THE DISTRIBUTION OF SQUAREFULL NUMBERS IN SHORT INTERVALS

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1. Introduction

A squarefull number is a positive integer n such that if p is a prime dividing n, then p^2 divides n. Let $Q(x)$ denote the number of $n \leq x$ which are squarefull. Then a result of Bateman and Grosswald [1] implies that

$$Q(x) = \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + \frac{\zeta(2/3)}{\zeta(2)} x^{1/3} + o(x^{1/6}).$$

It follows that there is a positive constant c_1 and arbitrarily large values of x for which the interval $(x, x + c_1 x^{1/2}]$ contains no squarefull numbers. On the other hand, since there is a square in $(x, x + 2\sqrt{x} + 1]$ for all $x \geq 0$, it follows that there is a constant c_2 such that for every $x \geq 1$, the interval $(x, x + c_2 x^{1/2}]$ contains a squarefull number. Thus, the order of the maximum size of gaps between squarefull numbers is known. However, a better understanding of the distribution of squarefull numbers in short intervals can still be obtained. The result of Bateman and Grosswald easily implies that

$$Q(x + x^{(1/2)+\theta}) - Q(x) \sim \frac{\zeta(3/2)}{2\zeta(3)} x^{\theta} \tag{1}$$

provided $1/6 \leq \theta < 1/2$. It is not too difficult to see that if (1) holds for some $\theta = \phi \in (0, 1/2)$, then it holds for every fixed $\theta \in [\phi, 1/2)$. Hence, we would like to determine the smallest value of $\theta \in (0, 1/2)$ for which (1) holds. It is likely that any $\theta > 0$ will do. Values of $\theta < 1/6$ which have been obtained include $0.1526\ldots$ by Shiu [10],$0.1507\ldots$ by

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P.G. Schmidt [8], $0.14903\ldots$ by Jia [6], $0.14259\ldots$ by P.G. Schmidt [9], $0.14254\ldots$ by Liu [7], and $0.1317\ldots$ by Heath-Brown [4]. In particular, Heath-Brown made a remarkable improvement on previous results by considering

$$S(x) = \sum_{a^2 b^3 \leq x} 1$$

and showing that asymptotic estimates for $S(x + x^{(1/2)}y) - S(x)$ imply similar asymptotic estimates for $Q(x + x^{(1/2)}y) - Q(x)$ (see Lemma 1 below). Using a result of P.G. Schmidt [9], it follows from Heath-Brown's work that one can take in (1) any θ satisfying $27/205 = 0.1317\cdots < \theta < 1/2$. We show how to take advantage of finite difference techniques to obtain

Theorem. *The asymptotic formula in (1) holds for*

$$\frac{5}{39} = 0.1282\cdots < \theta < \frac{1}{2}.$$

2. Preliminaries

We begin here with a slight reformulation of Heath-Brown's result in [4]. Although, we could proceed by using his results directly, we alter them slightly so as to give an elementary argument of the main result in this paper so that, in particular, our results do not depend on the use of exponential sums. We shall refer to the result of Heath-Brown in [4] and to a result of Huxley in [5], and note here that the arguments for both of these are obtained using elementary techniques.

Lemma 1. *Let $\theta \in (0, 1/2)$ and let $\epsilon \in (0, \theta)$. Suppose there is a $\delta = \delta(\theta, \epsilon) > 0$ such that*

$$S(x + x^{(1/2)+\gamma}) - S(x) = \frac{\zeta(3/2)}{2} x^{\gamma}(1 + O(x^{-\delta}))$$

uniformly for $\theta - \epsilon \leq \gamma \leq \theta$. Then there is an $\eta = \eta(\theta, \epsilon) > 0$ such that

$$Q(x + x^{(1/2)+\theta}) - Q(x) = \frac{\zeta(3/2)}{2\zeta(3)} x^{\theta}(1 + O(x^{-\eta})).$$

To clarify the statement of Lemma 1 and the role of γ throughout the paper, we note that the uniformity condition on γ indicates that the implied constant in the displayed formula involving $S(x)$ is absolute provided $x \geq x_0 = x_0(\theta, \epsilon)$. This formula is to hold for all γ with

$$x^{\theta-\epsilon} \leq x^{\gamma} \leq x^{\theta}$$

so that, for example, we will allow $\gamma = \theta - (\log\log x/\log x)$. The proof of Lemma 1 is essentially the same as the proof of the Theorem of Heath-Brown in [4]. Simply take

$$\theta_0 = \frac{(\theta-\epsilon)(1-2\theta)}{1-2\theta+2\epsilon}$$

in his argument (not in his Theorem). To estimate $S(x + x^{(1/2)+\gamma}) - S(x)$, we make use of the following.

Lemma 2. *Let x and z denote positive real numbers. Let $c \in (0, 1/4)$, and let $h = \sqrt{x}y$ where $y = x^{\gamma}$ and $c < \gamma < (1/2) - c$. Fix $\delta > 0$ with δ sufficiently small (possibly depending on c but not depending on γ). Then*

$$S(x+h) - S(x) = \frac{\zeta(3/2)}{2} y + O(S_1) + O(S_2) + O(yx^{-\delta}),$$

where

$$S_1 = S_1(z) = \sum_{yx^{-\delta} < n \leq z} \left(\left[\sqrt[3]{\frac{x+h}{n^2}} \right] - \left[\sqrt[3]{\frac{x}{n^2}} \right] \right)$$

and

$$S_2 = S_2(z) = \sum_{yx^{-\delta} < n \leq 2x^{1/3} z^{-2/3}} \left(\left[\sqrt{\frac{x+h}{n^3}} \right] - \left[\sqrt{\frac{x}{n^3}} \right] \right).$$

Proof. Let z_0 be defined by $yx^{-\delta} = 2x^{1/3} z_0^{-2/3}$ so that $yx^{-\delta} \leq 2x^{1/3} z^{-2/3}$ if and only if $z \leq z_0$. If $z > z_0$, then $S_1(z) \geq S_1(z_0)$ and $S_2(z) = S_2(z_0) = 0$. Thus, if the lemma holds for $z \leq z_0$, then the lemma holds for $z > z_0$ as well. Similarly, one checks that if the lemma holds for $z \geq yx^{-\delta}$, then it holds for $z < yx^{-\delta}$ as well. Therefore, we only consider

$$yx^{-\delta} \leq z \leq z_0.$$

Observe that

$$S(x+h) - S(x) = \sum_{x < a^2 b^3 \le x+h} 1.$$

On the other hand, if $x < a^2b^3 \le x+h$, then it is easy to see that either $a \le z$ or $b \le 2x^{1/3}z^{-2/3}$. First, we consider all such pairs (a,b) with $b \le 2x^{1/3}z^{-2/3}$. Note that for each $b \le 2x^{1/3}z^{-2/3}$, we have that

$$\sqrt{\frac{x}{b^3}} < a \le \sqrt{\frac{x+h}{b^3}};$$

hence, the total number of such pairs is

$$\sum_{n \le 2x^{1/3}z^{-2/3}} \left(\left[\sqrt{\frac{x+h}{n^3}}\right] - \left[\sqrt{\frac{x}{n^3}}\right] \right) = \sum_{n \le yx^{-\delta}} \left(\left[\sqrt{\frac{x+h}{n^3}}\right] - \left[\sqrt{\frac{x}{n^3}}\right] \right) + S_2$$

$$= \sum_{n \le yx^{-\delta}} \left(\sqrt{\frac{x+h}{n^3}} - \sqrt{\frac{x}{n^3}} \right) + S_2 + O(yx^{-\delta})$$

$$= \frac{\zeta(3/2)}{2} y - \frac{y}{2} \sum_{n > yx^{-\delta}} \frac{1}{n^{3/2}} + S_2 + O(yx^{-\delta})$$

$$= \frac{\zeta(3/2)}{2} y + S_2 + O(yx^{-\delta}).$$

Now, we count the pairs (a,b) with $a \le z$. We may be counting some pairs here that we have already counted, but observe that in any case we are through if we can show that the number of such pairs is $O(S_1) + O(yx^{-\delta})$. For each such pair, we have

$$\sqrt[3]{\frac{x}{a^2}} < b \le \sqrt[3]{\frac{x+h}{a^2}}.$$

Thus, the number of such pairs is bounded above by

$$\sum_{n \le z} \left(\left[\sqrt[3]{\frac{x+h}{n^2}}\right] - \left[\sqrt[3]{\frac{x}{n^2}}\right] \right) = \sum_{n \le yx^{-\delta}} \left(\left[\sqrt[3]{\frac{x+h}{n^2}}\right] - \left[\sqrt[3]{\frac{x}{n^2}}\right] \right) + S_1$$

$$= \sum_{n \le yx^{-\delta}} \left(\sqrt[3]{\frac{x+h}{n^2}} - \sqrt[3]{\frac{x}{n^2}} \right) + S_1 + O(yx^{-\delta})$$

$$= S_1 + O(yx^{-\delta}),$$

and the result follows. ■

It remains to estimate S_1 and S_2 for an appropriate choice of z. To obtain our Theorem form Lemma 1, we take $\theta = (5/39) + \epsilon'$ with $\epsilon' > 0$ and we take ϵ in Lemma 1 so that $0 < \epsilon < \epsilon'$. As mentioned in the Introduction, to obtain the full range of θ stated in the Theorem, it suffices to consider ϵ' sufficiently small. In particular, in Lemma 1, we need only consider

$$\frac{5}{39} + (\epsilon' - \epsilon) \le \gamma \le \frac{5}{39} + \epsilon' \le \frac{9}{67}.$$

This restriction on γ is only made to keep the arguments simpler and more self-contained; since $27/205 < 9/67$, we could have appealed to the work of P.G. Schmidt [9] to deal with $\gamma > 9/67$.

In Lemma 2, we take

$$c = \frac{5}{39}, \quad \delta = \epsilon' - \epsilon \quad \text{and} \quad z = x^{3/13}.$$

The Theorem will follow if we can show that $S_1(x^{3/13})$ and $S_2(x^{3/13})$ are each $\ll x^{5/39}$. We define

$$S_1(\alpha, \beta) = \sum_{\alpha < n \le \beta} \left(\left[\sqrt[3]{\frac{x+h}{n^2}} \right] - \left[\sqrt[3]{\frac{x}{n^2}} \right] \right).$$

To estimate S_1, we consider first $S_1(N, 2N)$ where $yx^{-\delta} \le N \le z$. One can interpret $S_1(N, 2N)$ as the number of pairs (a, b) for which

$$N < a \le 2N \quad \text{and} \quad x < a^2 b^3 \le x + h.$$

Since $h \le x$, the last inequality easily implies $b \le 2\sqrt[3]{x}$. Hence,

$$a^2(b+1)^3 - a^2 b^3 \ge 3a^2 b^2 = \frac{3a^2 b^3}{b} > \frac{3x}{b} > x^{2/3} > x^{(1/2)+\gamma} = h.$$

Therefore, for each $a \in (N, 2N]$, there is at most one pair (a, b) as above. In other words, $S_1(N, 2N)$ is the number of $a \in (N, 2N]$ with the property that there is an integer b between $\sqrt[3]{x/a^2}$ and $\sqrt[3]{(x+h)/a^2}$. For $a \in (N, 2N]$, we have

$$\sqrt[3]{\frac{x+h}{a^2}} - \sqrt[3]{\frac{x}{a^2}} \le \frac{h}{3x^{2/3} a^{2/3}} < (1/3) x^{\gamma - (1/6)} N^{-2/3}.$$

We set

$$\delta' = (1/3)x^{\gamma-(1/6)}N^{-2/3}.$$

Thus, $S_1(N, 2N)$ is bounded by the number of $a \in (N, 2N]$ such that $\sqrt[3]{x/a^2}$ is within δ' of an integer. We use Theorem 3 in Huxley's paper [5]. In the notation there, we set $F(x) = (x+1)^{-(2/3)}$, $L = M = N$, $\delta = \delta'$, $T = x^{1/3}M^{-2/3}$, and $\Delta = x^{1/3}N^{-8/3}$. One can easily check that all the conditions of the theorem are satisfied. We deduce from this result that

$$S_1(N, 2N) \ll x^{2/15}N^{-1/15} + x^{1/15}N^{4/15}$$

for $x^{5/39} \leq N \leq x^{3/13}$. Thus,

$$\begin{aligned} S_1(x^{3/13}) &= \sum_{yx^{-\delta} < n \leq x^{3/13}} \left(\left[\sqrt[3]{\frac{x+h}{n^2}} \right] - \left[\sqrt[3]{\frac{x}{n^2}} \right] \right) \\ &\ll x^{2/15}x^{(-1/15)(5/39)} + x^{1/15}x^{(4/15)(3/13)} \ll x^{5/39}. \end{aligned}$$

Our choice of $z = x^{3/13}$ focuses our attention on the problem at hand; we must now estimate $S_2(x^{3/13})$. This particular choice of z, however, is not so important. We could, for example, have chosen $z = 1$ so that $S_1(z) = 0$. Then we could use Theorem 2 of [5] to obtain an estimate for

$$\sum_{2x^{7/39} < n \leq 2x^{1/3}} \left(\left[\sqrt{\frac{x+h}{n^3}} \right] - \left[\sqrt{\frac{x}{n^3}} \right] \right),$$

thereby establishing part of an estimate for $S_2(1)$. The rest of the argument would be the same as what appears in the next section.

Before leaving this section, we note that it is possible to improve on the result of Heath-Brown for gaps between squarefull numbers by combining the above estimate for $S_1(x^{3/13})$ with an estimate for $S_2(x^{3/13})$ obtainable through the use of exponential sums. Using the current theory of exponential sums to estimate $S_2(x^{3/13})$, however, seems to lead to a weaker result than our Theorem.

3. The Use of Differences

We now use differencing techniques to estimate S_2. In fact, differencing techniques were already used in our estimate of S_1, as the result of Huxley that we used was based on differences. Our approach here is motivated by the authors' own work in [2] and [3].

Observe that

$$2x^{1/3}z^{-2/3} = 2x^{7/39}.$$

One can interpret the sum S_2 appearing in Lemma 2 as the number of positive integral pairs (a,b) for which $a^2b^3 \in (x, x+h]$ and

$$x^{5/39} \leq yx^{-\delta} < b \leq 2x^{7/39}.$$

For such (a,b), we have that

$$a \leq \frac{\sqrt{x+h}}{b^{3/2}} < 2\sqrt{x}x^{-5/26} = 2x^{4/13}.$$

Since

$$(a+1)^2b^3 - a^2b^3 > 2ab^3 = \frac{2a^2b^3}{a} > \frac{2x}{a} > x^{9/13} > x^{(1/2)+\gamma},$$

we deduce that each b in the range above corresponds to at most one pair (a,b) counted by S_2. Let S be the set of numbers b for which a^2b^3 is in the interval $(x, x+h]$ for some integer a, and let

$$S_2(\alpha,\beta) = \{u \in S : \alpha < u \leq \beta\}.$$

Then for $x^{5/39} \leq N \leq x^{7/39}$,

$$|S_2(N,2N)| = \sum_{N<n\leq 2N} \left(\left[\sqrt{\frac{x+h}{n^3}} \right] - \left[\sqrt{\frac{x}{n^3}} \right] \right).$$

Thus, to estimate the size of S_2, we can make use of estimates for the size of $|S_2(N,2N)|$. We consider $N = x^{\phi}$. We refer to $S_2(N,2N)$ in this case as S_ϕ. Our Theorem then is an easy consequence of the next lemma.

Lemma 3. *For $\gamma \leq 9/67$ and $\phi \leq 2/11$,*

$$|S_\phi| \ll x^{(3\phi+1)/12} + y^{1/4}x^{(1-2\phi)/8}.$$

Equivalently,

$$|S_\phi| \ll \begin{cases} x^{(3\phi+1)/12} & \text{for } \phi \geq (6\gamma+1)/12 \\ y^{1/4}x^{(1-2\phi)/8} & \text{for } \phi < (6\gamma+1)/12. \end{cases}$$

Proof. Let $T(a,b)$ denote the set of u for which u, $u+a$, and $u+a+b$ are consecutive elements of S_ϕ, and set $t(a,b) = |T(a,b)|$. Let $D = \gcd(a,b)$. Let

$$A = \min\{x^{(9\phi-1)/12}, y^{-1/4}x^{(10\phi-1)/8}\},$$

and observe that there are $\ll x^\phi/A$ occurences of consecutive u, $u+a$, and $u+a+b$ in S_ϕ with $\max\{a,b\} \geq A$. Now, suppose $\max\{a,b\} < A$. Note that $7\phi/24 \geq (9\phi-1)/12$ so that a and b are $< x^{7\phi/24}$. In particular, for $u \in S_\phi$, a and b may be viewed as being much smaller than u. We find a bound for $t(a,b)$. We use m_j to denote integers. For each $u \in S_\phi$, there is an integer m_1 for which $x < u^3m_1^2 \leq x+h$ so that

$$\frac{\sqrt{x}}{u^{3/2}} < m_1 \leq \frac{\sqrt{x+h}}{u^{3/2}} = \frac{\sqrt{x}}{u^{3/2}} + O(yx^{-3\phi/2}).$$

In other words,

$$\frac{\sqrt{x}}{u^{3/2}} = m_1 + O(yx^{-3\phi/2}).$$

Let $f(u) = \sqrt{x}/u^{3/2}$, and consider $u \in T(a,b)$. Thus, each of $f(u)$, $f(u+a)$, and $f(u+a+b)$ is within $O(yx^{-3\phi/2})$ of an integer. Now, there are $\xi_1 \in (u, u+a)$ and $\xi_2 \in (u+a, u+a+b)$ for which

$$f(u+a) - f(u) = af'(u) + \frac{a^2}{2}f''(u) + \frac{a^3}{6}f'''(u) + O(a^4 f^{(4)}(\xi_1)) \tag{2}$$

and

$$f(u+a+b) - f(u+a) = bf'(u+a) + \frac{b^2}{2}f''(u+a) + \frac{b^3}{6}f'''(u+a) + O(b^4 f^{(4)}(\xi_2)). \tag{3}$$

Since $u \in (x^{\phi}, 2x^{\phi}]$ and a and b are less than u, we get for each $j \in \{1, 2\}$,

$$f^{(4)}(\xi_j) \asymp x^{(1-11\phi)/2}.$$

Using similar expressions to (2) and (3) for the differences $f'(u+a)-f'(u)$, $f''(u+a)-f''(u)$, and $f'''(u+a) - f'''(u)$ and considering the sum of the product of (2) with the integer $-b/D$ with the product of (3) with the integer a/D, we obtain that

$$m_2 = \frac{15ab(a+b)\sqrt{x}}{8Du^{7/2}} - \frac{35ab(a+b)(2a+b)\sqrt{x}}{16Du^{9/2}} + O(ab(a+b)^3 x^{(1-11\phi)/2} D^{-1}) + O((a+b)D^{-1} y x^{-3\phi/2}), \tag{4}$$

for some integer m_2. Since m_2 is an integer, the first term on the right is at least $1/2$ if each other term on the right has absolute value less than $1/4$ of the first term. Hence,

$$ab(a+b) \gg x^{(7\phi-1)/2} D$$

provided

$$2\gamma + 4\phi < 1,$$

an inequality which holds for the range of γ and ϕ given in the lemma. Note that the above implies that at least one of every two consecutive differences between elements of S_ϕ must be $\gg x^{(7\phi-1)/6}$. By considering every other element of S_ϕ rather than every element of S_ϕ, we may use that

$$\min\{a, b\} \gg x^{(7\phi-1)/6};$$

in other words, if necessary, we may estimate instead of $|S_\phi|$ the size of the set

$$S'_\phi = \{u \in S_\phi : \text{ if } u + a \in S_\phi, \text{ then } a \geq \epsilon x^{(7\phi-1)/6}\}$$

for some appropriate $\epsilon > 0$ so that $|S_\phi| \leq 2|S'_\phi| + 1$ (and we estimate $|S'_\phi|$ by redefining $T(a, b)$ in terms of S'_ϕ rather than S_ϕ). Now, suppose that there is a c such that u and $u+c$ are in $T(a, b)$ and $c < \epsilon x^{\phi}$, where $\epsilon > 0$ is sufficiently small (depending on the arguments which follow). Since u, $u+a$, and $u+a+b$ are consecutive elements in S_ϕ and $u+c$ is in S_ϕ, we get that $c \geq a+b$. Considering $f(u+c+a)-f(u+c)-f(u+a)+f(u)$, we see that there is an integer m_3 such that

$$m_3 = \frac{15ac\sqrt{x}}{4u^{7/2}} - \frac{105ac(a+c)\sqrt{x}}{16u^{9/2}} + \frac{315ac(2a^2+3ac+2c^2)\sqrt{x}}{64u^{11/2}} + O\left(ac^4x^{(1-13\phi)/2}\right) + O(yx^{-3\phi/2}).$$

Similarly, considering $f(u+c+a+b)-f(u+c)-f(u+a+b)+f(u)$, there is an integer m_4 such that

$$\begin{aligned} m_4 = \frac{15(a+b)c\sqrt{x}}{4u^{7/2}} &- \frac{105(a+b)c(a+b+c)\sqrt{x}}{16u^{9/2}} \\ &+ \frac{315(a+b)c(2(a+b)^2+3(a+b)c+2c^2)\sqrt{x}}{64u^{11/2}} \\ &+ O\left((a+b)c^4x^{(1-13\phi)/2}\right) + O(yx^{-3\phi/2}). \end{aligned}$$

We use that there are integers k_1 and k_2 such that

$$ak_1 + (a+b)k_2 = D \quad \text{and} \quad \max\{|k_1|,|k_2|\} \leq (a+b)/D.$$

Observe that

$$k_1ac(a+c) + k_2(a+b)c(a+b+c) = Dc^2 + O\left(c(a+b)^3/D\right)$$

and

$$k_1ac(2a^2+3ac+2c^2) + k_2(a+b)c(2(a+b)^2+3(a+b)c+2c^2) = 2Dc^3 + O\left(c^2(a+b)^3/D\right).$$

We deduce that

$$\begin{aligned} (5) \qquad k_1m_3 + k_2m_4 = \frac{15Dc\sqrt{x}}{4u^{7/2}} &+ O\left(Dc^2x^{(1-9\phi)/2}\right) + O\left(c(a+b)^3D^{-1}x^{(1-9\phi)/2}\right) \\ &+ O\left(Dc^3x^{(1-11\phi)/2}\right) + O\left(c^2(a+b)^3D^{-1}x^{(1-11\phi)/2}\right) \\ &+ O\left(c^4(a+b)^2D^{-1}x^{(1-13\phi)/2}\right) + O((a+b)D^{-1}yx^{-3\phi/2}). \end{aligned}$$

Using that $c \geq a+b$, $a+b \leq x^{7\phi/24}$ and $c \leq \epsilon x^{\phi}$, it is easy to check that every term on the right-hand side of (5) after the first has absolute value $< 1/7$ of the first term on the right with the possible exception of $O\left(c^4(a+b)^2 D^{-1} x^{(1-13\phi)/2}\right)$. It follows that the remaining error term in absolute value is less than $1/7$ of the first term or

$$(a+b)^{2/3} c \gg x^{\phi}. \tag{6}$$

In the first case, since $k_1 m_3 + k_2 m_4$ is an integer, we deduce that

$$cD \gg x^{(7\phi-1)/2}. \tag{7}$$

Observe that if (6) holds, then since $a+b \leq 2A \ll x^{(9\phi-1)/12}$ and $\phi \leq 2/11$,

$$c \gg x^{\phi}(a+b)^{-2/3} \gg x^{(9\phi+1)/18} \gg x^{(7\phi-1)/2}.$$

Thus, in any case, (7) holds. Now, we obtain from our previous inequality for $ab(a+b)$ that

$$ab(a+b)c \gg x^{7\phi-1}.$$

Recall (4) holds for some integer m_2. In a similar manner, we obtain that there is an integer m_5 such that

$$\begin{aligned} m_5 = & \frac{15ab(a+b)\sqrt{x}}{8D(u+c)^{7/2}} - \frac{35ab(a+b)(2a+b)\sqrt{x}}{16D(u+c)^{9/2}} \\ & + O(ab(a+b)^3 x^{(1-11\phi)/2} D^{-1}) + O((a+b)D^{-1} y x^{-3\phi/2}), \end{aligned}$$

Therefore,

$$m_2 - m_5 = \frac{105ab(a+b)c\sqrt{x}}{16Du^{9/2}} + O(ab(a+b)c^2 x^{(1-11\phi)/2} D^{-1}) + O((a+b)D^{-1} y x^{-3\phi/2}).$$

Using that $c \leq \epsilon x^{\phi}$ and our lower bound for $ab(a+b)c$ above, we get that the absolute value of each of the error terms on the right is less than $1/3$ of the first term on the right unless

$$c \ll y x^{(6\phi-1)/2}/ab. \tag{8}$$

Thus, either (8) holds or

$$\frac{105ab(a+b)c\sqrt{x}}{16Du^{9/2}} \gg 1$$

so that

$$c \gg \frac{Dx^{(9\phi-1)/2}}{ab(a+b)}.$$

Recalling that $ab(a+b) \gg x^{(7\phi-1)/2}D$, we see that the elements of $T(a,b)$ can be separated into

$$\ll \frac{x^{\phi}ab(a+b)}{Dx^{(9\phi-1)/2}} + 1 \ll ab(a+b)D^{-1}x^{(1-7\phi)/2} + 1 \ll ab(a+b)D^{-1}x^{(1-7\phi)/2}$$

subintervals, with distinct elements in each subinterval being within $O(yx^{(6\phi-1)/2}/ab)$ of one another. On the other hand, we have already established that (7) holds so that distinct elements in each subinterval are also separated by $\gg x^{(7\phi-1)/2}/D$. In other words, each such subinterval contains

$$\ll \frac{yx^{(6\phi-1)/2}D}{abx^{(7\phi-1)/2}} + 1 \ll a^{-1}b^{-1}Dyx^{-\phi/2} + 1$$

elements. We deduce that

$$\begin{aligned} t(a,b) &\ll ab(a+b)D^{-1}x^{(1-7\phi)/2}\left(a^{-1}b^{-1}Dyx^{-\phi/2}+1\right) \\ &\ll (a+b)yx^{(1-8\phi)/2} + ab(a+b)D^{-1}x^{(1-7\phi)/2} \\ &\ll (a+b)yx^{(1-8\phi)/2} + ab(a+b)x^{(1-7\phi)/2}. \end{aligned}$$

Observe that $A \le x^{(9\phi-1)/12}$ implies $A \le x^{\phi}$. Since every u in S_ϕ, except at most 2, is in $T(a,b)$ for some positive integers a and b, we deduce that the number of elements of S_ϕ is

$$\begin{aligned} &\ll \sum_{a\le A}\sum_{b\le A} t(a,b) + \frac{x^{\phi}}{A} \\ &\ll yx^{(1-8\phi)/2}\sum_{a\le A}\sum_{b\le A}(a+b) + x^{(1-7\phi)/2}\sum_{a\le A}\sum_{b\le A} ab(a+b) + \frac{x^{\phi}}{A} \\ &\ll yx^{(1-8\phi)/2}A^3 + x^{(1-7\phi)/2}A^5 + x^{\phi}A^{-1}. \end{aligned}$$

Now,

$$A = \min\{x^{(9\phi-1)/12}, y^{-1/4}x^{(10\phi-1)/8}\} = \begin{cases} x^{(9\phi-1)/12} & \text{for } \phi \geq (6\gamma+1)/12 \\ y^{-1/4}x^{(10\phi-1)/8} & \text{for } \phi < (6\gamma+1)/12. \end{cases}$$

Thus, for $\phi \geq (6\gamma+1)/12$, we obtain that

$$|S_\phi| \ll yx^{(1-7\phi)/4} + x^{(3\phi+1)/12} \ll x^{(3\phi+1)/12},$$

and for $\phi < (6\gamma+1)/12$, we obtain that

$$|S_\phi| \ll y^{1/4}x^{(1-2\phi)/8} + y^{-5/4}x^{(22\phi-1)/8} \ll y^{1/4}x^{(1-2\phi)/8}.$$

Hence, the result follows. ■

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