, one takes $z = 1/9$ in the equation

$$P_r(z) - (1-z)^k Q_r(z) = z^{2r+1} E_r(z).$$

What’s Needed for the Method to Work:

What's Needed for the Method to Work:

One largely needs to be dealing with two primes (like 2 and 3) with a difference of powers of these primes being small (like $\mathbf{3^2 - 2^3 = 1}$).

Galois groups of classical polynomials:

Galois groups of classical polynomials:

- D. Hilbert (1892) used his now classical Hilbert's Irreducibility Theorem to show that for each integer $n \geq 1$, there is polynomial $f(x) \in \mathbb{Z}[x]$ such that the Galois group associated with $f(x)$ is the symmetric group S_n.

Galois groups of classical polynomials:

- D. Hilbert (1892) used his now classical Hilbert's Irreducibility Theorem to show that for each integer $n \geq 1$, there is polynomial $f(x) \in \mathbb{Z}[x]$ such that the Galois group associated with $f(x)$ is the symmetric group S_n. He also showed the analogous result in the case of the alternating group A_n.

Galois groups of classical polynomials:

- D. Hilbert (1892) used his now classical Hilbert's Irreducibility Theorem to show that for each integer $n \geq 1$, there is polynomial $f(x) \in \mathbb{Z}[x]$ such that the Galois group associated with $f(x)$ is the symmetric group S_n. He also showed the analogous result in the case of the alternating group A_n.
- Hilbert's work and work of E. Noether (1918) began what is now called Inverse Galois Theory.

Galois groups of classical polynomials:

- D. Hilbert (1892) used his now classical Hilbert's Irreducibility Theorem to show that for each integer $n \geq 1$, there is polynomial $f(x) \in \mathbb{Z}[x]$ such that the Galois group associated with $f(x)$ is the symmetric group S_n. He also showed the analogous result in the case of the alternating group A_n.
- Hilbert's work and work of E. Noether (1918) began what is now called Inverse Galois Theory.
- Van der Waerden showed that for "almost all" polynomials $f(x) \in \mathbb{Z}[x]$, the Galois group associated with $f(x)$ is the symmetric group S_n.

Galois groups of classical polynomials:

- Schur showed $L_n^{(0)}(x)$ has Galois group S_n.

Galois groups of classical polynomials:

- Schur showed $L_n^{(0)}(x)$ has Galois group S_n.
- Schur showed $L_n^{(1)}(x)$ has Galois group A_n (the alternating group) if n is odd.

Galois groups of classical polynomials:

- Schur showed $L_n^{(0)}(x)$ has Galois group S_n.
- Schur showed $L_n^{(1)}(x)$ has Galois group A_n (the alternating group) if n is odd.
- Schur showed $\displaystyle\sum_{j=0}^{n} \frac{x^j}{j!}$ has Galois group A_n if $4|n$.

Galois groups of classical polynomials:

- Schur showed $L_n^{(0)}(x)$ has Galois group S_n.
- Schur showed $L_n^{(1)}(x)$ has Galois group A_n (the alternating group) if n is odd.
- Schur showed $\displaystyle\sum_{j=0}^{n} \frac{x^j}{j!}$ has Galois group A_n if $4|n$.
- Schur did not find an explicit sequence of polynomials having Galois group A_n with $n \equiv 2 \pmod{4}$.

Galois groups of classical polynomials:

Theorem (R. Gow, 1989): If $n > 2$ is even and

$$L_n^{(n)}(x) = \sum_{j=0}^{n} \binom{2n}{n-j} \frac{(-x)^j}{j!}$$

is irreducible, then the Galois group of $L_n^{(n)}(x)$ is A_n.

Galois groups of classical polynomials:

Theorem (R. Gow, 1989): If $n > 2$ is even and

$$L_n^{(n)}(x) = \sum_{j=0}^{n} \binom{2n}{n-j} \frac{(-x)^j}{j!}$$

is irreducible, then the Galois group of $L_n^{(n)}(x)$ is A_n.

Theorem (joint work with R. Williams): For almost all positive integers n the polynomial $L_n^{(n)}(x)$ is irreducible (and, hence, has Galois group A_n for almost all even n).

Galois groups of classical polynomials:

Theorem (joint work with R. Williams): For almost all positive integers n the polynomial $L_n^{(n)}(x)$ is irreducible.

Galois groups of classical polynomials:

Theorem (joint work with R. Williams): For almost all positive integers n the polynomial $L_n^{(n)}(x)$ is irreducible.

Comment: The method had an ineffective component to it. We could show that if n is sufficiently large and $L_n^{(n)}(x)$ is reducible, then $L_n^{(n)}(x)$ has a linear factor. But we didn't know what sufficiently large was.

Galois groups of classical polynomials:

Theorem (joint work with R. Williams): For almost all positive integers n the polynomial $L_n^{(n)}(x)$ is irreducible.

Comment: The method had an ineffective component to it. We could show that if n is sufficiently large and $L_n^{(n)}(x)$ is reducible, then $L_n^{(n)}(x)$ has a linear factor. But we didn't know what sufficiently large was.

Theorem (Kidd, Trifonov, F.): If $n > 2$ and $n \equiv 2 \pmod 4$, then $L_n^{(n)}(x)$ is irreducible.

Galois groups of classical polynomials:

Theorem (joint work with R. Williams): For almost all positive integers n the polynomial $L_n^{(n)}(x)$ is irreducible.

Comment: The method had an ineffective component to it. We could show that if n is sufficiently large and $L_n^{(n)}(x)$ is reducible, then $L_n^{(n)}(x)$ has a linear factor. But we didn't know what sufficiently large was.

Theorem (Kidd, Trifonov, F.): If $n > 2$ and $n \equiv 2 \pmod 4$, then $L_n^{(n)}(x)$ has Galois group A_n.

The Ramanujan-Nagell equation:

The Ramanujan-Nagell equation:

Classical Ramanujan-Nagell Theorem: If x and n are positive integers satisfying

$$x^2 + 7 = 2^n,$$

then

The Ramanujan-Nagell equation:

Classical Ramanujan-Nagell Theorem: If x and n are positive integers satisfying

$$x^2 + 7 = 2^n,$$

then

$$x \in \{1, 3, 5, 11, 181\}.$$

The Ramanujan-Nagell equation:

Some Background: Beukers used a method "similar" to the approach for finding irrationality measures to show that $\sqrt{2}$ cannot be approximated too well by rationals a/b with b a power of 2. This implies bounds for solutions to the Diophantine equation $x^2 + D = 2^n$ with D fixed. He showed that if $D \neq 7$, then the equation has ≤ 4 solutions. Related work by Apéry, Beukers, and Bennett establishes that for odd primes p not dividing D, the equation $x^2 + D = p^n$ has at most 3 solutions. All of these are in some sense best possible (though more can and has been said).

The Ramanujan-Nagell equation:

Classical Ramanujan-Nagell Theorem: If x and n are positive integers satisfying

$$x^2 + 7 = 2^n,$$

then

$$x \in \{1, 3, 5, 11, 181\}.$$

The Ramanujan-Nagell equation:

Classical Ramanujan-Nagell Theorem: If x and n are positive integers satisfying

$$x^2 + 7 = 2^n,$$

then

$$x \in \{1, 3, 5, 11, 181\}.$$

Problem: If $x^2 + 7 = 2^n m$ and x is not in the set above, then can we say that m must be large?

Problem: If $x^2 + 7 = 2^n m$ and x is not in the set above, then can we say that m must be large?

Connection with $n(n+1)$ problem:

Problem: If $x^2 + 7 = 2^n m$ and x is not in the set above, then can we say that m must be large?

Connection with $n(n+1)$ problem:

$$x^2 + 7 = 2^n m$$

Problem: If $x^2 + 7 = 2^n m$ and x is not in the set above, then can we say that m must be large?

Connection with $n(n+1)$ problem:

$$x^2 + 7 = 2^n m$$

$$\left(\frac{x+\sqrt{-7}}{2}\right)\left(\frac{x-\sqrt{-7}}{2}\right) = \left(\frac{1+\sqrt{-7}}{2}\right)^{n-2}\left(\frac{1-\sqrt{-7}}{2}\right)^{n-2} m$$

Problem: If $x^2 + 7 = 2^n m$ and x is not in the set above, then can we say that m must be large?

Connection with $n(n+1)$ problem:

$$x^2 + 7 = 2^n m$$

$$\underset{\substack{\uparrow \\ \text{linear}}}{\left(\frac{x+\sqrt{-7}}{2}\right)}\underset{\substack{\uparrow \\ \text{linear}}}{\left(\frac{x-\sqrt{-7}}{2}\right)} = \left(\frac{1+\sqrt{-7}}{2}\right)^{n-2}\left(\frac{1-\sqrt{-7}}{2}\right)^{n-2} m$$

Problem: If $x^2 + 7 = 2^n m$ and x is not in the set above, then can we say that m must be large?

Connection with $n(n+1)$ problem:

$$x^2 + 7 = 2^n m$$

$$\underset{\substack{\uparrow \\ \text{linear}}}{\left(\frac{x+\sqrt{-7}}{2}\right)} \underset{\substack{\uparrow \\ \text{linear}}}{\left(\frac{x-\sqrt{-7}}{2}\right)} = \underset{\substack{\uparrow \\ \text{prime}}}{\left(\frac{1+\sqrt{-7}}{2}\right)}^{n-2} \underset{\substack{\uparrow \\ \text{prime}}}{\left(\frac{1-\sqrt{-7}}{2}\right)}^{n-2} m$$

Theorem (Bennett, F., Trifonov): If x, n and m are positive integers satisfying

$$x^2 + 7 = 2^n m \quad \text{and} \quad x \notin \{1, 3, 5, 11, 181\},$$

then

$$m \geq \text{???}$$

Theorem (Bennett, F., Trifonov): If x, n and m are positive integers satisfying

$$x^2 + 7 = 2^n m \quad \text{and} \quad x \notin \{1, 3, 5, 11, 181\},$$

then

$$m \geq x^{1/2}.$$

Theorem (Bennett, F., Trifonov): If x, n and m are positive integers satisfying

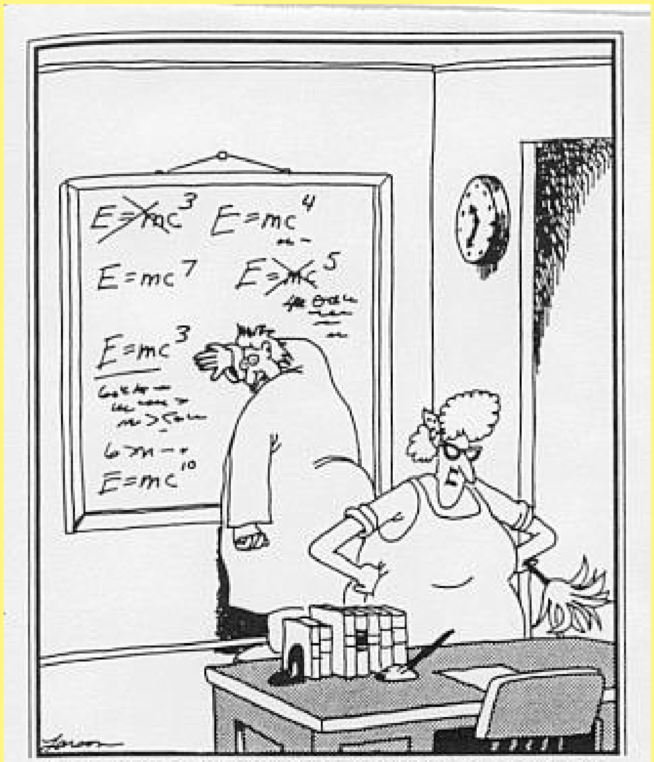
$$x^2 + 7 = 2^n m \quad \text{and} \quad x \notin \{1, 3, 5, 11, 181\},$$

then

$$m \geq x^{1/2}.$$

Comment: In the case of $x^2 + 7 = 2^n m$, the difference of the primes $(1 + \sqrt{-7})/2$ and $(1 - \sqrt{-7})/2$ each raised to the 13^{th} power has absolute value ≈ 2.65 and the powers themselves have absolute value ≈ 90.51.

k-free numbers in short intervals:



"*Now* that desk looks better. Everything's squared away, yessir, squaaaaaared away."

k-free numbers in short intervals:

"Now that desk looks better. Everything's squared away, yessir, squaaaaared away."

k-free numbers in short intervals:

Problem: Find $\theta = \theta(k)$ as small as possible such that, for x sufficiently large, the interval $(x, x + x^{\theta}]$ contains a k-free number.

k-free numbers in short intervals:

Problem: Find $\theta = \theta(k)$ as small as possible such that, for x sufficiently large, the interval $(x, x + x^{\theta}]$ contains a k-free number.

Main Idea: Show there are integers in $(x, x + x^{\theta}]$ not divisible by the k^{th} power of a prime. Consider primes in different size ranges. Deal with small primes and large primes separately.

Problem: Find $\theta = \theta(k)$ as small as possible such that, for x sufficiently large, the interval $(x, x + x^{\theta}]$ contains a k-free number.

Problem: Find $\theta = \theta(k)$ as small as possible such that, for x sufficiently large, the interval $(x, x + x^{\theta}]$ contains a k-free number.

Small Primes: $p \le z$

Problem: Find $\theta = \theta(k)$ as small as possible such that, for x sufficiently large, the interval $(x, x + x^{\theta}]$ contains a k-free number.

Small Primes: $p \le z$ where $z = x^{\theta}\sqrt{\log x}$

Problem: Find $\theta = \theta(k)$ as small as possible such that, for x sufficiently large, the interval $(x, x + x^{\theta}]$ contains a k-free number.

Small Primes: $p \le z$ where $z = x^{\theta}\sqrt{\log x}$

The number of integers $n \in (x, x + x^{\theta}]$ divisible by such a p^k is bounded by $(2/3)x^{\theta}$.

Large Primes: $p \in (N, 2N]$, $N \geq z = x^{\theta}\sqrt{\log x}$

Large Primes: $p \in (N, 2N],\ N \geq z = x^{\theta}\sqrt{\log x}$

$$x < p^k m \leq x + x^{\theta}$$

Large Primes: $p \in (N, 2N],\ N \geq z = x^{\theta}\sqrt{\log x}$

$$x < p^k m \leq x + x^{\theta} \implies \frac{x}{p^k} < m \leq \frac{x}{p^k} + \frac{x^{\theta}}{p^k}$$

Large Primes: $p \in (N, 2N],\ N \geq z = x^{\theta}\sqrt{\log x}$

$$x < p^k m \leq x + x^{\theta} \implies \frac{x}{p^k} < m \leq \frac{x}{p^k} + \frac{x^{\theta}}{p^k}$$

$$\implies \left\| \frac{x}{p^k} \right\| < \frac{x^{\theta}}{N^k}$$

Large Primes: $p \in (N, 2N]$, $N \geq z = x^\theta \sqrt{\log x}$

$$x < p^k m \leq x + x^\theta \implies \frac{x}{p^k} < m \leq \frac{x}{p^k} + \frac{x^\theta}{p^k}$$

$$\implies \left\| \frac{x}{p^k} \right\| < \frac{x^\theta}{N^k}$$

where $\|t\| = \min\{|t - \ell| : \ell \in \mathbb{Z}\}$

Large Primes: $p \in (N, 2N],\ N \geq z = x^{\theta}\sqrt{\log x}$

$$x < p^k m \leq x + x^{\theta} \implies \frac{x}{p^k} < m \leq \frac{x}{p^k} + \frac{x^{\theta}}{p^k}$$

$$\implies \left\|\frac{x}{p^k}\right\| < \frac{x^{\theta}}{N^k}$$

where $\|t\| = \min\{|t - \ell| : \ell \in \mathbb{Z}\}$

Idea: Show there are few primes $p \in (N, 2N]$ with x/p^k that close to an integer.

Large Primes: $p \in (N, 2N],\ N \geq z = x^{\theta}\sqrt{\log x}$

$$x < p^k m \leq x + x^{\theta} \implies \frac{x}{p^k} < m \leq \frac{x}{p^k} + \frac{x^{\theta}}{p^k}$$

$$\implies \left\|\frac{x}{p^k}\right\| < \frac{x^{\theta}}{N^k}$$

where $\|t\| = \min\{|t - \ell| : \ell \in \mathbb{Z}\}$

Idea: Show there are few integers $p \in (N, 2N]$ with x/p^k that close to an integer.

Large Primes: $p \in (N, 2N]$, $N \geq z = x^{\theta}\sqrt{\log x}$

$$x < p^k m \leq x + x^{\theta} \implies \frac{x}{p^k} < m \leq \frac{x}{p^k} + \frac{x^{\theta}}{p^k}$$

$$\implies \left\|\frac{x}{p^k}\right\| < \frac{x^{\theta}}{N^k}$$

where $\|t\| = \min\{|t - \ell| : \ell \in \mathbb{Z}\}$

Idea: Show there are few integers $u \in (N, 2N]$ with x/u^k that close to an integer.

$$\left\|\frac{x}{u^k}\right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta\sqrt{\log x}$$

$$\left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x}$$

Differences:

$$\left\|\frac{x}{u^k}\right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x}$$

Differences:

$$\left\|\frac{x}{u^k}\right\| < \frac{x^\theta}{N^k}, \quad \left\|\frac{x}{(u+a)^k}\right\| < \frac{x^\theta}{N^k}$$

$$\left\|\frac{x}{u^k}\right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x}$$

Differences:

$$\left\|\frac{x}{u^k}\right\| < \frac{x^\theta}{N^k}, \quad \left\|\frac{x}{(u+a)^k}\right\| < \frac{x^\theta}{N^k}$$

$$\frac{x}{u^k} - \frac{x}{(u+a)^k}$$

$$\left\|\frac{x}{u^k}\right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x}$$

Differences:

$$\left\|\frac{x}{u^k}\right\| < \frac{x^\theta}{N^k}, \quad \left\|\frac{x}{(u+a)^k}\right\| < \frac{x^\theta}{N^k}$$

$$\frac{x}{u^k} - \frac{x}{(u+a)^k} \asymp \frac{ax}{u^{k+1}}$$

$$\left\|\frac{x}{u^k}\right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x}$$

Differences:

$$\left\|\frac{x}{u^k}\right\| < \frac{x^\theta}{N^k}, \quad \left\|\frac{x}{(u+a)^k}\right\| < \frac{x^\theta}{N^k}$$

$$\frac{x}{u^k} - \frac{x}{(u+a)^k} \asymp \frac{ax}{u^{k+1}} \asymp \frac{ax}{N^{k+1}}$$

$$\left\|\frac{x}{u^k}\right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x}$$

Differences:

$$\left\|\frac{x}{u^k}\right\| < \frac{x^\theta}{N^k}, \quad \left\|\frac{x}{(u+a)^k}\right\| < \frac{x^\theta}{N^k}$$

$$\frac{x}{u^k} - \frac{x}{(u+a)^k} \asymp \frac{ax}{u^{k+1}} \asymp \frac{ax}{N^{k+1}}$$

$$\left\|\frac{x}{u^k}\right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x}$$

Differences:

$$\left\|\frac{x}{u^k}\right\| < \frac{x^\theta}{N^k}, \quad \left\|\frac{x}{(u+a)^k}\right\| < \frac{x^\theta}{N^k}$$

$$\frac{x}{u^k} - \frac{x}{(u+a)^k} \asymp \frac{ax}{u^{k+1}} \asymp \frac{ax}{N^{k+1}}$$

consider $N = x^{1/k}$

$$\left\|\frac{x}{u^k}\right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x}$$

Differences:

$$\left\|\frac{x}{u^k}\right\| < \frac{x^\theta}{N^k}, \quad \left\|\frac{x}{(u+a)^k}\right\| < \frac{x^\theta}{N^k}$$

$$\frac{x}{u^k} - \frac{x}{(u+a)^k} \asymp \frac{ax}{u^{k+1}} \asymp \frac{a}{x^{1/k}}$$

consider $N = x^{1/k}$

$$\left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x}$$

Differences:

$$\left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad \left\| \frac{x}{(u+a)^k} \right\| < \frac{x^\theta}{N^k}$$

$$\frac{x}{u^k} - \frac{x}{(u+a)^k} \asymp \frac{ax}{u^{k+1}} \asymp \frac{a}{x^{1/k}}$$

consider $N = x^{1/k}$, $a < x^{1/(2k)}$

$$\left\|\frac{x}{u^k}\right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x}$$

Differences:

$$\left\|\frac{x}{u^k}\right\| < \frac{x^\theta}{N^k}, \quad \left\|\frac{x}{(u+a)^k}\right\| < \frac{x^\theta}{N^k}$$

$$\frac{x}{u^k} - \frac{x}{(u+a)^k} \asymp \frac{ax}{u^{k+1}} \asymp \frac{a}{x^{1/k}}$$

consider $N = x^{1/k}$, $a < x^{1/(2k)}$, $\theta \approx 1/k$

$$\left\|\frac{x}{u^k}\right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta\sqrt{\log x}$$

Differences:

$$\left\|\frac{x}{u^k}\right\| < \frac{x^\theta}{N^k}, \quad \left\|\frac{x}{(u+a)^k}\right\| < \frac{x^\theta}{N^k}$$

$$\frac{x}{u^k} - \frac{x}{(u+a)^k} \asymp \frac{ax}{u^{k+1}} \asymp \frac{a}{x^{1/k}}$$

consider $N = x^{1/k}$, $a < x^{1/(2k)}$, $\theta \approx 1/k$

LHS small compared to RHS

$$\left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x}$$

“Modified” Differences:

$$\left\|\frac{x}{u^k}\right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x}$$

“Modified” Differences:

$$\left\|\frac{x}{u^k}\right\| < \frac{x^\theta}{N^k}, \quad \left\|\frac{x}{(u+a)^k}\right\| < \frac{x^\theta}{N^k}$$

$$\left\|\frac{x}{u^k}\right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x}$$

“Modified” Differences:

$$\left\|\frac{x}{u^k}\right\| < \frac{x^\theta}{N^k}, \quad \left\|\frac{x}{(u+a)^k}\right\| < \frac{x^\theta}{N^k}$$

$$\frac{x}{u^k}P - \frac{x}{(u+a)^k}Q \quad \text{small}$$

$$\left\|\frac{x}{u^k}\right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x}$$

“Modified” Differences:

$$\left\|\frac{x}{u^k}\right\| < \frac{x^\theta}{N^k}, \quad \left\|\frac{x}{(u+a)^k}\right\| < \frac{x^\theta}{N^k}$$

$$\frac{x}{u^k}P - \frac{x}{(u+a)^k}Q$$ small (but not too small)

$$\left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x}$$

“Modified” Differences:

$$\left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad \left\| \frac{x}{(u+a)^k} \right\| < \frac{x^\theta}{N^k}$$

$\dfrac{x}{u^k} P - \dfrac{x}{(u+a)^k} Q$ small (but not too small)

$(u+a)^k P - u^k Q$ small (but not too small)

$$\left\|\frac{x}{u^k}\right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x}$$

“Modified” Differences:

$$\left\|\frac{x}{u^k}\right\| < \frac{x^\theta}{N^k}, \quad \left\|\frac{x}{(u+a)^k}\right\| < \frac{x^\theta}{N^k}$$

$\frac{x}{u^k}P - \frac{x}{(u+a)^k}Q$ small (but not too small)

$(u+a)^k P - u^k Q$ small (but not too small)

consider $P_r(z) - (1-z)^k Q_r(z)$ with $z = \frac{a}{u+a}$

$$\left\|\frac{x}{u^k}\right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x}$$

“Modified” Differences:

Theorem (Halberstam & Roth):

$$\left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x}$$

“Modified” Differences:

Theorem (Halberstam & Roth & Nair):

$$\left\|\frac{x}{u^k}\right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \ge x^\theta \sqrt{\log x}$$

“Modified” Differences:

Theorem (Halberstam & Roth & Nair): For x large, there is a k-free number in $(x, x + x^{1/(2k)}]$.

$$\left\|\frac{x}{u^k}\right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x}$$

Modified Differences plus Divided Differences:

$$\left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \ge x^\theta \sqrt{\log x}$$

Modified Differences plus Divided Differences:

Theorem (F. & Trifonov): For x sufficiently large, there is a squarefree number in $(x, x + cx^{1/5} \log x]$.

$$\left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x}$$

Modified Differences plus Divided Differences:

Theorem (F. & Trifonov): For x sufficiently large, there is a squarefree number in $(x, x + cx^{1/5} \log x]$.

Theorem (Trifonov): For x sufficiently large, there is a k-free number in $(x, x + cx^{1/(2k+1)} \log x]$.

k-free values of polynomials and binary forms:

k-free values of polynomials and binary forms:

The method for obtaining results about gaps between k-free numbers generalizes to k-free values of polynomials.

k-free values of polynomials and binary forms:

The method for obtaining results about gaps between k-free numbers generalizes to k-free values of polynomials. Suppose $f(x) \in \mathbb{Z}[x]$ is irreducible and $\deg f = n$.

k-free values of polynomials and binary forms:

The method for obtaining results about gaps between k-free numbers generalizes to k-free values of polynomials. Suppose $f(x) \in \mathbb{Z}[x]$ is irreducible and $\deg f = n$. In what follows, we suppose further that f has no fixed k^{th} power divisors.

k-free values of polynomials and binary forms:

The method for obtaining results about gaps between k-free numbers generalizes to k-free values of polynomials. Suppose $f(x) \in \mathbb{Z}[x]$ is irreducible and $\deg f = n$. In what follows, we suppose further that f has no fixed k^{th} power divisors.

Theorem (Nair): Let $k \geq n + 1$. For x sufficiently large, there is an integer m such that $f(m)$ is k-free with

$$x < m \leq x + cx^{\frac{n}{2k-n+1}}.$$

Theorem (Nair): Let $k \geq n+1$. For x sufficiently large, there is an integer m such that $f(m)$ is k-free with

$$x < m \leq x + cx^{\frac{n}{2k-n+1}}.$$

Theorem (Nair): Let $k \geq n+1$. For x sufficiently large, there is an integer m such that $f(m)$ is k-free with

$$x < m \leq x + cx^{\frac{n}{2k-n+1}}.$$

Theorem: Let $k \geq n+1$. For x sufficiently large, there is an integer m such that $f(m)$ is k-free with

$$x < m \leq x + cx^{\frac{n}{2k-n+r}},$$

where $r =$

Theorem (Nair): Let $k \geq n+1$. For x sufficiently large, there is an integer m such that $f(m)$ is k-free with

$$x < m \leq x + cx^{\frac{n}{2k-n+1}}.$$

Theorem: Let $k \geq n+1$. For x sufficiently large, there is an integer m such that $f(m)$ is k-free with

$$x < m \leq x + cx^{\frac{n}{2k-n+r}},$$

where $r = \sqrt{2n} - \frac{1}{2}$.

Basic Idea: One works in a number field where $f(x)$ has a linear factor. As in the case $f(x) = x$, one wants to show certain u (in the ring of algebraic integers in the field) are not close by considering

$$(u + a)^k P - u^k Q$$

arising from Padé approximations. One uses that this expression is an integer and, hence, either 0 or ≥ 1.

Basic Idea: One works in a number field where $f(x)$ has a linear factor. As in the case $f(x) = x$, one wants to show certain u (in the ring of algebraic integers in the field) are not close by considering

$$(u + a)^k P - u^k Q$$

arising from Padé approximations. One uses that this expression is an integer and, hence, either 0 or ≥ 1.

Difficulty: An "integer" in this context can be small without being 0.

Basic Idea: One works in a number field where $f(x)$ has a linear factor. As in the case $f(x) = x$, one wants to show certain u (in the ring of algebraic integers in the field) are not close by considering

$$(u + a)^k P - u^k Q$$

arising from Padé approximations. One uses that this expression is an integer and, hence, either 0 or ≥ 1.

Difficulty: An "integer" in this context can be small without being 0.

Solution: If it's small, work with a conjugate instead.

Comment: In the case that $k \leq n$, one can *try* the same methods. The gap size becomes “bad” in the sense that one obtains $m \in (x, x+h]$ where $f(m)$ is k-free but h increases as k decreases. There is a point where h exceeds x itself and the method fails (the size of $f(m)$ is no longer of order x^n). Nair took the limit of what can be done with $k \leq n$ and obtained

Comment: In the case that $k \le n$, one can *try* the same methods. The gap size becomes "bad" in the sense that one obtains $m \in (x, x+h]$ where $f(m)$ is k-free but h increases as k decreases. There is a point where h exceeds x itself and the method fails (the size of $f(m)$ is no longer of order x^n). Nair took the limit of what can be done with $k \le n$ and obtained

Theorem (Nair): If $f(x)$ is irreducible of degree n and $k \ge \left(2\sqrt{2}-1\right)n/2$, then there are infinitely many integers m for which $f(m)$ is k-free.

Theorem (Nair): If $f(x)$ is irreducible of degree n and $k \geq (2\sqrt{2} - 1)n/2$, then there are infinitely many integers m for which $f(m)$ is k-free.

Theorem (Nair): If $f(x)$ is irreducible of degree n and $k \geq \left(2\sqrt{2}-1\right)n/2$, then there are infinitely many integers m for which $f(m)$ is k-free.

Theorem: If $f(x, y)$ is an irreducible binary form of degree n and $k \geq \left(2\sqrt{2}-1\right)n/4$, then there are infinitely many integer pairs (a, b) for which $f(a, b)$ is k-free.

The abc-conjecture:

The abc-conjecture:

Notation: $Q(n) = \prod_{p|n} p$

The abc-conjecture:

Notation: $Q(n) = \prod_{p \mid n} p$

The abc-Conjecture: For a and b in $\mathbb{Z}^{+}$, define

$$L_{a,b} = \frac{\log(a+b)}{\log Q\big(ab(a+b)\big)}$$

and

$$\mathcal{L} = \{L_{a,b} : a \geq 1, b \geq 1, \gcd(a,b) = 1\}.$$

The abc-conjecture:

Notation: $Q(n) = \prod_{p|n} p$

The abc-Conjecture: For a and b in $\mathbb{Z}^+$, define

$$L_{a,b} = \frac{\log(a+b)}{\log Q\big(ab(a+b)\big)}$$

and

$$\mathcal{L} = \{L_{a,b} : a \geq 1, b \geq 1, \gcd(a,b) = 1\}.$$

The set of limit points of $\mathcal{L}$ is the interval $[1/3, 1]$.

$$L_{a,b} = \frac{\log(a+b)}{\log Q\big(ab(a+b)\big)}$$

$$\mathcal{L} = \{L_{a,b} : a \geq 1, b \geq 1, \gcd(a,b) = 1\}$$

$$L_{a,b} = \frac{\log(a+b)}{\log Q\big(ab(a+b)\big)}$$

$$\mathcal{L} = \{L_{a,b} : a \geq 1, b \geq 1, \gcd(a,b) = 1\}$$

Theorem: The set of limit points of $\mathcal{L}$ includes the interval $[1/3, 36/37]$.

$$L_{a,b} = \frac{\log(a+b)}{\log Q\big(ab(a+b)\big)}$$

$$\mathcal{L} = \{L_{a,b} : a \geq 1, b \geq 1, \gcd(a,b) = 1\}$$

Theorem: The set of limit points of $\mathcal{L}$ includes the interval $[1/3, 36/37]$.

(work of Browkin, Greaves, F., Nitaj, Schinzel)

Approach: Makes use of a preliminary result about squarefree values of binary forms.

Approach: Makes use of a preliminary result about squarefree values of binary forms. In particular, for

$$\begin{aligned} f(x,y) &= xy(x+y)(x-y)(x^2+y^2)(2x^2+y^2)(x^2+2y^2) \\ &\quad \times (x^4-x^2y^2+y^4)(3x^4+3x^2y^2+y^4)(x^4+3x^2y^2+3y^4) \end{aligned}$$

the number $f(x,y)/6$ takes on the right proportion of squarefree values for

$$X < x \le 2X, \quad Y < y \le 2Y, \quad X = Y^{\alpha},$$

where $\alpha \in (1,3)$.

Polynomial Identity:

Polynomial Identity:

$$P_3(z) - (1-z)^7 Q_3(z) = z^7 E_3(z)$$

where

$$P_3(z) = (2z-1)(3z^2 - 3z + 1),$$
$$Q_3(z) = -(z+1)(z^2 + z + 1),$$

and

$$E_3(z) = -(z-2)(z^2 - 3z + 3)$$

Polynomial Identity:

$$\boldsymbol{P_3(z) - (1-z)^7 Q_3(z) = z^7 E_3(z)}$$

Polynomial Identity:

$$P_3(z) - (1-z)^7 Q_3(z) = z^7 E_3(z)$$

$$z = \frac{x}{x+y} \Longrightarrow$$

Polynomial Identity:

$$P_3(z) - (1-z)^7 Q_3(z) = z^7 E_3(z)$$

$$z = \frac{x}{x+y} \Longrightarrow \left\{ \begin{array}{l} (x+y)^7(x-y)(x^2-xy+y^2) \\ \quad + y^7(2x+y)(3x^2+3xy+y^2) \\ = x^7(x+2y)(x^2+3xy+3y^2) \end{array} \right.$$

$$
\begin{aligned}
&(x+y)^7(x-y)(x^2-xy+y^2) \\
&\qquad + y^7(2x+y)(3x^2+3xy+y^2) \\
&\quad = x^7(x+2y)(x^2+3xy+3y^2)
\end{aligned}
$$

$$(x+y)^7(x-y)(x^2-xy+y^2)$$
$$+y^7(2x+y)(3x^2+3xy+y^2)$$
$$=x^7(x+2y)(x^2+3xy+3y^2)$$

$$(x^2+y^2)^7(x^2-y^2)(x^4-x^2y^2+y^4)$$
$$+y^{14}(2x^2+y^2)(3x^4+3x^2y^2+y^4)$$
$$=x^{14}(x^2+2y^2)(x^4+3x^2y^2+3y^4)$$

$$
\begin{aligned}
&(x+y)^7(x-y)(x^2-xy+y^2) \\
&\qquad + y^7(2x+y)(3x^2+3xy+y^2) \\
&= x^7(x+2y)(x^2+3xy+3y^2)
\end{aligned}
$$

$$
\begin{aligned}
&(x^2+y^2)^7(x^2-y^2)(x^4-x^2y^2+y^4) \\
&\qquad + y^{14}(2x^2+y^2)(3x^4+3x^2y^2+y^4) \\
&= x^{14}(x^2+2y^2)(x^4+3x^2y^2+3y^4)
\end{aligned}
$$

$$
\begin{aligned}
f(x,y) = {}& xy(x+y)(x-y)(x^2+y^2)(2x^2+y^2)(x^2+2y^2) \\
&\times (x^4-x^2y^2+y^4)(3x^4+3x^2y^2+y^4)(x^4+3x^2y^2+3y^4)
\end{aligned}
$$

$$
\begin{aligned}
&(x+y)^7(x-y)(x^2-xy+y^2) \\
&\qquad + y^7(2x+y)(3x^2+3xy+y^2) \\
&= x^7(x+2y)(x^2+3xy+3y^2)
\end{aligned}
$$

$$
\begin{aligned}
&(x^2+y^2)^7(x^2-y^2)(x^4-x^2y^2+y^4) \\
&\qquad + y^{14}(2x^2+y^2)(3x^4+3x^2y^2+y^4) \\
&= x^{14}(x^2+2y^2)(x^4+3x^2y^2+3y^4)
\end{aligned}
$$

$$
\begin{aligned}
f(x,y) &= xy(x+y)(x-y)(x^2+y^2)(2x^2+y^2)(x^2+2y^2) \\
&\quad \times (x^4-x^2y^2+y^4)(3x^4+3x^2y^2+y^4)(x^4+3x^2y^2+3y^4)
\end{aligned}
$$

$$
\begin{aligned}
&(x+y)^7(x-y)(x^2-xy+y^2)\\
&\qquad + y^7(2x+y)(3x^2+3xy+y^2)\\
&= x^7(x+2y)(x^2+3xy+3y^2)
\end{aligned}
$$

$$
\begin{aligned}
&(x^2+y^2)^7(x^2-y^2)(x^4-x^2y^2+y^4)\\
&\qquad + y^{14}(2x^2+y^2)(3x^4+3x^2y^2+y^4)\\
&= x^{14}(x^2+2y^2)(x^4+3x^2y^2+3y^4)
\end{aligned}
$$

$$
\begin{aligned}
f(x,y) &= xy(x+y)(x-y)(x^2+y^2)(2x^2+y^2)(x^2+2y^2)\\
&\quad \times (x^4-x^2y^2+y^4)(3x^4+3x^2y^2+y^4)(x^4+3x^2y^2+3y^4)
\end{aligned}
$$

$$a = (x^2 + y^2)^7 (x^2 - y^2)(x^4 - x^2 y^2 + y^4)$$
$$b = y^{14}(2x^2 + y^2)(3x^4 + 3x^2 y^2 + y^4)$$
$$X = Y^{\alpha}, \quad 1 < \alpha < 3$$
$$a + b = x^{14}(x^2 + 2y^2)(x^4 + 3x^2 y^2 + 3y^4)$$

$$a = (x^2 + y^2)^7 (x^2 - y^2)(x^4 - x^2 y^2 + y^4)$$
$$b = y^{14}(2x^2 + y^2)(3x^4 + 3x^2 y^2 + y^4)$$
$$X = Y^{\alpha}, \quad 1 < \alpha < 3$$
$$a + b = x^{14}(x^2 + 2y^2)(x^4 + 3x^2 y^2 + 3y^4)$$

$$\begin{aligned} f(x,y) = xy(x+y)(x-y)(x^2+y^2)(2x^2+y^2)(x^2+2y^2) \\ \times (x^4 - x^2y^2 + y^4)(3x^4 + 3x^2y^2 + y^4)(x^4 + 3x^2y^2 + 3y^4) \end{aligned}$$

$$a = (x^2+y^2)^7(x^2-y^2)(x^4-x^2y^2+y^4)$$
$$b = y^{14}(2x^2+y^2)(3x^4+3x^2y^2+y^4)$$
$$X = Y^{\alpha}, \quad 1 < \alpha < 3$$
$$a+b = x^{14}(x^2+2y^2)(x^4+3x^2y^2+3y^4)$$

$$\begin{aligned} f(x,y) = xy(x+y)(x-y)(x^2+y^2)(2x^2+y^2)(x^2+2y^2) \\ \times (x^4-x^2y^2+y^4)(3x^4+3x^2y^2+y^4)(x^4+3x^2y^2+3y^4) \end{aligned}$$

$$L_{a,b} = \frac{\log(a+b)}{\log Q\big(ab(a+b)\big)}$$

$$a = (x^2+y^2)^7(x^2-y^2)(x^4-x^2y^2+y^4)$$

$$b = y^{14}(2x^2+y^2)(3x^4+3x^2y^2+y^4)$$

$$X = Y^{\alpha}, \quad 1 < \alpha < 3$$

$$a+b = x^{14}(x^2+2y^2)(x^4+3x^2y^2+3y^4)$$

$$\begin{aligned} f(x,y) = {} & xy(x+y)(x-y)(x^2+y^2)(2x^2+y^2)(x^2+2y^2) \\ & \times (x^4-x^2y^2+y^4)(3x^4+3x^2y^2+y^4)(x^4+3x^2y^2+3y^4) \end{aligned}$$

$$L_{a,b} = \frac{\log(a+b)}{\log Q\big(ab(a+b)\big)}$$

$$a = (x^2 + y^2)^7 (x^2 - y^2)(x^4 - x^2y^2 + y^4)$$
$$b = y^{14}(2x^2 + y^2)(3x^4 + 3x^2y^2 + y^4)$$
$$X = Y^{\alpha}, \quad 1 < \alpha < 3$$
$$a + b = x^{14}(x^2 + 2y^2)(x^4 + 3x^2y^2 + 3y^4)$$

$$\begin{aligned} f(x,y) = xy(x+y)(x-y)(x^2+y^2)(2x^2+y^2)(x^2+2y^2) \\ \times (x^4 - x^2y^2 + y^4)(3x^4+3x^2y^2+y^4)(x^4+3x^2y^2+3y^4) \end{aligned}$$

$$L_{a,b} = \frac{\log(a+b)}{\log Q\big(ab(a+b)\big)} \approx \frac{20\alpha \log Y}{\qquad}$$

$$a = (x^2 + y^2)^7 (x^2 - y^2)(x^4 - x^2 y^2 + y^4)$$
$$b = y^{14}(2x^2 + y^2)(3x^4 + 3x^2 y^2 + y^4)$$
$$X = Y^{\alpha}, \quad 1 < \alpha < 3$$
$$a + b = x^{14}(x^2 + 2y^2)(x^4 + 3x^2 y^2 + 3y^4)$$

$$\begin{aligned} f(x,y) = {} & xy(x+y)(x-y)(x^2+y^2)(2x^2+y^2)(x^2+2y^2) \\ & \times (x^4 - x^2y^2 + y^4)(3x^4 + 3x^2y^2 + y^4)(x^4 + 3x^2y^2 + 3y^4) \end{aligned}$$

$$L_{a,b} = \frac{\log(a+b)}{\log Q\big(ab(a+b)\big)} \approx \frac{20\alpha \log Y}{}$$

$$a = (x^2 + y^2)^7 (x^2 - y^2)(x^4 - x^2 y^2 + y^4)$$
$$b = y^{14} (2x^2 + y^2)(3x^4 + 3x^2 y^2 + y^4)$$
$$X = Y^\alpha, \quad 1 < \alpha < 3$$
$$a + b = x^{14} (x^2 + 2y^2)(x^4 + 3x^2 y^2 + 3y^4)$$

$$\begin{aligned} f(x,y) = xy(x+y)(x-y)(x^2+y^2)(2x^2+y^2)(x^2+2y^2) \\ \times (x^4 - x^2y^2 + y^4)(3x^4 + 3x^2y^2 + y^4)(x^4 + 3x^2y^2 + 3y^4) \end{aligned}$$

$$L_{a,b} = \frac{\log(a+b)}{\log Q\big(ab(a+b)\big)} \approx \frac{20\alpha \log Y}{(21\alpha + 1)\log Y}$$

$$a = (x^2 + y^2)^7 (x^2 - y^2)(x^4 - x^2 y^2 + y^4)$$

$$b = y^{14} (2x^2 + y^2)(3x^4 + 3x^2 y^2 + y^4)$$

$$X = Y^{\alpha}, \quad 1 < \alpha < 3$$

$$a + b = x^{14} (x^2 + 2y^2)(x^4 + 3x^2 y^2 + 3y^4)$$

$$\begin{aligned} f(x, y) = xy(x + y)(x - y)(x^2 + y^2)(2x^2 + y^2)(x^2 + 2y^2) \\ \times (x^4 - x^2 y^2 + y^4)(3x^4 + 3x^2 y^2 + y^4)(x^4 + 3x^2 y^2 + 3y^4) \end{aligned}$$

$$L_{a,b} = \frac{\log(a + b)}{\log Q\big(ab(a + b)\big)} \approx \frac{20\alpha}{21\alpha + 1}$$

$$L_{a,b} = \frac{\log(a+b)}{\log Q\big(ab(a+b)\big)} \approx \frac{20\alpha}{21\alpha+1}$$

$$L_{a,b} = \frac{\log(a+b)}{\log Q\big(ab(a+b)\big)} \approx \frac{20\alpha}{21\alpha+1}$$

$$1 < \alpha < 3 \implies$$

$$L_{a,b} = \frac{\log(a+b)}{\log Q\big(ab(a+b)\big)} \approx \frac{20\alpha}{21\alpha+1}$$

$$1 < \alpha < 3 \implies ?? < L_{a,b} < ??$$

$$L_{a,b} = \frac{\log(a+b)}{\log Q\bigl(ab(a+b)\bigr)} \approx \frac{20\alpha}{21\alpha+1}$$

$$1 < \alpha < 3 \implies \frac{10}{11} < L_{a,b} < \frac{15}{16}$$

$$L_{a,b} = \frac{\log(a+b)}{\log Q\big(ab(a+b)\big)} \approx \frac{20\alpha}{21\alpha+1}$$

$$1 < \alpha < 3 \implies \frac{10}{11} < L_{a,b} < \frac{15}{16}$$

Comment: This shows $[10/11, 15/16]$ is contained in the set of limit points of $L_{a,b}$. A similar argument is given for other subintervals of $[1/3, 36/37]$ (not all involving Padé approximations).

The End