ON THE IRREDUCIBILITY OF THE GENERALIZED LAGUERRE POLYNOMIALS

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1 Introduction

The generalized Laguerre polynomials are defined by

$$L_m^{(\alpha)}(x) = \sum_{j=0}^{m} \frac{(m+\alpha)(m-1+\alpha)\cdots(j+1+\alpha)(-x)^j}{(m-j)!j!},$$

where m is a positive integer and α is an arbitrary complex number. In 1929, I. Schur [4] established the irreducibility over the rationals of $L_m^{(0)}(x)$, the classical Laguerre polynomials, for every m. In 1931, I. Schur [5] considered $L_m^{(\alpha)}(x)$ in general and showed that $L_m^{(1)}(x)$ is irreducible over the rationals for every m. The case $\alpha \notin \{0,1\}$ remained open. The purpose of this paper is to establish the following:

Theorem 1. *Let α be a rational number which is not a negative integer. Then for all but finitely many positive integers m, the polynomial $L_m^{(\alpha)}(x)$ is irreducible over the rationals.*

Before going to the proof, it is worth noting that reducible $L_m^{(\alpha)}(x)$ do exist even with $\alpha = 2$. In particular, we give the following examples:

$$L_2^{(2)}(x) = \frac{1}{2}(x-2)(x-6)$$

$$L_2^{(23)}(x) = \frac{1}{2}(x-20)(x-30)$$

$$L_4^{(23)}(x) = \frac{1}{24}(x-30)(x^3-78x^2+1872x-14040)$$

$$L_4^{(12/5)}(x) = \frac{1}{15000}(25x^2-420x+1224)(25x^2-220x+264)$$

$$L_5^{(39/5)}(x) = \frac{-1}{375000}(5x-84)\big(625x^4-29500x^3 + 448400x^2-2662080x+5233536\big).$$

It is not difficult to show that in fact there are infinitely many positive integers α for which $L_2^{(\alpha)}(x)$ is reducible (a product of two linear polynomials).

Theorem 1 is a direct consequence of the following more general result:

Theorem 2. *Let α be a rational number which is not a negative integer. Then for all but finitely many positive integers m, the polynomial*

$$\sum_{j=0}^{m} a_j \frac{(m+\alpha)(m-1+\alpha)\cdots(j+1+\alpha)x^j}{(m-j)!j!}$$

is irreducible over the rationals provided only that $a_j \in \mathbb{Z}$ for $0 \le j \le m$ and $|a_0| = |a_m| = 1$.

I. Schur obtained his irreducibility results for $L_m^{(0)}(x)$ and $L_m^{(1)}(x)$ through general results similar to the above (except also for all $m \ge 1$). Recent work of a similar nature has been done by Filaseta [1, 2] and by Filaseta and Trifonov [3]. We note also that the above results can be made effective so that for any fixed $\alpha \in \mathbb{Q}$ it is possible to determine a finite set $S = S(\alpha)$ of m such that the polynomial in Theorem 2 is irreducible (for a_j as stated there) provided $m \notin S$.

2 A Proof of Theorem 2

For a prime p and a non-zero integer a, we define $\nu(a) = \nu_p(a) = e$ where $p^e || a$. We set $\nu(0) = +\infty$. We define the Newton polygon of a polynomial $f(x) = \sum_{j=0}^{n} a_j x^j$ with respect to a prime p, where $a_n a_0 \neq 0$ as the lower convex hull of the points $(j, \nu(a_{n-j}))$. Thus, the slopes of the edges of the Newton polygon of $f(x)$ with respect to p are increasing from left to right. We begin with the following preliminary results.

Lemma 1. *Let k and ℓ be integers with $k > \ell \ge 0$. Suppose $g(x) = \sum_{j=0}^{n} b_j x^j \in \mathbb{Z}[x]$ and p is a prime such that $p \nmid b_n$, $p | b_j$ for all $j \in \{0, 1, \dots, n-\ell-1\}$, and the right-most edge of the Newton polygon for $g(x)$ with respect to p has slope $< 1/k$. Then for any integers $a_0, a_1, \dots, a_n$ with $|a_0| = |a_n| = 1$, the polynomial $f(x) = \sum_{j=0}^{n} a_j b_j x^j$ cannot have a factor with degree in the interval $[\ell+1, k]$.*

Lemma 2. *Let a, b, c and d be integers with $bc - ad \neq 0$. Then the largest prime factor of $(am+b)(cm+d)$ tends to infinity as the integer m tends to infinity.*

Lemma 1 is given as Lemma 2 in [1]. Lemma 2 above is a fairly easy consequence of the fact that the Thué equation $ux^3 - vy^3 = w$ has finitely many solutions in integers x and y where u, v, and w are fixed integers with $w \neq 0$. It also immediately follows from Corollary 1.2 of [6]. We omit the proofs.

Fix α now as in Theorem 2. Throughout the argument we suppose as we may that m is large. Define

$$c_j = \binom{m}{j}(m+\alpha)(m-1+\alpha)\cdots(j+1+\alpha) \qquad \text{for } 0 \le j \le m.$$

We want to show that for all but finitely many positive integers m, the polynomial $f(x) = \sum_{j=0}^{m} a_j c_j x^j$ is irreducible over the rationals, where a_j are arbitrary integers with $|a_0| = |a_n| = 1$. Motivated by Lemma 1, we consider instead $g(x) = \sum_{j=0}^{m} c_j x^j$. Let u and v be relatively prime integers with $v > 0$ such that $\alpha = u/v$. The condition that α is not a negative integer implies that for each $j \in \{0, 1, \ldots, m-1\}$, $m - j + \alpha$ and, hence, $v(m-j) + u$ cannot be zero. We assume that $g(x)$ has a factor in $\mathbb{Z}[x]$ of degree $k \in [1, m/2]$ and establish the theorem by obtaining a contradiction to Lemma 1. We divide the argument into cases depending on the size of k.

Case 1. $k > m/\log^2 m$.

For a and b integers with $b > 0$, let $\pi(x; b, a)$ denote the number of primes $\le x$ which are $\equiv a \pmod{b}$. Then the Prime Number Theorem for Arithmetic Progressions implies that if $\gcd(a, b) = 1$, then

$$\begin{aligned}\pi(x; b, a) &= \frac{1}{\phi(b)} \int_2^x \frac{dt}{\log t} + O\left(\frac{x}{\log^4 x}\right) \\ &= \frac{1}{\phi(b)} \left(\frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + O\left(\frac{x}{\log^4 x}\right)\right).\end{aligned}$$

By considering $\pi(x; b, a) - \pi(x-h; b, a)$, it follows that for a and b fixed, the interval $(x-h, x]$ contains a prime $\equiv a \pmod{b}$ if $h = x/(2\log^2 x)$ and if x is sufficiently large. Taking $a = u$, $b = v$, and $x = vm + u$, we deduce that for some integer $j \in [0, k)$, the number $v(m-j) + u$ is prime. Call such a prime p, and observe that $p \ge 2vm/3$ (since v is a positive integer and m is large). We deduce that p does not divide v. Observe that

$$c_\ell = \binom{m}{\ell} \frac{(vm+u)(v(m-1)+u)\cdots(v(\ell+1)+u)}{v^{m-\ell}} \qquad \text{for } 0 \le \ell \le m.$$

For $j \in \{0, 1, \ldots, k-1\}$, the numbers $v(m-j) + u$ appear in the numerator of the fraction on the right-hand side above whenever $0 \le \ell \le m - k$. Therefore,

$$\nu_p(c_\ell) \ge 1 \qquad \text{for } 0 \le \ell \le m - k. \tag{1}$$

Since $c_m = \pm 1$, $\nu_p(c_m) = 0$. To obtain a contradiction from Lemma 1 for the case under consideration, we show that $\nu_p(c_0) = 1$; the contradiction will be achieved since then it will follow that the right-most edge of the Newton polygon of $g(x)$ with respect to p has slope $< 1/(m-k) < 1/k$. Recall that $p \nmid v$ and that $p \geq 2vm/3$. For $j \in \{0, 1, \ldots, m-1\}$, we deduce the inequality

$$2p > vm + u \geq v(m-j) + u \geq v + u > -p.$$

The condition that α is not a negative integer implies that none of $v(m-j)+u$ can be zero. Hence, p itself is the only multiple of p among the numbers $v(m-j)+u$ with $0 \leq j \leq m-1$. Since $c_0 = (vm+u)(v(m-1)+u)\cdots(v+u)/v^m$, we obtain $\nu_p(c_0) = 1$.

Case 2. $k_0 \leq k \leq m/\log^2 m$ with $k_0 = k_0(u,v)$ a sufficiently large integer.

Let $z = k(\log\log k)^{1/2}$. We first show that there is a prime $p > z$ that divides $v(m-j)+u$ for some $j \in \{0, 1, \ldots, k-1\}$. Then (1) follows as before, and we will obtain a contradiction to Lemma 1 by showing that the right-most edge of the Newton polygon of $g(x)$ with respect to p has slope $< 1/k$.

Let

$$T = \{v(m-j)+u : 0 \leq j \leq k-1\}.$$

Since m is large, we deduce that the elements of T are each $\geq m/2$. Also, observe that $\gcd(u,v) = 1$ implies that each element of T is relatively prime to v. For each prime $p \leq z$, we consider an element $a_p = v(m-j)+u \in T$ with $\nu_p(a_p)$ as large as possible. We let

$$S = T - \{a_p : p \nmid v, p \leq z\}.$$

By the Prime Number Theorem,

$$\pi(z) \leq \frac{2k(\log\log k)^{1/2}}{\log k}.$$

We combine this estimate momentarily with $|S| \geq k - \pi(z)$. Since $k \leq m/\log^2 m$, we obtain $m \geq k\log^2 k$. Consider a prime $p \leq z$ with p not dividing v, and let $r = \nu_p(a_p)$. By the definition of a_p, if $j > r$, then there are no multiples of p^j in T (and, hence, in S). For $1 \leq j \leq r$, there are $\leq [k/p^j] + 1$ multiples of p^j in T and, hence, at most $[k/p^j]$ multiples of p^j in S. Therefore,

$$\nu_p\left(\prod_{s \in S} s\right) \leq \sum_{j=1}^{r} \left[\frac{k}{p^j}\right] \leq \nu_p(k!),$$

and

$$\prod_{s\in S}\prod_{p\le z} p^{\nu_p(s)} \le k! \le k^k.$$

On the other hand,

$$\prod_{s\in S} s \ge \left(\frac{m}{2}\right)^{|S|} \ge \left(\frac{k\log^2 k}{2}\right)^{k-\pi(z)}.$$

Recalling our bound on $\pi(z)$, we obtain

$$\begin{aligned}
\log\Big(\prod_{s\in S} s\Big) &\ge (k-\pi(z))\big(\log k + 2\log\log k - \log 2\big) \\
&\ge \left(k - \frac{2k\sqrt{\log\log k}}{\log k}\right)\big(\log k + 2\log\log k - \log 2\big) \\
&\ge k\log k + 2k\log\log k + O\big(k\sqrt{\log\log k}\big).
\end{aligned}$$

Since $k \ge k_0$ where k_0 is sufficiently large,

$$\log\Big(\prod_{s\in S} s\Big) > k\log k \ge \log\Big(\prod_{s\in S}\prod_{p\le z} p^{\nu_p(s)}\Big).$$

It follows that there is a prime $p > z$ that divides some element of S and, hence, divides some element of T.

Fix a prime $p > z$ that divides an element in T, and let $\nu = \nu_p$. The right-most edge of the Newton polygon of $g(x)$ with respect to p is

$$\max_{1\le j\le m}\left\{\frac{\nu(c_0)-\nu(c_j)}{j}\right\}.$$

Fix $j \in \{1, 2, \dots, m\}$. To complete the case under consideration, we want to show that the fraction above is $< 1/k$. Observe that

$$\begin{aligned}
\nu(c_0)-\nu(c_j) &\le \nu\left((vj+u)(v(j-1)+u)\cdots(v+u)\right) \\
&\le \nu((vj+|u|)!) = \sum_{j=1}^{\infty}\left[\frac{vj+|u|}{p^j}\right] \\
&< \sum_{j=1}^{\infty}\frac{vj+|u|}{p^j} = \frac{vj+|u|}{p-1}.
\end{aligned}$$

Since $p > z = k(\log\log k)^{1/2}$ and $k \geq k_0$, we easily deduce that the right-most edge of the Newton polygon of $g(x)$ with respect to p has slope $< 1/k$ as desired. Hence, as indicated at the beginning of this case, we obtain a contradiction to Lemma 1.

Case 3. $2 \leq k < k_0$.

By Lemma 2 (with $a = v$, $b = u$, $c = v$, and $d = u - v$), the largest prime factor of the product $(vm+u)(v(m-1)+u)$ tends to infinity. Since m is large, we deduce that there is a prime $p > (v+|u|)k_0$ that divides $(vm+u)(v(m-1)+u)$. The argument now follows as in the previous case. In particular,

$$\frac{\nu(c_0) - \nu(c_j)}{j} < \frac{vj + |u|}{j(p-1)} \leq \frac{v + |u|}{p-1} \leq \frac{1}{k_0} < \frac{1}{k} \qquad \text{for } 1 \leq j \leq m,$$

and the right-most edge of the Newton polygon of $g(x)$ with respect to p has slope $< 1/k$. Hence, in this case, we also obtain a contradiction.

Case 4. $k = 1$.

From Lemma 2, the largest prime factor of $m(vm+u)$ tends to infinity with m. We consider a large prime factor p of this product. In particular, we suppose that $p > v + |u|$. Note this implies $p \nmid v$. As in the previous case, we are through if $p|(vm+u)$. So suppose $p|m$. The binomial coefficient $\binom{m}{j}$ appears in the definition of c_j, and this is sufficient to guarantee that $\nu(c_j) \geq 1$ and $\nu(c_{m-j}) \geq 1$ for $1 \leq j \leq p-1$. On the other hand,

$$c_j = \binom{m}{j} \frac{(vm+u)(v(m-1)+u)\cdots(v(j+1)+u)}{v^{m-j}}.$$

For $j \leq m - p$, the numerator of the fraction on the right is a product of $\geq p$ consecutive terms in the arithmetic progression $vt + u$ with $\gcd(p, v) = 1$; thus, $\nu(c_{m-j}) \geq 1$ for $j \geq p$. This implies that (1) holds with $k = 1$. It follows in the same manner as before that the slope of the right-most edge is < 1. A contradiction to Lemma 1 is again obtained (and the proof of the theorem is complete).

References

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