Lecture 4: Ljunggren's Approach to Specific Lacunary Results

Theorem (Ljunggren): Let n and m be integers with $n > m > 0$, and let $\varepsilon_j \in \{1, -1\}$ for $j \in \{1, 2\}$. Then the non-cyclotomic part of $x^n + \varepsilon_1 x^m + \varepsilon_2$ is irreducible or identically 1.

Proof:

- The non-reciprocal part of $f(x) = x^n + \varepsilon_1 x^m + \varepsilon_2$ is the same as the non-cyclotomic part of $f(x)$ (consider $\varepsilon_2 f(x) - \tilde{f}(x)$).
- Suppose $w(x) \in \mathbb{Z}[x]$ with $w(x)\widetilde{w}(x) = f(x)\tilde{f}(x)$. The goal is to show $w(x) = \pm f(x)$ or $w(x) = \pm \tilde{f}(x)$. This will imply the non-reciprocal (equivalently, non-cyclotomic) part of $f(x)$ is irreducible or 1.
- We can suppose $w(x)$ has positive leading coefficient and $m \le n - m$ (the latter by using $\tilde{f}$ instead of f if necessary).
- Observe that $w(0) \ne 0$ and $w(x)$, $\widetilde{w}(x)$, $f(x)$, and $\tilde{f}(x)$ have the same degree, namely n.
- Since $\|w\|^2 = \|f\|^2 = 3$, each coefficient of $w(x)$ is either 1 or -1. Write $w(x) = x^n + \varepsilon_1' x^k + \varepsilon_2'$ where $\varepsilon_j' \in \{1, -1\}$.
- We can suppose $k \le n - k$.
- Note that

  $$\begin{aligned} f(x)\tilde{f}(x) &= (x^n + \varepsilon_1 x^m + \varepsilon_2)(\varepsilon_2 x^n + \varepsilon_1 x^{n-m} + 1) \\ &= \varepsilon_2 + \varepsilon_1 x^m + \varepsilon_1\varepsilon_2 x^{n-m} + 3x^n + \varepsilon_1\varepsilon_2 x^{n+m} + \varepsilon_1 x^{2n-m} + \varepsilon_2 x^{2n} \end{aligned}$$

  and

  $$\begin{aligned} w(x)\widetilde{w}(x) &= (x^n + \varepsilon_1' x^k + \varepsilon_2')(\varepsilon_2' x^n + \varepsilon_1' x^{n-k} + 1) \\ &= \varepsilon_2' + \varepsilon_1' x^k + \varepsilon_1'\varepsilon_2' x^{n-k} + 3x^n + \varepsilon_1'\varepsilon_2' x^{n+k} + \varepsilon_1' x^{2n-k} + \varepsilon_2' x^{2n}. \end{aligned}$$

- Comparing the least two exponents above, $\varepsilon_2' = \varepsilon_2$, $\varepsilon_1' = \varepsilon_1$, and $k = m$. Thus, $w(x) = f(x)$.

Theorem (F. & Solan): Let $f(x) = x^n + x^m + x^p + x^q + 1$ be a polynomial with $n > m > p > q > 0$. Then the non-reciprocal part of $f(x)$ is either irreducible or 1.

Proof:

- Suppose $w(x) \in \mathbb{Z}[x]$ with $w(x)\widetilde{w}(x) = f(x)\tilde{f}(x)$. The goal is to show $w(x) = \pm f(x)$ or $w(x) = \pm \tilde{f}(x)$.
- In this case, we may further suppose $w(x)$ is a $0,1$-polynomial (and do so). Write $w(x) = x^n + x^{k_3} + x^{k_2} + x^{k_1} + 1$ with $0 < k_1 < k_2 < k_3 < n$.
- By considering reciprocal polynomials if necessary, we consider $m + q \le n$ and $k_1 + k_3 \le n$.

- The condition $w(x)\widetilde{w}(x) = f(x)\tilde{f}(x)$ implies

$$x^{2n} + x^{2n-q} + x^{2n-p} + x^{2n-m} + x^{n+m} \\ + x^{n+p} + x^{n+q} + x^{n+m-q} + x^{n+m-p} + x^{n+p-q} + 5x^n + \cdots$$

$$= x^{2n} + x^{2n-k_1} + x^{2n-k_2} + x^{2n-k_3} + x^{n+k_3} + x^{n+k_2} \\ + x^{n+k_1} + x^{n+k_3-k_1} + x^{n+k_3-k_2} + x^{n+k_2-k_1} + 5x^n + \cdots .$$

- Deduce $2n - k_1 = 2n - q$ so that $k_1 = q$.
- By adding exponents, deduce $14n + 2k_3 - 2k_1 = 14n + 2m - 2q$ so $k_3 = m$.
- Substitute and compare exponents to obtain

$$\{2n - p, n + p, n + m - p, n + p - q\} = \{2n - k_2, n + k_2, n + k_3 - k_2, n + k_2 - k_1\}.$$

- Comparing largest elements of these sets, deduce one of $2n - p$ and $n + p$ must equal one of $2n - k_2$ and $n + k_2$.
- If $2n - p = 2n - k_2$ or $n + p = n + k_2$, $k_2 = p$ and $w(x) = f(x)$.
- If $2n - p = n + k_2$ or $n + p = 2n - k_2$, then $k_2 = n - p$. Substituting and comparing exponents, deduce

$$\{n + m - p, n + p - q\} = \{n + k_3 - k_2, n + k_2 - k_1\} = \{m + p, 2n - p - q\}.$$

  If $n + m - p = m + p$, then $n = 2p$ so that $k_2 = n - p = p$ and $w(x) = f(x)$. If $n + m - p = 2n - p - q$, then $n = m + q$ so that $k_3 = m = n - q$, $k_1 = q = n - m$, and $w(x) = \tilde{f}(x)$.