Factoring Lacunary Polynomials

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Factoring Sparse Polynomials

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FACTORING SPARSE POLYNOMIALS

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Definitions and Notations:

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- *irreducibility* will be over the integers
- if $f(x) = \sum_{j=0}^{n} a_j x^j$, then $\|f\|^2 = \sum_{j=0}^{n} a_j^2$
- $\tilde{f}(x) = x^{\deg f} f(1/x)$
- $\tilde{f}(x)$ will be called the *reciprocal of* $f(x)$
- $f(x)$ *reciprocal* means $\tilde{f}(x) = \pm f(x)$
- the *non-reciprocal part of* $f(x)$ is $f(x)$ removed of its irreducible reciprocal factors (sort of)

BASIC QUESTION 1

Are lacunary polynomials easier to factor than non-lacunary polynomials?

Recall:

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- There exists $w(x)$ with $f(x)\tilde{f}(x) = w(x)\widetilde{w}(x)$, $w(x) \neq f(x)$, and $w(x) \neq \tilde{f}(x)$ if and only if the non-reciprocal part of $f(x)$ is reducible.

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- $\|w\| = \|f\|$

- If $f(x)$ is a $0, 1$-polynomial, then $w(x)$ is also.

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- $\|w\| = \|f\|$
- If $f(x)$ is a $0, 1$-polynomial, then $w(x)$ is also.

Example: Factor (if possible)

$$f(x) = 1 + x^{211} + x^{517} + x^{575} + x^{1245} + x^{1398}.$$

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$$\begin{aligned} f(x)\tilde{f}(x) = 1 &+ x^{153} + x^{211} + x^{364} + x^{517} \\ &+ x^{575} + x^{670} + x^{728} + x^{823} + x^{881} \\ &+ x^{1034} + x^{1092} + x^{1187} + x^{1245} \\ &+ x^{1340} + 6x^{1398} + \cdots \end{aligned}$$

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$$\begin{aligned} f\tilde{f} \rightarrow [&0, 153, 211, 364, 517, 575, 670, 728, 823, 881, \\ &1034, 1092, 1187, 1245, 1340, 1398, \ldots] \end{aligned}$$

$$\boldsymbol{f(x) = 1 + x^{211} + x^{517} + x^{575} + x^{1245} + x^{1398}}$$

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$$w = 1 + \cdots + x^{1398}$$

$$\widetilde{w} = 1 + \cdots + x^{1398}$$

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$$w = 1 + \cdots + x^{1398}$$

$$\widetilde{w} = 1 + \cdots + x^{1398}$$

Question: How can we get the exponent $\mathbf{153}$ in $w\widetilde{w}$?

$$f(x) = 1 + x^{211} + x^{517} + x^{575} + x^{1245} + x^{1398}$$

$$f\tilde{f} \longrightarrow [0, 153, 211, 364, 517, 575, 670, 728, 823, 881, 1034, 1092, 1187, 1245, 1340, 1398, \ldots]$$

$$w = 1 + x^{153} + \cdots + x^{1398}$$

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$$f\tilde{f} \to [0, 153, 211, 364, 517, 575, 670, 728, 823, 881,$$
$$1034, 1092, 1187, 1245, 1340, 1398, \ldots]$$

$$w = 1 + x^{153} + \cdots + x^{1398}$$

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$$f\tilde{f} \longrightarrow [0, 153, 211, 364, 517, 575, 670, 728, 823, 881, 1034, 1092, 1187, 1245, 1340, 1398, \dots]$$

$$w \longrightarrow [0, 153, \dots, 1398]$$

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$$\boldsymbol{w \longrightarrow [0, 153, \ldots, 1398]}$$

$$\boldsymbol{\widetilde{w} \longrightarrow [0, \ldots, 1245, 1398]}$$

$$\boldsymbol{w\widetilde{w}} \text{ (so far)} \longrightarrow \boldsymbol{[0, 153, 1245, 1398, 1551, 2643, 2796]}$$

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$w\widetilde{w}$ (so far) $\longrightarrow [0, 153, 1245, 1398, 1551, 2643, 2796]$

Question: How can we get $\mathbf{211}$ in $w\widetilde{w}$?

$$f\,\tilde{f} \longrightarrow [\mathbf{0, 153, 211, 364, 517, 575, 670, 728, 823, 881, 1034, 1092, 1187, 1245, 1340, 1398, \ldots}]$$

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$$f\tilde{f} \longrightarrow [\mathbf{0}, \mathbf{153}, \mathbf{211}, \mathbf{364}, \mathbf{517}, \mathbf{575}, \mathbf{670}, \mathbf{728}, \mathbf{823}, \mathbf{881}, \mathbf{1034}, \mathbf{1092}, \mathbf{1187}, \mathbf{1245}, \mathbf{1340}, \mathbf{1398}, \ldots]$$

$$w \longrightarrow [\mathbf{0}, \mathbf{153}, \mathbf{211}, \ldots, \mathbf{1398}]$$

$$\widetilde{w} \longrightarrow [\mathbf{0}, \ldots, \mathbf{1187}, \mathbf{1245}, \mathbf{1398}]$$

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$$f\tilde{f} \longrightarrow [\mathbf{0, 153, 211, 364, 517, 575, 670, 728, 823, 881, 1034, 1092, 1187, 1245, 1340, 1398, \ldots}]$$

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$$w \longrightarrow [\mathbf{0}, \mathbf{153}, \mathbf{211}, \ldots, \mathbf{1398}]$$

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$$w\widetilde{w} \longrightarrow [\mathbf{0}, \mathbf{153}, \mathbf{211}, \mathbf{1187}, \mathbf{1245}, \mathbf{1340}, \mathbf{1398}, \ldots]$$

Note: Everything appears fine.

Question: How can we get $\mathbf{364}$ in $w\widetilde{w}$?

$$\boldsymbol{f\tilde{f} \longrightarrow [0, 153, 211, 364, 517, 575, 670, 728, 823, 881,}$$
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$$\boldsymbol{w \longrightarrow [0, 153, 211, \ldots, 1398]}$$

$$\boldsymbol{\widetilde{w} \longrightarrow [0, \ldots, 1187, 1245, 1398]}$$

$$f\tilde{f} \longrightarrow [0, 153, 211, 364, 517, 575, 670, 728, 823, 881, \\ 1034, 1092, 1187, 1245, 1340, 1398, \ldots]$$

$$w \longrightarrow [0, 153, 211, 364, \ldots, 1398]$$

$$\widetilde{w} \longrightarrow [0, \ldots, 1034, 1187, 1245, 1398]$$

$$f\tilde{f} \longrightarrow [\mathbf{0, 153, 211, 364, 517, 575, 670, 728, 823, 881,}$$
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$$w \longrightarrow [\mathbf{0, 153, 211, 364, \ldots, 1398}]$$

$$\widetilde{w} \longrightarrow [\mathbf{0, \ldots, 1034, 1187, 1245, 1398}]$$

$$w\widetilde{w} \text{ (so far)} \longrightarrow [\mathbf{0, 153, 211, 364, 1034, 1187, 1245,}$$
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$$\boldsymbol{w} \longrightarrow [\mathbf{0, 153, 211, 364, \ldots, 1398}]$$

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Note: Everything appears fine

$$f\tilde{f} \longrightarrow [\mathbf{0, 153, 211, 364, 517, 575, 670, 728, 823, 881,}\\ \mathbf{1034, 1092, 1187, 1245, 1340, 1398, \ldots}]$$

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Note: Everything appears fine ... or does it?

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$$w\widetilde{w} \text{ (so far)} \longrightarrow [0, 153, 211, 364, 1034, 1187, 1245,$$
$$1340, 1398, \ldots]$$

Problem: The coefficient of x^{1187} is 2
(since $0 + 1187 = 153 + 1034 = 1187$).

$$f\tilde{f} \longrightarrow [\mathbf{0, 153, 211, 364, 517, 575, 670, 728, 823, 881, 1034, 1092, 1187, 1245, 1340, 1398, \ldots}]$$

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Problem: The coefficient of $x^{\mathbf{1187}}$ is $\mathbf{2}$ (since $\mathbf{0 + 1187 = 153 + 1034 = 1187}$).

How to Proceed: Since $\mathbf{364}$ cannot be an exponent in w, we backtrack and consider $\mathbf{364}$ as an exponent in $\widetilde{w}$.

$$f\tilde{f} \longrightarrow [0, 153, 211, 364, 517, 575, 670, 728, 823, 881, 1034, 1092, 1187, 1245, 1340, 1398, \ldots]$$

$$w \longrightarrow [0, 153, 211, \ldots, 1034, 1398]$$

$$\widetilde{w} \longrightarrow [0, 364, \ldots, 1187, 1245, 1398]$$

$$f\tilde{f} \longrightarrow [\mathbf{0, 153, 211, 364, 517, 575, 670, 728, 823, 881,} \\ \mathbf{1034, 1092, 1187, 1245, 1340, 1398, \ldots}]$$

$$w \longrightarrow [\mathbf{0, 153, 211, \ldots, 1034, 1398}]$$

$$\widetilde{w} \longrightarrow [\mathbf{0, 364, \ldots, 1187, 1245, 1398}]$$

$$w\widetilde{w} \text{ (so far)} \longrightarrow [\mathbf{0, 153, 211, 364, 517, 575, 1034,} \\ \mathbf{1187, 1245, 1340, 1398, \ldots}]$$

$$f\tilde{f} \longrightarrow [\mathbf{0, 153, 211, 364, 517, 575, 670, 728, 823, 881,}$$
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$$\boldsymbol{w} \longrightarrow [\mathbf{0, 153, 211, \ldots, 1034, 1398}]$$

$$\widetilde{\boldsymbol{w}} \longrightarrow [\mathbf{0, 364, \ldots, 1187, 1245, 1398}]$$

$$\boldsymbol{w}\widetilde{\boldsymbol{w}} \text{ (so far)} \longrightarrow [\mathbf{0, 153, 211, 364, 517, 575, 1034,}$$
$$\mathbf{1187, 1245, 1340, 1398, \ldots}]$$

Note: Everything appears fine.

$$f\tilde{f} \longrightarrow [\mathbf{0, 153, 211, 364, 517, 575, 670, 728, 823, 881,}$$
$$\mathbf{1034, 1092, 1187, 1245, 1340, 1398, \ldots}]$$

$$w \longrightarrow [\mathbf{0, 153, 211, \ldots, 1034, 1398}]$$

$$\widetilde{w} \longrightarrow [\mathbf{0, 364, \ldots, 1187, 1245, 1398}]$$

$$w\widetilde{w} \text{ (so far)} \longrightarrow [\mathbf{0, 153, 211, 364, 517, 575, 1034,}$$
$$\mathbf{1187, 1245, 1340, 1398, \ldots}]$$

Note: Everything appears fine.

$$f\tilde{f} \longrightarrow [\mathbf{0, 153, 211, 364, 517, 575, 670, 728, 823, 881,}\\ \mathbf{1034, 1092, 1187, 1245, 1340, 1398, \ldots}]$$

$$w \longrightarrow [\mathbf{0, 153, 211, \ldots, 1034, 1398}]$$

$$\widetilde{w} \longrightarrow [\mathbf{0, 364, \ldots, 1187, 1245, 1398}]$$

$$w\widetilde{w} \text{ (so far)} \longrightarrow [\mathbf{0, 153, 211, 364, 517, 575, 1034,}\\ \mathbf{1187, 1245, 1340, 1398, \ldots}]$$

Note: Everything appears fine.

Question: How can we get $\mathbf{670}$ in $w\widetilde{w}$?

$$f\tilde{f} \longrightarrow [\mathbf{0, 153, 211, 364, 517, 575, 670, 728, 823, 881, 1034, 1092, 1187, 1245, 1340, 1398, \ldots}]$$

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$$w \longrightarrow [\mathbf{0, 153, 211, \ldots, 1034, 1398}]$$

$$\widetilde{w} \longrightarrow [\mathbf{0, 364, \ldots, 1187, 1245, 1398}]$$

$$f\,\tilde{f} \longrightarrow [\mathbf{0, 153, 211, 364, 517, 575, 670, 728, 823, 881, 1034, 1092, 1187, 1245, 1340, 1398, \ldots}]$$

$$w \longrightarrow [\mathbf{0, 153, 211, 670, \ldots, 1034, 1398}]$$

$$f\tilde{f} \longrightarrow [\mathbf{0, 153, 211, 364, 517, 575, 670, 728, 823, 881, 1034, 1092, 1187, 1245, 1340, 1398, \ldots}]$$

$$w \longrightarrow [\mathbf{0, 153, 211, 670, 1034, 1398}]$$

$$f\tilde{f} \longrightarrow [0, 153, 211, 364, 517, 575, 670, 728, 823, 881, 1034, 1092, 1187, 1245, 1340, 1398, \ldots]$$

$$w \longrightarrow [0, 153, 211, 670, 1034, 1398]$$

$$\widetilde{w} \longrightarrow [0, 364, 728, 1187, 1245, 1398]$$

$$f\tilde{f} \longrightarrow [\mathbf{0, 153, 211, 364, 517, 575, 670, 728, 823, 881, 1034, 1092, 1187, 1245, 1340, 1398, \ldots}]$$

$$w \longrightarrow [\mathbf{0, 153, 211, 670, 1034, 1398}]$$

$$\widetilde{w} \longrightarrow [\mathbf{0, 364, 728, 1187, 1245, 1398}]$$

$$w\widetilde{w} \longrightarrow [\mathbf{0, 153, 211, 364, 517, 575, 670, 728, 881, 939, 1034, 1187, 1245, 1340, 1398, \ldots}]$$

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$$w \longrightarrow [0, 153, 211, 670, 1034, 1398]$$

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$$w\widetilde{w} \longrightarrow [0, 153, 211, 364, 517, 575, 670, 728, 881, 939, 1034, 1187, 1245, 1340, 1398, \ldots]$$

Problem: The coefficient of x^{1034} is 2 (since $1034 + 0 = 670 + 364 = 1034$).

$$f\tilde{f} \longrightarrow [0, 153, 211, 364, 517, 575, 670, 728, 823, 881, 1034, 1092, 1187, 1245, 1340, 1398, \ldots]$$

$$w \longrightarrow [0, 153, 211, 670, 1034, 1398]$$

$$\widetilde{w} \longrightarrow [0, 364, 728, 1187, 1245, 1398]$$

$$w\widetilde{w} \longrightarrow [0, 153, 211, 364, 517, 575, 670, 728, 881, 939, 1034, 1187, 1245, 1340, 1398, \ldots]$$

Problem: The coefficient of x^{1034} is 2 (since $1034 + 0 = 670 + 364 = 1034$). Also, the exponent 939 should not be in $w\widetilde{w}$.

$$f\tilde{f} \longrightarrow [\mathbf{0}, \mathbf{153}, \mathbf{211}, \mathbf{364}, \mathbf{517}, \mathbf{575}, \mathbf{670}, \mathbf{728}, \mathbf{823}, \mathbf{881}, \mathbf{1034}, \mathbf{1092}, \mathbf{1187}, \mathbf{1245}, \mathbf{1340}, \mathbf{1398}, \ldots]$$

$$w \longrightarrow [\mathbf{0}, \mathbf{153}, \mathbf{211}, \mathbf{670}, \mathbf{1034}, \mathbf{1398}]$$

$$\widetilde{w} \longrightarrow [\mathbf{0}, \mathbf{364}, \mathbf{728}, \mathbf{1187}, \mathbf{1245}, \mathbf{1398}]$$

$$w\widetilde{w} \longrightarrow [\mathbf{0}, \mathbf{153}, \mathbf{211}, \mathbf{364}, \mathbf{517}, \mathbf{575}, \mathbf{670}, \mathbf{728}, \mathbf{881}, \mathbf{939}, \mathbf{1034}, \mathbf{1187}, \mathbf{1245}, \mathbf{1340}, \mathbf{1398}, \ldots]$$

How to Proceed: Since $\mathbf{670}$ cannot be an exponent in w, we backtrack and consider $\mathbf{670}$ as an exponent in $\widetilde{w}$.

$$\boldsymbol{f\tilde{f} \longrightarrow [0, 153, 211, 364, 517, 575, 670, 728, 823, 881,}$$
$$\boldsymbol{1034, 1092, 1187, 1245, 1340, 1398, \ldots]}$$

$$\boldsymbol{w \longrightarrow [0, 153, 211, 728, 1034, 1398]}$$

$$\boldsymbol{\widetilde{w} \longrightarrow [0, 364, 670, 1187, 1245, 1398]}$$

$$\boldsymbol{f\tilde{f} \longrightarrow [0, 153, 211, 364, 517, 575, 670, 728, 823, 881,}$$
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$$\boldsymbol{w \longrightarrow [0, 153, 211, 728, 1034, 1398]}$$

$$\boldsymbol{\widetilde{w} \longrightarrow [0, 364, 670, 1187, 1245, 1398]}$$

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$$w \longrightarrow [\mathbf{0, 153, 211, 728, 1034, 1398}]$$

$$\widetilde{w} \longrightarrow [\mathbf{0, 364, 670, 1187, 1245, 1398}]$$

$$w\widetilde{w} \longrightarrow [\mathbf{0, 153, 211, 364, 517, 575, 670, 728, 823, 881,}$$
$$\mathbf{1034, 1092, 1187, 1245, 1340, 1398, \ldots}\,]$$

Note: We now have $f\tilde{f} = w\widetilde{w}$!!

$$f\tilde{f} \longrightarrow [0, 153, 211, 364, 517, 575, 670, 728, 823, 881, 1034, 1092, 1187, 1245, 1340, 1398, \dots]$$

$$w \longrightarrow [0, 153, 211, 728, 1034, 1398]$$

$$\widetilde{w} \longrightarrow [0, 364, 670, 1187, 1245, 1398]$$

$$w\widetilde{w} \longrightarrow [0, 153, 211, 364, 517, 575, 670, 728, 823, 881, 1034, 1092, 1187, 1245, 1340, 1398, \dots]$$

Note: We now have $f\tilde{f} = w\widetilde{w}$!!

How to Proceed: We check that $w(x) \neq f(x)$ and that $w(x) \neq \tilde{f}(x)$. Since $w(x) \neq f(x)$ and $w(x) \neq \tilde{f}(x)$, the non-reciprocal part of $f(x)$ is reducible.

Recall

$$f(x) = 1 + x^{211} + x^{517} + x^{575} + x^{1245} + x^{1398}.$$

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A factor of $f(x)$ is

$$\gcd(f(x), w(x)) = 1 + x^{211} - x^{364} + x^{881}.$$

Recall

$$f(x) = 1 + x^{211} + x^{517} + x^{575} + x^{1245} + x^{1398}.$$

A factor of $f(x)$ is

$$\gcd(f(x), w(x)) = 1 + x^{211} - x^{364} + x^{881}.$$

We have the factorization

$$f(x) = (1 + x^{211} - x^{364} + x^{881})(1 + x^{364} + x^{517}).$$

Recall

$$f(x) = 1 + x^{211} + x^{517} + x^{575} + x^{1245} + x^{1398}.$$

A factor of $f(x)$ is

$$\gcd(f(x), w(x)) = 1 + x^{211} - x^{364} + x^{881}.$$

We have the factorization

$$f(x) = (1 + x^{211} - x^{364} + x^{881})(1 + x^{364} + x^{517}).$$

MAPLE: `irreduc(f);`

took > 3 hours on an Ultra 5 Workstation

Recall

$$f(x) = 1 + x^{211} + x^{517} + x^{575} + x^{1245} + x^{1398}.$$

A factor of $f(x)$ is

$$\gcd(f(x), w(x)) = 1 + x^{211} - x^{364} + x^{881}.$$

We have the factorization

$$f(x) = (1 + x^{211} - x^{364} + x^{881})(1 + x^{364} + x^{517}).$$

MAPLE: `irreduc(f);`

took > 3 hours on an Ultra 5 Workstation

Comment: $x^2 + 1$ is a factor of $f(x)$

Comparison of Three Algorithms

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Comparison of Three Algorithms

`irreduc`: MAPLE command for testing irreducibility of an *arbitrary* polynomial in $\mathbb{Z}[x]$

`irreduc01NR`: program, using MAPLE, based on the example just given, for testing the irreducibility of the non-reciprocal part of a $0, 1$-polynomial

`irreduc01`: program that combines `irreduc01NR` with a check for reciprocal factors that tests the irreducibility of a $0, 1$-polynomial

Comparison of Three Algorithms

`irreduc`   `irreduc01NR`   `irreduc01`

Comparison of Three Algorithms

`irreduc`   `irreduc01NR`   `irreduc01`

We ran tests of random polynomials

of degree $\leq M$

and consisting

of r non-zero terms.

Comparison of Three Algorithms

`irreduc` `irreduc01NR` `irreduc01`

We ran tests of random polynomials

of degree $\leq M$

and consisting

of $r+1$ non-zero terms.

Comparison of Three Algorithms

M	r	# tests	Average CPU time (sec)		
			`irreduc`	`irreduc01NR`	`irreduc01`
100	10	50	0.584	0.015	0.041
200	10	50	4.692	0.015	0.039
300	10	50	20.663	0.014	0.043
400	10	30	48.145	0.013	0.048
500	10	30	72.717	0.015	0.057
1000	10	10	475.771	0.014	0.132
10000	10	1/30/30	89.105 hrs	0.013	8.943
50000	30	0/50/10	NA	0.192	309.574
100000	30	0/50/10	NA	0.191	1373.840

Computational Complexity of `irreduc01`

M	r	# tests	# irred	CPU time (sec) Range	Mean
20000	10	30	22	0.010 – 54.550	28.796
20000	20	30	26	0.010 – 125.110	46.693
20000	30	30	26	0.010 – 212.180	54.694
20000	40	30	26	0.010 – 130.830	53.728
20000	50	30	28	0.020 – 136.140	59.368
50000	30	10	9	0.010 – 390.780	309.574
100000	30	10	9	0.020 – 2872.090	1373.840
100000	50	10	9	0 020 – 1562.580	1342.016
100000	100	10	9	0.040 – 1642.850	1424.939

Computational Complexity of `irreduc01NR`

M	r	# tests	# irreduc	CPU time (sec)	
				Range	Mean
$\mathbf{10^{10}}$	30	50	50	0.170 – 0.210	0.198

Computational Complexity of `irreduc01NR`

M	r	# tests	# irreduc	CPU time (sec) Range	Mean
10^{10}	30	50	50	0.170 – 0.210	0.198
10^{100}	30	50	50	0.210 – 0.360	0.261
10^{100}	50	50	50	1.170 – 1.800	1.462
10^{100}	100	50	50	19.630 – 26.730	23.981

Computational Complexity of `irreduc01NR`

M	r	# tests	# irreduc	CPU time (sec) Range	Mean
10^{10}	30	50	50	0.170 – 0.210	0.198
10^{100}	30	50	50	0.210 – 0.360	0.261
10^{100}	50	50	50	1.170 – 1.800	1.462
10^{100}	100	50	50	19.630 – 26.730	23.981
10^{10^5}	30	10	10	20.090 – 21.210	20.856
10^{10^5}	50	10	10	79.560 – 85.080	82.454

THEOREM (F. & SCHINZEL): There is an algorithm with the following property: Given a non-reciprocal $f(x) \in \mathbb{Z}[x]$ with N non-zero terms and height H, the algorithm determines whether $f(x)$ is irreducible in time

$$c(N, H)(\log \deg f)^{c'(N)}$$

where $c(N, H)$ depends only on N and H and $c'(N)$ depends only on N.

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$$c(N,H)(\log \deg f)^{c'(N)}$$

where $c(N,H)$ depends only on N and H and $c'(N)$ depends only on N.

Time For A Demo

BASIC QUESTION 2

Can we categorize the polynomials having small Euclidean norm that are reducible?

Selmer (1956): Investigated the irreducibility of

$$x^a \pm x^b \pm 1.$$

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$$\boldsymbol{x^a \pm x^b \pm 1.}$$

Ljunggren (1960): Investigated the irreducibility of

$$\boldsymbol{x^a \pm x^b \pm 1} \quad \text{and} \quad \boldsymbol{x^a \pm x^b \pm x^c \pm 1.}$$

Selmer (1956): Investigated the irreducibility of

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Ljunggren (1960): Investigated the irreducibility of

$$x^a \pm x^b \pm 1 \quad \text{and} \quad x^a \pm x^b \pm x^c \pm 1.$$

Theorem: The non-cyclotomic parts of $x^a \pm x^b \pm 1$ (if $a > b > 0$) and of $x^a \pm x^b \pm x^c \pm 1$ (if $a > b > c > 0$) are always irreducible.

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Theorem: The non-cyclotomic parts of $x^a \pm x^b \pm 1$ (if $a > b > 0$) and of $x^a \pm x^b \pm x^c \pm 1$ (if $a > b > c > 0$) are always irreducible.

Mills (1985): Noted that

$$x^8 + x^7 + x - 1 = (x^2 + 1)(x^3 + x^2 - 1)(x^3 - x + 1).$$

$$x^8+x^7+x-1=(x^2+1)(x^3+x^2-1)(x^3-x+1)$$
$$x^8+x^4+x^2-1=(x^2+1)(x^3+x^2-1)(x^3-x^2+1)$$
$$x^8-x^6-x^4-1=(x^2+1)(x^3-x-1)(x^3-x+1)$$
$$x^8-x^7-x-1=(x^2+1)(x^3-x-1)(x^3-x^2+1)$$

$$x^8+x^7+x-1=(x^2+1)(x^3+x^2-1)(x^3-x+1)$$
$$x^8+x^4+x^2-1=(x^2+1)(x^3+x^2-1)(x^3-x^2+1)$$
$$x^8-x^6-x^4-1=(x^2+1)(x^3-x-1)(x^3-x+1)$$
$$x^8-x^7-x-1=(x^2+1)(x^3-x-1)(x^3-x^2+1)$$

Call these “variations” of each other.

Mills' Theorem: Suppose

$$f(x) = x^a \pm x^b \pm 1 \quad \text{with } a > b > 0$$

or

$$f(x) = x^a \pm x^b \pm x^c \pm 1 \quad \text{with } a > b > c > 0.$$

Then the non-cyclotomic part of $f(x)$ is irreducible unless $f(x)$ is a variation of

$$x^{8k} + x^{7k} + x^k - 1$$
$$= (x^{2k} + 1)(x^{3k} + x^{2k} - 1)(x^{3k} - x^k + 1).$$

Theorem (Schinzel): Fix $a_0, \ldots, a_r \in \mathbb{Z} - \{0\}$. Then it is possible to classify the polynomials of the form

$$a_r x^{d_r} + \cdots + a_1 x^{d_1} + a_0$$

that have reducible non-reciprocal part.

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$$a_r x^{d_r} + \cdots + a_1 x^{d_1} + a_0$$

that have reducible non-reciprocal part.

Theorem (F. & Solan): If $a > b > c > d > 0$, then the non-reciprocal part of $x^a + x^b + x^c + x^d + 1$ is irreducible.

Theorem: If $a > b > c > d > e > 0$, then the non-reciprocal part of

$$f(x) = x^a + x^b + x^c + x^d + x^e + 1$$

is irreducible unless $f(x)$ is a variation of

$$\begin{aligned} f(x) &= x^{5s+3t} + x^{4s+2t} + x^{2s+2t} + x^t + x^s + 1 \\ &= (x^{3s+2t} - x^{s+t} + x^t + 1)(x^{2s+t} + x^s + 1). \end{aligned}$$

Theorem (F. & Murphy): If $n > c > b > a > 0$, then the non-reciprocal part of

$$f(x) = x^n \pm x^c \pm x^b \pm x^a \pm 1$$

is irreducible unless $f(x)$ is a variation of one of the following:

Theorem (F. & Murphy): If $n > c > b > a > 0$, then the non-reciprocal part of

$$f(x) = x^n \pm x^c \pm x^b \pm x^a \pm 1$$

is irreducible unless $f(x)$ is a variation of one of the following:

$$x^{8t} - x^{7t} - x^{4t} + x^{2t} - 1 = (x^{3t} - x^t - 1)(x^{5t} - x^{4t} + x^{3t} - x^t + 1)$$

$$x^{8t} - x^{6t} + x^{4t} - x^t - 1 = (x^{3t} - x^{2t} + 1)(x^{5t} + x^{4t} - x^{2t} - x^t - 1)$$

$$x^{9t} - x^{7t} + x^{6t} - x^t - 1 = (x^{3t} - x^{2t} + 1)(x^{6t} + x^{5t} - x^{2t} - x^t - 1)$$

$$x^{10t} - x^{7t} - x^{6t} - x^{4t} - 1 = (x^{3t} - x^t - 1)(x^{7t} + x^{5t} + x^{2t} - x^t + 1)$$

$$x^{10t} - x^{9t} + x^{8t} - x^t - 1 = (x^{3t} - x^{2t} + 1)(x^{7t} + x^{5t} - x^{2t} - x^t - 1)$$

$$x^{10t} - x^{6t} - x^{5t} + x^{4t} - 1 = (x^{5t} - x^{4t} + x^{3t} - x^t + 1)(x^{5t} + x^{4t} - x^{2t} - x^t - 1)$$

$$x^{10t} - x^{9t} - x^{6t} + x^{3t} - 1 = (x^{3t} - x^t - 1)(x^{7t} - x^{6t} + x^{5t} - x^{3t} + x^{2t} - x^t + 1)$$

$$x^{10t} + x^{7t} + x^{4t} - x^t - 1 = (x^{3t} - x^{2t} + 1)(x^{7t} + x^{6t} + x^{5t} + x^{4t} - x^{2t} - x^t - 1)$$

$$x^{11t} - x^{8t} - x^{6t} - x^{5t} - 1 = (x^{4t} - x^{t} + 1)(x^{7t} - x^{3t} - x^{2t} - x^{t} - 1)$$

$$x^{11t} + x^{8t} + x^{6t} - x^{t} - 1 = (x^{3t} - x^{2t} + 1)(x^{8t} + x^{7t} + x^{6t} + x^{5t} - x^{2t} - x^{t} - 1)$$

$$x^{13t} - x^{11t} - x^{9t} - x^{4t} - 1 = (x^{3t} - x^{t} - 1)(x^{10t} + x^{7t} - x^{6t} + x^{5t} + x^{2t} - x^{t} + 1)$$

$$x^{13t} - x^{11t} + x^{10t} - x^{2t} - 1 = (x^{5t} - x^{4t} + x^{2t} - x^{t} + 1)(x^{8t} + x^{7t} - x^{2t} - x^{t} - 1)$$

$$\begin{aligned} x^{14t} - x^{11t} + x^{9t} - x^{3t} - 1 &= (x^{7t} - x^{6t} + x^{3t} - x^{t} + 1) \\ &\quad \times (x^{7t} + x^{6t} + x^{5t} - x^{3t} - x^{2t} - x^{t} - 1) \end{aligned}$$

$$\begin{aligned} x^{14t} - x^{9t} - x^{8t} + x^{7t} - 1 &= (x^{7t} - x^{6t} + x^{5t} - x^{3t} + x^{2t} - x^{t} + 1) \\ &\quad \times (x^{7t} + x^{6t} - x^{4t} - x^{t} - 1) \end{aligned}$$

$$x^{2t+u} - x^{t+2u} + x^{2u} - x^{t} - 1 = (x^{t} - x^{u} + 1)(x^{t+u} - x^{u} - 1)$$

$$x^{5t+2u} - x^{4t+2u} - x^{t+u} - x^{t} - 1 = (x^{2t+u} - x^{t+u} - 1)(x^{3t+u} + x^{t} + 1)$$

$$x^{5t+3u} - x^{4t+2u} - x^{t+u} - x^{t} - 1 = (x^{2t+u} - x^{t} - 1)(x^{3t+2u} + x^{t+u} + 1)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

How to Prove Such Theorems:

$$f = x^n - x^c - x^b + x^a + 1$$

$$f = x^n - x^c - x^b + x^a + 1$$
$$\tilde{f} = x^n + x^{n-a} - x^{n-b} - x^{n-c} + 1$$

$$f = x^n - x^c - x^b + x^a + 1$$
$$\tilde{f} = x^n + x^{n-a} - x^{n-b} - x^{n-c} + 1$$

$$f\tilde{f} = 1 + x^a - x^b - x^c - x^{n-c} - x^{n-b} + x^{n-a} - x^{n+a-c} - x^{n+a-b} + x^{n+b-c} + \cdots$$

$$\boldsymbol{f = x^n - x^c - x^b + x^a + 1}$$
$$\boldsymbol{\tilde{f} = x^n + x^{n-a} - x^{n-b} - x^{n-c} + 1}$$

$$\boldsymbol{f\tilde{f} = 1 + x^a - x^b - x^c - x^{n-c} - x^{n-b} + x^{n-a} - x^{n+a-c} - x^{n+a-b} + x^{n+b-c} + \cdots}$$

$$\boldsymbol{w = x^n + x^t - x^s - x^r + 1}$$
$$\boldsymbol{\widetilde{w} = x^n - x^{n-r} - x^{n-s} + x^{n-t} + 1}$$

$$\begin{aligned} f &= x^n - x^c - x^b + x^a + 1 \\ \tilde{f} &= x^n + x^{n-a} - x^{n-b} - x^{n-c} + 1 \end{aligned}$$

$$\begin{aligned} f\tilde{f} = 1 + x^a - x^b - x^c - x^{n-c} - x^{n-b} + x^{n-a} \\ - x^{n+a-c} - x^{n+a-b} + x^{n+b-c} + \cdots \end{aligned}$$

$$\begin{aligned} w &= x^n + x^t - x^s - x^r + 1 \\ \widetilde{w} &= x^n - x^{n-r} - x^{n-s} + x^{n-t} + 1 \end{aligned}$$

$$\begin{aligned} w\tilde{w} = 1 - x^r - x^s + x^t + x^{n-t} - x^{n-s} - x^{n-r} \\ - x^{n+r-t} + x^{n+r-s} - x^{n+s-t} + \cdots \end{aligned}$$

$$f\tilde{f} = 1 + x^a - x^b - x^c - x^{n-c} - x^{n-b} + x^{n-a} - x^{n+a-c} - x^{n+a-b} + x^{n+b-c} + \cdots$$

$$w\tilde{w} = 1 - x^r - x^s + x^t + x^{n-t} - x^{n-s} - x^{n-r} - x^{n+r-t} + x^{n+r-s} - x^{n+s-t} + \cdots$$

$$f\tilde{f} = 1 + x^a - x^b - x^c - x^{n-c} - x^{n-b} + x^{n-a} - x^{n+a-c} - x^{n+a-b} + x^{n+b-c} + \cdots$$

$$w\tilde{w} = 1 - x^r - x^s + x^t + x^{n-t} - x^{n-s} - x^{n-r} - x^{n+r-t} + x^{n+r-s} - x^{n+s-t} + \cdots$$

Basic Idea: We want to equate exponents. But there may be cancellation of terms.

$$
\begin{aligned}
&x^{a} + x^{n-a} + x^{n+b-c} + x^{r} + x^{s} \\
&\qquad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
&= x^{c} + x^{n-c} + x^{b} + x^{n-b} + x^{n+a-c} \\
&\qquad + x^{n+a-b} + x^{t} + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

$$
\begin{aligned}
&x^{a} + x^{n-a} + x^{n+b-c} + x^{r} + x^{s} \\
&\qquad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
&= x^{c} + x^{n-c} + x^{b} + x^{n-b} + x^{n+a-c} \\
&\qquad + x^{n+a-b} + x^{t} + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

Basic Idea: Solve the resulting systems of equations obtained by equating exponents.

$$
\begin{aligned}
&x^{a} + x^{n-a} + x^{n+b-c} + x^{r} + x^{s} \\
&\qquad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
&= x^{c} + x^{n-c} + x^{b} + x^{n-b} + x^{n+a-c} \\
&\qquad + x^{n+a-b} + x^{t} + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

$$
\begin{aligned}
&x^{a} + x^{n-a} + x^{n+b-c} + x^{r} + x^{s} \\
&\qquad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
&= x^{c} + x^{n-c} + x^{b} + x^{n-b} + x^{n+a-c} \\
&\qquad + x^{n+a-b} + x^{t} + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

$$0 < a < b < c < n$$

$$0 < r < s < t < n$$

$$
\begin{aligned}
& x^{a} + x^{n-a} + x^{n+b-c} + x^{r} + x^{s} \\
& \qquad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
= {} & x^{c} + x^{n-c} + x^{b} + x^{n-b} + x^{n+a-c} \\
& \qquad + x^{n+a-b} + x^{t} + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:

a

$$
\begin{aligned}
x^{a} &+ x^{n-a} + x^{n+b-c} + x^{r} + x^{s} \\
&\quad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
= x^{c} &+ x^{n-c} + x^{b} + x^{n-b} + x^{n+a-c} \\
&\quad + x^{n+a-b} + x^{t} + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:

$a, n-a$

$$
\begin{aligned}
&x^{a}+x^{n-a}+x^{n+b-c}+x^{r}+x^{s}\\
&\qquad +x^{n-s}+x^{n-r}+x^{n+r-t}+x^{n+s-t}\\
&=x^{c}+x^{n-c}+x^{b}+x^{n-b}+x^{n+a-c}\\
&\qquad +x^{n+a-b}+x^{t}+x^{n-t}+x^{n+r-s}
\end{aligned}
$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:
$a, n-a, n+b-c$

$$
\begin{aligned}
&x^{a} + x^{n-a} + x^{n+b-c} + x^{r} + x^{s} \\
&\qquad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
&= x^{c} + x^{n-c} + x^{b} + x^{n-b} + x^{n+a-c} \\
&\qquad + x^{n+a-b} + x^{t} + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:

$a, n-a, n+b-c, r$

$$
\begin{aligned}
&x^{a} + x^{n-a} + x^{n+b-c} + x^{r} + x^{s} \\
&\qquad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
&= x^{c} + x^{n-c} + x^{b} + x^{n-b} + x^{n+a-c} \\
&\qquad + x^{n+a-b} + x^{t} + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:
$a, n-a, n+b-c, r, n-s$

$$
\begin{aligned}
&x^{a} + x^{n-a} + x^{n+b-c} + x^{r} + x^{s} \\
&\quad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
&= x^{c} + x^{n-c} + x^{b} + x^{n-b} + x^{n+a-c} \\
&\quad + x^{n+a-b} + x^{t} + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:
$a, n-a, n+b-c, r, n-s, n+r-t$

$$
\begin{aligned}
&x^{a}+x^{n-a}+x^{n+b-c}+x^{r}+x^{s}\\
&\qquad +x^{n-s}+x^{n-r}+x^{n+r-t}+x^{n+s-t}\\
&=x^{c}+x^{n-c}+x^{b}+x^{n-b}+x^{n+a-c}\\
&\qquad +x^{n+a-b}+x^{t}+x^{n-t}+x^{n+r-s}
\end{aligned}
$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:
$a, n-a, n+b-c, r, n-s, n+r-t$

Possible Least Exponent on the Right:
$n-c$

$$
\begin{aligned}
&x^{a} + x^{n-a} + x^{n+b-c} + x^{r} + x^{s} \\
&\quad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
= {}&x^{c} + x^{n-c} + x^{b} + x^{n-b} + x^{n+a-c} \\
&\quad + x^{n+a-b} + x^{t} + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:
$a, n-a, n+b-c, r, n-s, n+r-t$

Possible Least Exponent on the Right:
$n-c, b$

$$
\begin{aligned}
&x^{a} + x^{n-a} + x^{n+b-c} + x^{r} + x^{s} \\
&\quad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
= {} & x^{c} + x^{n-c} + x^{b} + x^{n-b} + x^{n+a-c} \\
&\quad + x^{n+a-b} + x^{t} + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:

$a, n-a, n+b-c, r, n-s, n+r-t$

Possible Least Exponent on the Right:

$n-c, b, t$

$$\begin{aligned} & x^{a}+x^{n-a}+x^{n+b-c}+x^{r}+x^{s} \\ & \quad +x^{n-s}+x^{n-r}+x^{n+r-t}+x^{n+s-t} \\ = \; & x^{c}+x^{n-c}+x^{b}+x^{n-b}+x^{n+a-c} \\ & \quad +x^{n+a-b}+x^{t}+x^{n-t}+x^{n+r-s} \end{aligned}$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:

$a, n-a, n+b-c, r, n-s, n+r-t$

Possible Least Exponent on the Right:

$n-c, b, t, n-t$

$$
\begin{aligned}
&x^{a} + x^{n-a} + x^{n+b-c} + x^{r} + x^{s} \\
&\qquad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
&= x^{c} + x^{n-c} + x^{b} + x^{n-b} + x^{n+a-c} \\
&\qquad + x^{n+a-b} + x^{t} + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:
$a, n-a, n+b-c, r, n-s, n+r-t$

Possible Least Exponent on the Right:
$n-c, b, t, n-t$

$$
\begin{aligned}
& x^{a}+x^{n-a}+x^{n+b-c}+x^{r}+x^{s} \\
& \quad +x^{n-s}+x^{n-r}+x^{n+r-t}+x^{n+s-t} \\
= {} & x^{c}+x^{n-c}+x^{b}+x^{n-b}+x^{n+a-c} \\
& \quad +x^{n+a-b}+x^{t}+x^{n-t}+x^{n+r-s}
\end{aligned}
$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:

$a, n-a, n+b-c, r, n-s, n+r-t$

Possible Least Exponent on the Right:

$n-c, \quad t, n-t$

$$
\begin{aligned}
& x^{a} + x^{n-a} + x^{n+b-c} + x^{r} + x^{s} \\
& \quad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
= {} & x^{c} + x^{n-c} + x^{b} + x^{n-b} + x^{n+a-c} \\
& \quad + x^{n+a-b} + x^{t} + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:

$a, \qquad n+b-c, r, n-s, n+r-t$

Possible Least Exponent on the Right:

$n-c, \quad t, n-t$

$$
\begin{aligned}
& x^a + x^{n-a} + x^{n+b-c} + x^r + x^s \\
&\qquad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
&= x^c + x^{n-c} + x^b + x^{n-b} + x^{n+a-c} \\
&\qquad + x^{n+a-b} + x^t + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:

$a, \qquad r, n-s, n+r-t$

Possible Least Exponent on the Right:

$n-c, \quad t, n-t$

$$\begin{aligned} &x^a + x^{n-a} + x^{n+b-c} + x^r + x^s \\ &\quad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\ &= x^c + x^{n-c} + x^b + x^{n-b} + x^{n+a-c} \\ &\quad + x^{n+a-b} + x^t + x^{n-t} + x^{n+r-s} \end{aligned}$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:

$a,$ $\qquad\qquad$ $r, n-s, n+r-t$

Possible Least Exponent on the Right:

$n-c,$ $\qquad$ $n-t$

$$\begin{aligned}
&x^a + x^{n-a} + x^{n+b-c} + x^r + x^s \\
&\qquad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
&= x^c + x^{n-c} + x^b + x^{n-b} + x^{n+a-c} \\
&\qquad + x^{n+a-b} + x^t + x^{n-t} + x^{n+r-s}
\end{aligned}$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:

$a,\qquad r,\qquad n+r-t$

Possible Least Exponent on the Right:

$n-c,\qquad n-t$

$$
\begin{aligned}
&x^{a} + x^{n-a} + x^{n+b-c} + x^{r} + x^{s} \\
&\qquad + x^{n-s} + x^{n-r} + x^{n+r-t} + x^{n+s-t} \\
&= x^{c} + x^{n-c} + x^{b} + x^{n-b} + x^{n+a-c} \\
&\qquad + x^{n+a-b} + x^{t} + x^{n-t} + x^{n+r-s}
\end{aligned}
$$

Modified Idea: Proceed as suggested but make use of the ordering of the exponents.

Possible Least Exponent on the Left:

a, $\qquad$ r

Possible Least Exponent on the Right:

$n - c$, $\qquad$ $n - t$

Possible Least Exponent on the Left:

$a,$ r

Possible Least Exponent on the Right:

$n - c,$ $n - t$

Possible Least Exponent on the Left:

$a, \qquad\qquad\qquad\qquad r$

Possible Least Exponent on the Right:

$n - c, \qquad\qquad n - t$

One of the Following Holds:

$$a = n - c$$
$$a = n - t$$
$$r = n - c$$
$$r = n - t$$