Lecture 1: An Example Concerning the Irreducibility of $x^n + g(x)$

Problem (posed by Charles Nicol): Does this ever end?

$$1 + x^3 + x^{15} + x^{16} + x^{32} + x^{33} + x^{34} + x^{35} + \cdots$$

Goal: Justify the answer (whatever it is).

Definitions and Notation: Given $f(x) \in \mathbb{C}[x]$ with $f \not\equiv 0$, $\tilde{f}(x) = x^{\deg f} f(1/x)$ is the *reciprocal* of $f(x)$. If $f = \pm\tilde{f}$, then f is called *reciprocal*.

Comment: If f is reciprocal and α is a root of f, then $1/\alpha$ is a root of f.

Two-Step Approach: 1. Handle reciprocal factors (there are none).
2. Handle non-reciprocal factors (there is no more than one).

Step 1: Take $g(x) = 1 + x^3 + x^{15} + x^{16} + x^{32} + x^{33} + x^{34} + x^{35}$.

- If f is an irreducible reciprocal factor of $F(x) = x^n + g(x)$, then it divides $\widetilde{F}(x)$.
- So it divides $g(x)\tilde{g}(x) - x^{\deg g}$.
- So it is either $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ or

$$x^{64} + x^{61} - x^{60} + x^{54} - \cdots - x^{43} + 2x^{42} + x^{41} - \cdots + x^{10} - x^4 + x^3 + 1.$$

- In the first case, check $0 \le n \le 6$. Done.
- In the second case, f has a root $\alpha = 0.58124854 - 0.96349774i$ with $1.125 < |\alpha| < 1.126$. Observe that $|g(\alpha)| < g(1.126) < 231 < 1.125^{47} < |\alpha|^{47}$. So $F(\alpha) \neq 0$ for all $n \geq 1$.

Step 2: Assume $F(x) = x^n + g(x)$ is reducible. Let $a(x)$ be an irreducible non-reciprocal factor. If $\tilde{a}(x)$ divides F, write $F(x) = u(x)v(x)$ where $\tilde{a}(x) \nmid u(x)$ and $a(x) \nmid v(x)$. If $\tilde{a}(x)$ does not divide F, consider an irreducible non-reciprocal $b(x)$ such that $a(x)b(x)$ divides F. If $\tilde{b}(x)$ divides F, write $F(x) = u(x)v(x)$ where $\tilde{b}(x) \nmid u(x)$ and $b(x) \nmid v(x)$. If $\tilde{a}(x)$ and $\tilde{b}(x)$ do not divide F, write $F(x) = u(x)v(x)$ where $a(x)|u(x)$ and $b(x)|v(x)$. In all cases, we may take both u and v to have a positive leading coefficient.

- Can F have a reciprocal factor? Maybe, but u and v are non-reciprocal.
- **Lemma.** The polynomial $w(x) = u(x)\tilde{v}(x)$ has the following properties:
  (i) $w \neq \pm F$ and $w \neq \pm\widetilde{F}$.
  (ii) $w\widetilde{w} = F\widetilde{F}$.
  (iii) $w(1) = F(1)$.
  (iv) $\|w\| = \|F\|$.
  (v) w is a $0,1$-polynomial with the same number of non-zero terms as F.

Proof of (v). If $F(x) = \sum_{j=1}^{r} a_j x^{d_j}$ and $w(x) = \sum_{j=1}^{s} b_j x^{e_j}$, then

$$\left(\sum_{j=1}^{s} b_j\right)^2 \le \left(\sum_{j=1}^{s} b_j^2\right)^2 = \left(\sum_{j=1}^{s} a_j^2\right)^2 = \left(\sum_{j=1}^{s} a_j\right)^2 = \left(\sum_{j=1}^{s} b_j\right)^2. \qquad \blacksquare$$

- If $n \ge 83$, then $F\widetilde{F} = 1 + x^3 + x^{15} + x^{16} + x^{32} + x^{33} + x^{34} + x^{35} + x^m + \cdots$ where $m \ge 48$.
- What can w and $\widetilde{w}$ be given (v), (ii), and $n \ge 83$?

$$\begin{aligned}
w(x) &= 1 + x^3 + \cdots + x^n & \widetilde{w}(x) &= 1 + \cdots + x^{n-3} + x^n \\
w(x) &= 1 + x^3 + x^{15} + \cdots + x^n & \widetilde{w}(x) &= 1 + \cdots + x^{n-15} + x^{n-3} + x^n \\
w(x) &= 1 + x^3 + x^{15} + x^{16} + \cdots + x^n & \widetilde{w}(x) &= 1 + \cdots + x^{n-16} + x^{n-15} + x^{n-3} + x^n \\
&\vdots & &\vdots
\end{aligned}$$

- Given (i), "the non-reciprocal part is irreducible".

Comment: In general, consider a $0, 1$-polynomial $g(x)$ with the property that $g(x)$ is irreducible over the set of $0, 1$-polynomials (that is, $g(x)$ is not the product of two $0, 1$-polynomials of degree > 0). Then the non-reciprocal part of $F(x) = x^n + g(x)$ is irreducible if $n > 3 \deg g$.