A RESULT ON THE DIGITS OF a^n

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Dedicated to the memory of David R. Richman.

1. Introduction

Let $d_r d_{r-1} \dots d_1 d_0$ be the base b representation of a positive integer m. We refer to a block (of digits) of m base b as being a successive sequence of equal digits $d_i d_{i-1} \dots d_j$ of maximal length. For example, the base 10 number 8037776589 consists of 8 blocks: 8, 0, 3, 777, 6, 5, 8, and 9. We may view the number of blocks of m base b as one more than the number of $k \in \{0, 1, \dots, r-1\}$ for which $d_k \neq d_{k+1}$, and we denote the number of blocks by $B(m, b)$. Thus, in the example above, $B(8037776589, 10) = 8$. If the base b is understood, we may omit any reference to it.

It is reasonable to suspect, from a probabilistic point of view, that whenever a is a positive integer and a is not a power of 10, then the number of blocks of a^n tends to infinity as n goes to infinity. For an arbitrary base $b > 1$, it is not difficult to show that $B(a^n, b)$ is bounded whenever $\log a / \log b$ is rational, and for other values of a, we would like to conclude that $B(a^n, b)$ tends to infinity with n. We show in fact that this is a consequence of a certain transcendence result.

Theorem 1. *Let a and b be integers ≥ 2. If $\log a / \log b$ is irrational, then*

$$\lim_{n \to \infty} B(a^n, b) = \infty. \tag{1}$$

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Theorem 1 can be improved whenever b is not a prime power and a is a prime divisor of the base b.

Theorem 2. *Let b be a positive integer which is not a prime power and let p be a prime. Then p divides b if and only if*

$$\lim_{n\to\infty} \min_{\substack{k\in Z^+ \\ b\nmid p^n k}} B(p^n k, b) = \infty. \tag{2}$$

We will give an elementary proof of Theorem 2, so it is worth noting that Theorem 2 implies that (1) holds with $b = 10$ for $a = 2, 4, 5, 6, 8, 12, \ldots$ and, in general, whenever the exponent of 2 in the largest power of 2 dividing a differs from the exponent of 5 in the largest power of 5 dividing a.

We make one further observation. Theorem 2 implies that there is a positive integer n such that every multiple of 2^n which is relatively prime to 5 contains two blocks formed from the same digit. We were able to establish computationally that $n = 53$ is the smallest such n. Similarly, any odd multiple of 5^{13} contains two blocks formed from the same digit, and the exponent 13 is best possible in this case. In particular, if $\mathcal{B}$ is the set of all numbers not ending in the digit 0 base 10 and consisting of blocks formed from distinct digits, then there are exactly two numbers in $\mathcal{B}$ divisible by 2^{52}. They are $3\underbrace{\cdots}_{9}37\underbrace{\cdots}_{16}70049999996\underbrace{\cdots}_{11}688512$ and $76\underbrace{\cdots}_{9}62\underbrace{\cdots}_{16}29950000003\underbrace{\cdots}_{11}311488$. On the other hand, there are infinitely many numbers in $\mathcal{B}$ divisible by 5^{12} and these are given by the elements of $\mathcal{B}$ ending in 336669921875 or 663330078125.

2. The Proof of Theorem 1

We first show that Theorem 1 follows from

Lemma 1. *Let a and b be integers > 1 such that $\log a/\log b$ is irrational. Let $a_1, a_2, \ldots, a_m$ be arbitrary integers. Then there are finitely many $(m+1)$−tuples $(k_1, k_2, \ldots, k_m, n)$ of non-negative integers satisfying* *(i)* $k_1 < k_2 < \cdots < k_m$,

(ii) $\sum_{j=r}^{m} a_j b^{k_j} > 0$ *for* $1 \le r \le m$, *and*

(iii) $\sum_{j=1}^{m} a_j b^{k_j} = (b-1)a^n$.

To prove Theorem 1, it suffices to show that for any positive integer M, there are only finitely many n for which $B(a^n, b) \le M$. Given $M \in \mathbb{Z}^+$, consider any n such that $B(a^n, b) \le M$. Let $m = B(a^n, b) + \epsilon$, where $\epsilon = 0$ if $b \mid a^n$ and $\epsilon = 1$ otherwise. Define d_1 as the first right-most nonzero digit of a^n base b and take k_1 to be the number of right-most consecutive zero digits of a^n. Let d_2 be the next right-most digit of a^n satisfying $d_2 \ne d_1$ and continue in this manner, defining d_{j+1} as the next digit of a^n such that $d_{j+1} \ne d_j$, until d_{m-1} has been defined. There exist positive integers $l_2, \ldots, l_m$ with $l_2 < l_3 < \cdots < l_m$ such that

$$a^n = b^{k_1}\Big[(d_1 - d_2)\frac{b^{l_2} - 1}{b-1} + \cdots + (d_{m-2} - d_{m-1})\frac{b^{l_{m-1}} - 1}{b-1} + d_{m-1}\frac{b^{l_m} - 1}{b-1}\Big].$$

Condition (iii) of Lemma 1 holds with $a_1 = -d_1$, $a_j = d_{j-1} - d_j$ for $j \in \{2, \ldots, m-1\}$, $a_m = d_{m-1}$, and $k_j = k_1 + l_j$ for $j \in \{2, \ldots, m\}$. Note that regardless of the value of n, we have that $a_j \ne 0$ and $|a_j| \le b - 1$ for every $j \in \{1, \ldots, m\}$. Thus, each n produces a solution to at most one of $(2b-2)^m \le (2b-2)^{M+1}$ equations of the form given in (iii). Moreover, with the k_j and a_j defined as above, (i) is clearly satisfied and (ii) holds since $a_m = d_{m-1} \ge 1$ and

$$\sum_{j=r}^{m} a_j b^{k_j} \ge b^{k_m} - \sum_{j=r}^{m-1} |a_j| b^{k_j} \ge b^{k_m} - \sum_{j=r}^{m-1} (b-1) b^{k_j} > 0.$$

We deduce from Lemma 1 that there are only finitely many n for which $B(a^n, b) \le M$. Theorem 1 follows.

Instead of applying Lemma 1 above, we could have appealed to the following result of Revuz [2]: If $\lambda_1, \ldots, \lambda_M, \mu_1, \ldots, \mu_N$ are algebraic numbers, then the equation $\sum_{i=1}^{M} \lambda_i \theta^{m_i} = \sum_{j=1}^{N} \mu_j \phi^{n_j} \ne 0$ holds for only a finite number of rational integer $(m+n)$-

tuples (m_i, n_j), provided $\log\theta/\log\phi$ is irrational. It appears, however, that counterexamples exist to this statement, although perhaps the conditions of the theorem can be modified to make a correct verifiable result. For example, if θ is the positive real root of $x^2 - x - 1$, one can conclude from this statement that

$$\theta^{k_5} - \theta^{k_4} - \theta^{k_3} + \theta^{k_2} - \theta^{k_1} = 2^m$$

has finitely many solutions in integers $m, k_1, \ldots, k_5$; however, the equation is satisfied whenever $(m, k_1, \ldots, k_5) = (0, 1, 2, k, k+1, k+2)$ where k is an arbitrary integer. Note that we could replace 2^m on the right-hand side of this example with i^m and then take $m = 4n$, thereby introducing a second integer parameter.

We say that an algebraic number α has degree d and height A if α satisfies an irreducible polynomial $f(x) = \sum_{j=0}^{d} a_j x^j \in \mathbb{Z}[x]$ with $a_d \neq 0$, $\gcd(a_d, \ldots, a_1, a_0) = 1$, and $\max_{0 \leq j \leq d} |a_j| = A$. To prove Lemma 1, we make use of the following result which can be found in [1]. (See Theorem 3.1 and the comments following it. Note that a stronger result could have been stated.)

Lemma 2. *Let $\alpha_1, \ldots, \alpha_r$ be non-zero algebraic numbers with degrees at most d and heights at most A. Let $\beta_0, \beta_1, \ldots, \beta_r$ be algebraic numbers with degrees at most d and heights at most $B > 1$. Suppose that*

$$\Lambda = \beta_0 + \beta_1 \log\alpha_1 + \cdots + \beta_r \log\alpha_r \neq 0.$$

Then there are numbers $C = C(r, d) > 0$ and $w = w(r) \geq 1$ such that

$$|\Lambda| > B^{-C(\log A)^w}.$$

Proof of Lemma 1. Throughout the proof, we will make use of the notation $f \ll g$ which will mean that $|f| \leq cg$ for some constant $c = c(m, a, b, a_1, \ldots, a_m)$ and for all $k_1, \ldots, k_m$, and n being considered. We also will add to the conditions (i), (ii), and (iii) of the lemma, a fourth condition:

(iv) $\sum_{j=1}^{r} a_j b^{k_j} \neq 0$ for $1 \leq r \leq m$.

We justify being able to do so by showing that if Lemma 1 is true with the additonal condition (iv), then it is true without it. Suppose that Lemma 1 with (iv) holds. If $(k_1, k_2, \ldots, k_m, n)$ satisfies conditions (i), (ii), and (iii) of Lemma 1, but not (iv), then let $r \in \{1, 2, \ldots, m\}$ be as large as possible such that $\sum_{j=1}^{r} a_j b^{k_j} = 0$. Note by (ii) that $r < m$. Observe now that $(k_{r+1}, k_2, \ldots, k_m, n)$ satisfies $k_{r+1} < \cdots < k_m$, $\sum_{j=t}^{m} a_j b^{k_j} > 0$ for $r+1 \leq t \leq m$, $\sum_{j=r+1}^{m} a_j b^{k_j} = (b-1)a^n$, and $\sum_{j=r+1}^{t} a_j b^{k_j} \neq 0$ for $r+1 \leq t \leq m$. One can then appeal to Lemma 1 with (iv) to conclude that there are only finitely many such $(k_{r+1}, k_2, \ldots, k_m, n)$. But for each such solution $(k_{r+1}, \ldots, k_m, n)$, there is only a finite number of choices for $(k_1, \ldots, k_r)$ satisfying $0 \leq k_1 < \cdots < k_r < k_{r+1}$. Since there are at most $m-1$ possible values of r, we see that if Lemma 1 holds under condition (iv), then it must hold in general.

Assume that $(k_1, k_2, \ldots, k_m, n)$ satisfies conditions (i) - (iv). If $m = 1$, then (iii) becomes

$$a_1 b^{k_1} = (b-1)a^n.$$

Observe that if k_1 and n satisfy the above equation and k_1' and n' are integers for which $a_1 b^{k_1'} = (b-1)a^{n'}$, then $b^{k_1 - k_1'} = a^{n-n'}$. Since $\log a / \log b$ is irrational, we could then deduce that $n' = n$ and $k_1' = k_1$. In other words, the above equation has at most one solution in integers k_1 and n. Lemma 1 follows immediately, in this case.

Suppose now that $m > 1$. We make some preliminary estimates. Since $a^n \leq M b^{k_m}$, where

$$M = \sum_{j=1}^{m} |a_j| \geq 1,$$

we have that

$$n \ll k_m.$$

We improve this estimate to

$$n \ll k_m - k_1.$$

This is just the previous bound on n if $k_1 = 0$. Suppose now that $k_1 > 0$. Then conditions (i) and (iii) of the lemma imply that every prime divisor of b divides a. Let $p_1, \dots, p_t$ be the distinct prime divisors of a. Write

$$a = \prod_{j=1}^{t} p_j^{e_j} \quad \text{and} \quad b = \prod_{j=1}^{t} p_j^{f_j},$$

where $e_j \geq 1$ and $f_j \geq 0$ for each $j \in \{1, \dots, t\}$. We show that for some u and v in $\{1, \dots, t\}$,

$$e_u f_v < e_v f_u. \tag{3}$$

If some $f_v = 0$, then (3) holds upon taking p_u to be any prime divisor of b. On the other hand, if each $f_j > 0$, then the values of e_j/f_j for $j \in \{1, \dots, t\}$ cannot all be the same, since otherwise $\log a / \log b$ would equal this common value and, hence, would be rational. Thus, there are u and v in $\{1, \dots, t\}$ for which $e_u/f_u < e_v/f_v$, so (3) holds in this case. Fix u and v as in (3) and consider equation (iii). Note that $f_u > 0$. The largest power of p_u dividing the right-hand side of (iii) is $p_u^{e_u n}$. Since $p_u^{f_u}$ divides b and b^{k_1} divides the left-hand side of (iii), we obtain $k_1 f_u \leq e_u n$. Now divide both sides of (iii) by b^{k_1}. Then the left-hand side becomes

$$\sum_{j=1}^{m} a_j b^{k_j - k_1} \leq M b^{k_m - k_1} \ll b^{k_m - k_1},$$

while the right-hand side $(b-1)a^n/b^{k_1}$ will be a positive integer divisible by p_v^w, where

$$w = e_v n - k_1 f_v \geq (e_v f_u - e_u f_v)\, n / f_u \geq \frac{n}{f_u}.$$

It follows that

$$p_v^{(n/f_u)-1} \ll b^{k_m - k_1}.$$

Since p_v and f_u depend only on a and b, we deduce the inequality $n \ll k_m - k_1$, as desired.

We will also want

$$k_m \ll n+1, \tag{4}$$

so we show next that this is a consequence of (i), (ii), and (iii). For $r \in \{2,3,\dots,m\}$, we obtain

$$\begin{aligned}(b-1)a^n &= \sum_{j=1}^{m} a_j b^{k_j} = \Big(\sum_{j=r}^{m} a_j b^{k_j - k_r}\Big) b^{k_r} + \sum_{j=1}^{r-1} a_j b^{k_j} \\ &\geq b^{k_r} - \Big(\sum_{j=1}^{r-1} |a_j|\Big) b^{k_{r-1}} \geq b^{k_r - k_{r-1}} - \sum_{j=1}^{r-1} |a_j|,\end{aligned}$$

provided that this last expression is positive. Since $b^{k_r - k_{r-1}} \ll 1$ if this last expression is nonpositive, it follows that in either case

$$k_r - k_{r-1} \ll n+1 \quad \text{for } r \in \{2,3,\dots,m\}.$$

Therefore,

$$k_m - k_1 = (k_m - k_{m-1}) + (k_{m-1} - k_{m-2}) + \cdots + (k_2 - k_1) \ll n+1.$$

From (iii), we obtain that $b^{k_1} | a^n$ so that $k_1 \ll n+1$. Hence, (4) follows.

The basic idea now is to use Lemma 2 to strengthen these estimates. More precisely, we consider $n > 2$ and show that

$$k_{m-i+1} - k_{m-i} \ll (\log n)^{w^{i-1} i} \quad \text{for } 1 \leq i \leq m-1, \tag{5}$$

where $w = w(4)$ is as in Lemma 2. This will imply that

$$n \ll k_m - k_1 = (k_m - k_{m-1}) + (k_{m-1} - k_{m-2}) + \cdots + (k_2 - k_1) \ll (\log n)^{w^{m-1} m}. \tag{6}$$

Since m and w are fixed, we can conclude that n is bounded. By (4) and (i), we have that all the k_i are bounded, thereby completing the proof.

It remains to establish (5), which we now prove by induction on i. Assume $n > 2$ and consider first the case when $i = 1$. Using (ii) with $r = m$, we see that $a_m > 0$. We get from (iii) that

$$a_m b^{k_m} (1 + D) = (b-1)a^n, \tag{7}$$

where from (i)

$$|D| = \left| \sum_{j=1}^{m-1} \frac{a_j}{a_m} b^{k_j - k_m} \right| \le M\, b^{k_{m-1}-k_m}.$$

If $k_m - k_{m-1} \le \log(2M)/\log b$, then since $n \ge 3$, we have immediately that $k_m - k_{m-1} \ll \log n$, which is (5) for the case $i = 1$. So suppose $k_m - k_{m-1} > \log(2M)/\log b$. It follows that $|D| < 1/2$ and hence

$$|\log(1+D)| \le \sum_{j=1}^{\infty} (|D|^j/j) \le |D| + \frac{|D|^2}{2(1-|D|)} < (1+|D|)|D| < \frac{3}{2}|D| \ll b^{k_{m-1}-k_m}.$$

Taking the logarithm of both sides of (7) gives

$$\log a_m + k_m \log b - \log(b-1) - n \log a \ll b^{k_{m-1}-k_m}. \tag{8}$$

We use Lemma 2 with $d = 1$, $r = 4$, $A = \max\{b, a, a_m\} \ll 1$, and $B = \max\{k_m, n\} \ll n$, where the last inequality follows from (4). Observe that the left-hand side of (8) is zero if and only if $D = 0$. But $D = 0$ implies that $\sum_{j=1}^{m-1} a_j b^{k_j} = 0$, contradicting (iv) since $m \ge 2$. So $D \ne 0$ and, therefore, the left-hand side of (8) is non-zero. It follows from Lemma 2 that

$$b^{k_{m-1}-k_m} \gg B^{-C(\log A)^w},$$

where $C = C(4,1)$ and $w = w(4)$. Thus,

$$k_m - k_{m-1} \ll C(\log A)^w \log B \ll \log n,$$

proving that (5) holds for $i = 1$. Now fix i in the range $2 \le i \le m-1$ and suppose that (5) holds for each positive integer $j < i$. Then from (iii), we obtain that

$$D_1 b^{k_{m-i+1}} (1 + D_2) = (b-1)a^n,$$

where from (ii) with $r = m - i + 1$,

$$0 < D_1 = a_m b^{k_m - k_{m-i+1}} + a_{m-1} b^{k_{m-1} - k_{m-i+1}} + \cdots + a_{m-i+1} \ll b^{k_m - k_{m-i+1}}$$

and

$$|D_2| = \left| \sum_{j=1}^{m-i} \frac{a_j}{D_1} b^{k_j - k_{m-i+1}} \right| \le M b^{k_{m-i} - k_{m-i+1}} \ll b^{k_{m-i} - k_{m-i+1}}.$$

The induction hypothesis implies that

$$k_m - k_{m-i+1} = (k_m - k_{m-1}) + \cdots + (k_{m-i+2} - k_{m-i+1}) \ll (\log n)^{w^{i-2}(i-1)}$$

so that

$$\log D_1 \ll (\log n)^{w^{i-2}(i-1)}. \tag{9}$$

If $k_{m-i+1} - k_{m-i} \le \log(2M)/\log b$, then $k_{m-i+1} - k_{m-i} \ll (\log n)^{w^{i-1}i}$, as desired. So suppose $k_{m-i+1} - k_{m-i} > \log(2M)/\log b$. As in the above case for $i = 1$ we have $|D_2| < \frac{1}{2}$ and hence $|\log(1 + D_2)| < \dfrac{3}{2}|D_2|$. Thus,

$$\log D_1 + k_{m-i+1} \log b - \log(b-1) - n \log a \ll b^{k_{m-i} - k_{m-i+1}}. \tag{10}$$

We use Lemma 2 with $d = 1$, $r = 4$, $A = \max\{b, a, D_1\}$, and $B = \max\{k_{m-i+1}, n\} \ll n$. Observe that the left-hand side of (10) is zero if and only if $D_2 = 0$. But $D_2 = 0$ implies $\sum_{j=1}^{m-i} a_j b^{k_j} = 0$, which contradicts (iv) since $m \ge m - i \ge 1$. Hence the left-hand side of (10) is non-zero. Therefore, from Lemma 2,

$$b^{k_{m-i} - k_{m-i+1}} \gg B^{-C(\log A)^w}$$

where $C = C(4, 1)$ and $w = w(4)$. Note that (9) implies that

$$\log A \ll (\log n)^{w^{i-2}(i-1)}.$$

Thus, we easily deduce that

$$k_{m-i+1} - k_{m-i} \ll C(\log A)^w \log B \ll (\log n)^{w^{i-1}i},$$

which completes the induction and the proof. □

3. The Proof of Theorem 2

Fix b not a prime power, and let p be a prime. If p does not divide b, then for each positive integer m, p^n divides $b^{m\phi(p^n)}-1$, a number having exactly one block, and so (2) does not hold. Conversely, suppose p divides b. To prove (2), it suffices to show that for each positive integer k, there is a positive integer n such that every multiple of p^n not ending in the digit 0 base b has $> k$ blocks base b. Assume to the contrary that there exists a positive integer k such that for each positive integer n there is a multiple m_n of p^n which does not end in 0 and which has $\leq k$ blocks. Since $\{m_n\}_{n=1}^{\infty}$ is an infinite sequence, some infinite subsequence S_1 satisfies the condition that every $m \in S_1$ ends in the same non-zero digit d_1 base b. There must now exist an infinite subsequence S_2 of S_1 such that every $m \in S_2$ ends in the same two digits $d_2 d_1$ base b. Continue in this manner so that for $j \geq 2$, S_j is a subsequence of S_{j-1} such that every $m \in S_j$ ends in the same j digits $d_j d_{j-1} \dots d_1$ base b. We now have an infinite sequence $\{d_j\}_{j=1}^{\infty}$, where $d_1 \neq 0$, such that for each positive integer n, there is a multiple m of p^n such that the last n digits of m are $d_n d_{n-1} \dots d_1$ and $B(m,b) \leq k$. Since each such m has $\leq k$ blocks, there are at most $k-1$ integers $j \geq 2$ such that $d_j \neq d_{j-1}$. Hence, there exists an integer $J \geq 2$ and a $d \in \{0,1,2,\dots,b-1\}$ such that $d_j = d$ for every $j \geq J$. Write

$$(d_{J-1} d_{J-2} \dots d_1)_b = p^{n_1} u \qquad \text{and} \qquad b^{J-1} d = p^{n_2} v,$$

where the integers u and v are relatively prime to p. We consider two cases, arriving at a contradiction in each case.

Case 1. $n_1 \neq n_2$.

Since the b–ary number $111\dots11_b$ is congruent to $1 \pmod{b}$, we get that $111\dots11_b \equiv 1 \pmod{p}$. Thus, $(dd\dots dd_{J-1}\dots d_1)_b = b^{J-1}d(11\dots1)_b + (d_{J-1}\dots d_1)_b$ is a sum of two numbers, the first exactly divisible by p^{n_2} and the second exactly divisible by p^{n_1}. Let

$t = \min\{n_1, n_2\}$. Since $n_1 \neq n_2$, we have that

$$p^t \mid\mid (dd\ldots dd_{J-1}\ldots d_1)_b, \tag{11}$$

for any positive number of d's. By the definition of S_{J+t}, there is an $m \in S_{J+t}$ such that p^{t+1} divides m. Also, we may write m in the form $b^{J+t}m' + (dd\ldots dd_{J-1}\ldots d_1)_b$, where m' is a positive integer and $t{+}1$ d's occur to the left of d_{J-1}. The fact that p^{t+1} divides both m and $b^{J+t}m'$ implies p^{t+1} divides $(dd\ldots dd_{J-1}\ldots d_1)_b$, contradicting (11).

Case 2. $n_1 = n_2$.

Let $w = v - u(b-1)$. First, we show that $w \neq 0$. For suppose $w = 0$. Since $d_1 \neq 0$, we deduce that $b^{J-1}d = p^{n_2}v = p^{n_1}v = p^{n_1}u(b-1) = (d_{J-1}\ldots d_1)_b(b-1)$ is not divisible by b. This contradicts the fact that b divides $b^{J-1}d$, since J was chosen ≥ 2. Thus $w \neq 0$. Let t be the nonnegative integer for which p^t exactly divides w. Pick $m \in S_{J+t+n_1+1}$ such that p^{J+t+n_1+1} divides m and write m in the form $b^{J+t+n_1+1}m' + (dd\ldots dd_{J-1}\ldots d_1)_b$, where m' is an integer and $t{+}n_1{+}2$ digits d occur to the left of d_{J-1}. We obtain

$$p^{J+t+n_1+1} \mid (dd\ldots dd_{J-1}\ldots d_1)_b = b^{J-1}d\left(\frac{b^{t+n_1+2}-1}{b-1}\right) + (d_{J-1}\ldots d_1)_b.$$

Hence,

$$p^{n_1}v\left(b^{t+n_1+2}-1\right) \equiv -p^{n_1}u(b-1) \pmod{p^{J+t+n_1+1}}.$$

Since p^{t+1} divides b^{t+n_1+2}, we get that $v \equiv u(b-1) \pmod{p^{t+1}}$. This contradicts the fact that p^t exactly divides $w = v - u(b-1)$. $\square$

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